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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) the golden section; (2) the geometry of the pentagon and the golden section; (3) meet Mr. Tau; and (4) the golden section, Phyllotaxis, and Wythoff's game. (MP)

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Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

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The Golden Measure

FOREWORD

Among early Greek mathematicians, the Pythagoreans firmly believed that virtually everything could be explained in terms of number. Thus they recognized the relation of number to tone intervals in music. They also insisted that there was always some law involving numbers which characterized works of art, or the creations of Man, as well as living forms, or the creations of Nature.

In applying this line of thought the Greeks leaned heavily upon the notion of a proportion, as in $a/b = c/d$, which they called *analogia*. Arranging a series of equal ratios in "harmonious" sequence led to the Greek idea of symmetry. This is not our modern technical meaning of the word "symmetry" as reflection on opposite sides of an axis or a plane. Rather, in their own words, symmetry denoted "the correlation by measurement between the various elements of a plan or pattern, and between each of these elements and the plan as a whole."

One of the best known of these symmetries was the Law of the Golden Mean, or the Golden Section. We meet this proportion in geometry when we divide a given line segment into mean and extreme ratio, i.e., into two parts, a and b, such that $a:b = b:(a+b)$, where $a<b$. Luca Pacioli, a fifteenth-century Italian mathematician and friend of Leonardo da Vinci, called this the "Divine Proportion"; the celebrated astronomer Kepler called it "one of the two Jewels of Geometry."

The reader can easily verify the fact that the "golden number"

$$\frac{b}{a} = \frac{1+\sqrt{5}}{2} = 1.618\cdots;$$

it is often called ϕ (phi) or τ (tau). Also,

$$\frac{a}{b} = \frac{\sqrt{5}-1}{2} = 0.618\cdots = \frac{1}{\phi}.$$

The Golden Mean appears at many unexpected turns, and as one pursues the ramifications of the golden measure one quickly comes upon a variety of interesting and curious mathematical relationships involving, for example, the regular decagon and the pentagram, Fibonacci numbers, continued fractions, regular polyhedrons, dynamic symmetry, phyllotaxis, whirling rectangles, and spirals.

Thus in Nature, among many plant and animal forms, we find the Golden Measure involved in phyllotaxy, or the leaf arrangement on stems; we find pentagonal symmetry in flowers and marine animals; and there is pentadactylism in vertebrates. Many characteristic measurements of the human body exemplify the Golden Measure: for example, the length of the hand up to the wrist joint multiplied by 1.618 gives the length of the forearm.

Man has also apparently imitated Nature, consciously or otherwise, in many of his art forms — in sculpture, ceramics, painting, and architecture. The use of dynamic symmetry may be seen in early Greek vases, statues and temples, and in some Egyptian pyramids. The use of the Golden Mean is also to be found in examples of Gothic architecture and in classical Renaissance paintings. In modern times, some authorities insist that the principles of dynamic symmetry are useful to printers in the design of books; to advertising men in making their "layouts"; to photographers; in radio design; in the design of reinforced concrete structures; and, generally, in related visual arts and crafts.

Here, then, is a wonderful new world for you to explore — the geometry of art and of life, as exemplified by the Golden Measure.

The Golden Section

R. F. GRAESSER

The division of a line segment into extreme and mean ratio is called the Golden Section. By this we mean so dividing the segment that the whole segment shall have the same ratio to its larger part that its larger part has to its smaller part. In other words, the larger part is the mean proportional between the whole segment and its smaller part. The Golden Section has some interesting geometric implications. The same is true if we treat it algebraically. I shall devote about half of this paper, then, to the mathematical aspects of the Golden Section, but I shall use only very simple mathematics. In the remainder of the paper I shall try to explain why it is called the Golden Section. Most of the writers on the Golden Section have been interested in this latter aspect; i.e., its association with metaphysics, magic, natural science, and the graphic and plastic arts.

Geometry of the Golden Section

In our school days we learned a ruler and compass construction for the Golden Section. I should like to show one of these constructions. Let AB be the

