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A NEW RHOMBIC HEXECONTAHEDRON

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A few months ago Professor G. Hasenjaeger wrote to me asking whether I know who was the first one to discover the interesting polyhedron shown below in Figure 1. This *rhombic hexecontahedron* is an extremely pleasing and geometrically interesting polyhedron: it has all the symmetries of the regular (Platonic) dodecahedron (or icosahedron), its sixty faces are all equivalent under symmetries of the polyhedron, and each face is a "golden rhombus" whose diagonals have lengths in the ratio of the golden section $\tau = (1+\sqrt{5})/2 = 1.618...$. The convex hull of this polyhedron is the regular dodecahedron, and its kernel is the rhombic

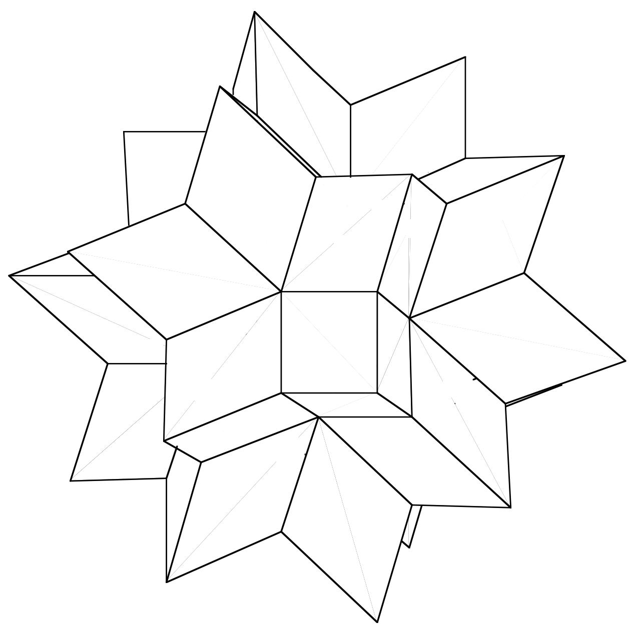

Figure 1. A rhombic hexecontahedron of very pleasing appearance and interesting geometric properties.

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triacontahedron. Moreover, twelve rhombic triacontahedra can fit snugly into the twelve "crowns", for a very pleasing configuration. (The reader not familiar with the names of the various polyhedra and their properties mentioned here should consult some of the excellent books on polyhedra, such as Cundy & Rollett [4], Wenninger [6], or, in particular, the most interesting new text Cromwell [3].) I do not know who was the first to find the polyhedron in Figure 1 — it is not mentioned in the classical "source" of polyhedra Brückner [1], nor in any of the books listed above; however, it appears in the very attractive and unusual book by Holden [5], published in 1971.

While looking at the various sources of images of polyhedra preparing to answer Professor Hasenjaeger's question, I began to wonder whether there exists a polyhedron which is "inverse" of the one in Figure 1, in the following sense. The outermost vertices of the polyhedron in Figure 1 are formed by some of the acute corners of its faces; the obtuse corners meet at vertices that are closer to the center, and the remaining acute corners are still closer to the center. Does there exist a polyhedron with the same symmetries, but in which the outermost vertices are formed by obtuse corners of the faces ? It is not hard to verify that no such polyhedron exists if one insists that it also be free of selfintersections. However, if selfintersections of the kind that occur in the Kepler-Poinsot regular polyhedra are allowed, a rather remarkable polyhedron can be constructed. It is another rhombic hexecontahedron with "golden rhombi" as faces, shown in Figure 2; its faces are all equivalent under symmetries of the polyhedron. Naturally, the illustration in Figure 2 is somewhat misleading, as are all presentations of selfintersecting polyhedra. It shows the parts of the faces that are visible from the outside of the polyhedron, while those parts that are covered by other faces are not shown. This is best seen by considering the three faces which have been highlighted by shading. The visible part of each faces consists of two triangles, which meet only at the outside, obtuse corner. At each such vertex of the polyhedron, five faces meet, arranged in a pentagrammatic way, at vertices which coincide with those of a regular dodecahedron. The acute corners of the faces meet by sixes at vertices which coincide with the vertices of an icosahedron; in the polyhedron, these vertices have nonconvex vertex figures. The hidden obtuse corners of the faces meet by threes at hidden vertices of the polyhedron, each of which is beneath one of the six-valent vertices. One 3-valent vertex, common to the three shaded faces, is indicated in Figure 2 by the dotted lines.