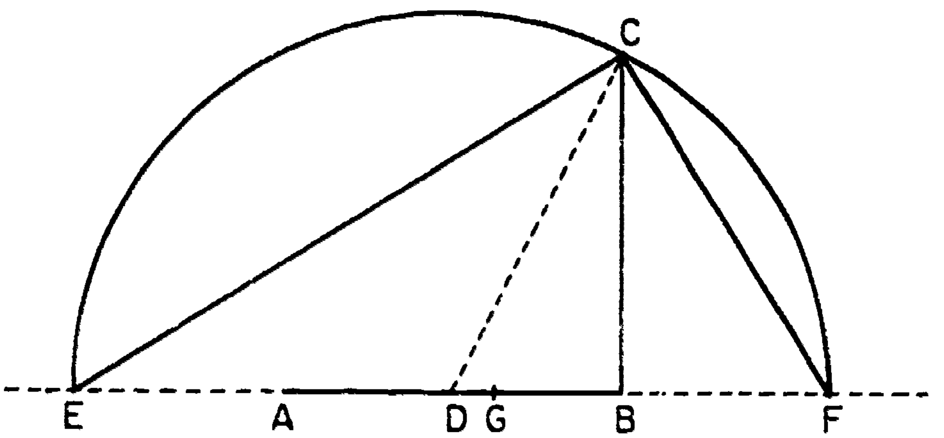

Fig. 1

segment to be divided by the Golden Section (Fig. 1). At B erect BC equal and perpendicular to AB. Let D be the center of AB, and with DC as a radius draw an arc cutting AB produced in E and F. Then BF laid off on AB gives G, the Golden Section. To prove this, recall that the perpendicular to the diameter from a point on the circumference is the mean proportional between the segments of the diameter; that is, $EB/BC = BC/BF$. Subtracting unity from both members, we have $(EB-BC)/BC = (BC-BF)/BF$. Hence, substituting equals, $AG/AB = GB/AG$, or, inverting, $AB/AG = AG/GB$.

If we take AB as the radius of a circle, then AG can be stepped off as a chord exactly ten times. In other words, AG is the side of an inscribed decagon. In order to establish this, lay off AG along AB (Fig. 2). Draw CG. Then $AC/CB = CB/GB$ by the Golden Section, and angle 1 is equal to angle 2 as base angles of an isosceles triangle. Then $\triangle ACB$ is similar to $\triangle CGB$, and hence angle 3 is

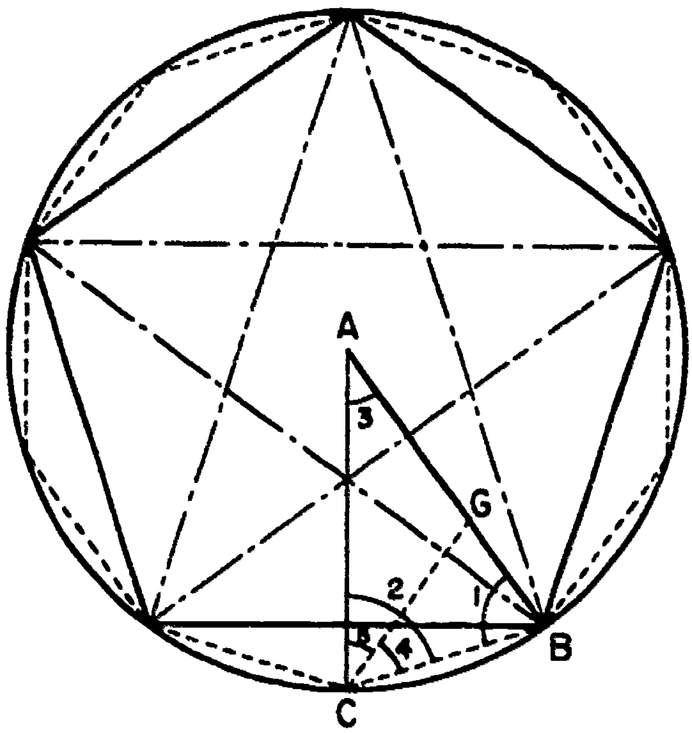

Fig. 2

equal to angle 4 being corresponding angles of similar triangles. Also, angle 5 equals angle 3 since they are base angles of an isosceles triangle. Hence, the sum of the angles of $\triangle ABC$ is equal to five times angle 3, so that angle 3 is equal to 36°. Thus, CB is the side of a regular inscribed decagon.

Return to Figure 1. I wish to state some facts which I shall not take time to prove. Consider a circle of radius AB. Then BF is the side of a rectangular inscribed decagon, BC is the radius or also the side of a regular inscribed hexagon, and CF is the side of a regular inscribed pentagon. From the Pythagorean theorem we have: *Given a circle, the sum of the square of the side of the regular inscribed decagon and the square of the side of the regular inscribed hexagon equals the square of the side of the regular inscribed pentagon.* This rather pretty theorem is due to Eudoxus, a contemporary of Plato, in the fourth century before Christ. Proclus, whose works inform us concerning the history of Greek geometry, says that Eudoxus "greatly added to the number of the theorems which Plato originated regarding the section," meaning, of course, the Golden Section. Again, EC is the side of the regular inscribed star pentagon or pentagram; BF is the Golden Section of CB; CB is the Golden Section of EB; and finally CF is the Golden Section of EF.