The search of the literature accessible to me did not reveal this "new" rhombic hexecontahedron, and I believe that it has not been described previously. One of the remarkable features of this polyhedron is that its sixty faces come in thirty coplanar pairs; one such pair is shaded in Figure 3. The 30 rhombi obtained by extending the free edges of each coplanar pair of faces form the "great stellated triacontahedron" ([4], p. 126; [7], p. 54; [5], pp. 117, 118). It is possible that the existence of these pairs of faces may be responsible in part for the absence of the polyhedron from the literature. Another possible explanation is that the three rhombic triacontahedra described in the literature (one convex, and two selfintersecting) are usually obtained as polars of suitable uniform polyhedra (see, for example, [2], pp. 102 - 103; [4], p. 123 - 126; [7], pp. 41, 54). However, the "new" rhombic hexecontahedron is not the polar of any uniform polyhedron, since its six-valent vertices are not regular.

It would be interesting to determine whether there are any additional rhombohedra in which symmetries act transitively on the faces. I conjecture that there are none, but I do not see how to prove this.

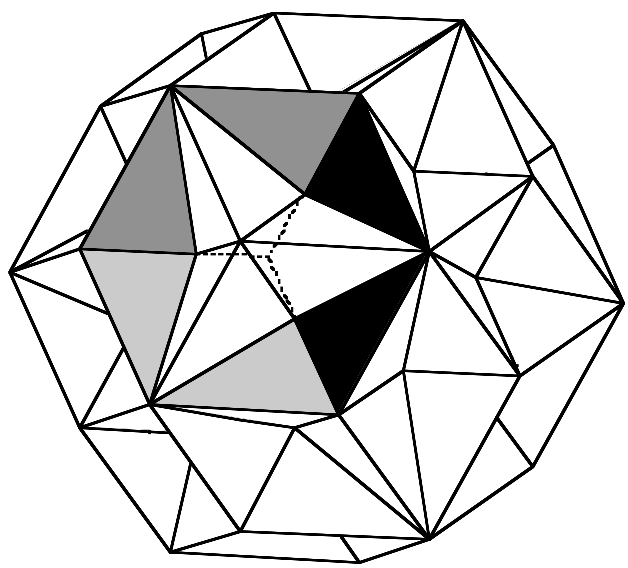

Figure 2. The "new" rhombic hexecontahedron. Three of its faces are shaded; they have a common vertex which is hidden by other faces.

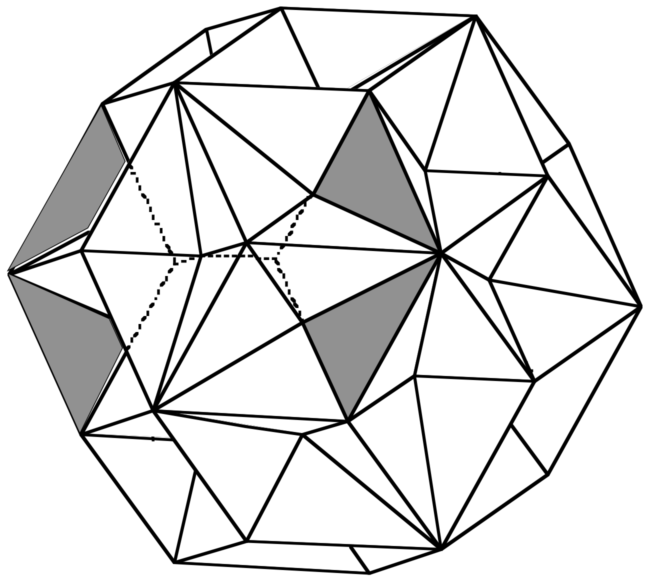

Figure 3. The "new" rhombic hexecontahedron, with a pair of coplanar faces shaded.

References.

[1 M. Brückner, *Vielecke und Vielflache*. Teubner, Leipzig 1900.

[2] H. S. M. Coxeter, *Regular Polytopes*. 3rd ed. Dover, New York 1973.

[3] P. Cromwell, *The Story of Polyhedra*. Cambridge University Press, 1996.

[4] H. M. Cundy and A. P. Rollett, *Mathematical Models*. 2nd ed., Clarendon Press, Oxford 1961.

[5] A. Holden, *Shapes, Space, and Symmetry*. Columbia University Press, New York 1971.

[6] M. J. Wenninger, *Polyhedron Models*. Cambridge University Press 1971.

[7] M. J. Wenninger, *Dual Models*. Cambridge University Press 1983.