Later we want to discuss the so-called Golden Rectangle so perhaps I should define it before leaving geometric considerations. If the sides of a rectangle are in the ratio of the Golden Section, then we have a Golden Rectangle. Such a rectangle can be divided into a square and another Golden Rectangle which later may be again divided into a square and another Golden Rectangle, and so on ad infinitum. For this reason a Golden Rectangle forms in a sense a complete unit.

ALGEBRA OF THE GOLDEN SECTION

We now consider the Golden Section algebraically. Let a line be divided in the Golden Section, and let x be the ratio of the longer segment to the whole line, the so-called ratio of the Golden Section. If we let the length of the line be unity when x is the longer segment, then $1 - x$ is the shorter segment, and $1/x = x/(1-x)$, or $x^2 + x = 1$. Solving this equation we find the ratio of the Golden Section to be $x = \frac{1}{2}(-1 \pm \sqrt{5})$. Since x is positive, we select the plus sign, and $x = \sqrt{1.25} - 0.5 = 0.618$, approximately. If we write $x^2 = 1 - x$ so that $x = \sqrt{(1-x)}$, and then replace x in the right member by its value, $\sqrt{(1-x)}$, we obtain $x = \sqrt{[1 - \sqrt{(1-x)}]}$. Continuing this process indefinitely we secure the value of x as an infinite radical,

$$x = \sqrt{1 - \sqrt{1 - \sqrt{1 - \cdots}}}.$$

The validity of this result I shall not stop to prove, but shall present it only as a formal result. This seems to be the simplest infinite radical obtainable, and it is curious that this should give the Golden Section.

Starting again with $x^2 + x = 1$, we can write

$$x(x+1) = 1, \text{ or } x = 1/(1+x).$$

Replacing the x in the right member by its value, $1/(1+x)$, and continuing this process we obtain

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}},$$

which for convenience we write

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots.$$

This is called an infinite continued fraction. Again it is hard to imagine any simpler such fraction, and it is striking that the simplest infinite continued fraction, like the simplest infinite radical, should give us the Golden Section. If as a sequence of approximations to this infinite continued fraction we take

$$\frac{1}{1}.\quad \frac{1}{1+}\,\frac{1}{1}.\quad \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1}.\quad \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1}.\cdots$$

and simplify these complex fractions, we obtain $1/1$, $1/2$, $2/3$, $3/5$, etc. The nth fraction may be obtaned by adding unity to the $(n-1)$st fraction and taking its reciprocal. This sequence of fractions is technically known as the successive convergents of the infinite continued fraction, but I am not assuming that my readers know anything of the theory of continued fractions. Another law of formation for these convergents is readily seen. Add two consecutive numerators for the numerator of the next fraction, and do the same with the two consecutive denominators to obtain the next denominator. The sequence of numerators is the same as the sequence of denominators; viz., 1, 1, 2, 3, 5, 8, 13, $\cdots$. Each term of the sequence is obtained by adding the two preceding terms. Any term divided by the one following gives an approximation to the Golden Section, and the further out we go the better the approximation. This sequence is well known and is named for the Italian who is considered by some to be the greatest mathematician of the Middle Ages, Leonardo Fibonacci, sometimes called Leonard Pisano or Leonardo the Pisan. It is called the Fibonacci series, also sometimes the Lame series. Among the miscellaneous arithmetical problems in his *Liber Abaci*, Fibonacci gives the following: How many pairs of rabbits can be produced from a single pair in a year if it is supposed (1) that every month each pair begets a new pair which, from the second month on, becomes productive, and (2) no deaths occur? This problem leads to Fibonacci's series. We shall notice this series further in our applications.

THE GOLDEN SECTION IN HISTORY AND PHILOSOPHY

What we have just been discussing seems to me to be an interesting topic in elementary mathematics, but there is nothing mysterious nor miraculous about it. Yet, from time immemorial it has been given an enchantment of mystery. It has been called the Golden Section, the Golden Mean, the Divine Proportion, the Divine Section. Perhaps we can now understand a little of the reason for this. Mankind seems possessed of an innate urge to associate the mystic and the supernatural with phenomena that he cannot explain. From this urge arises the significances attributed to mathematical forms. The belief in such significances commenced with the ancient Babylonians and Egyptians and became a leading principle for the explanation of the universe with the members of the Pythagorean brotherhood.

Pythagoras of Samos is perhaps the most picturesque and interesting figure in Greek mathematics. He lived in the sixth century B. C. and established at Crotona, in Magna Grecia, a secret brotherhood, the so-called Pythagoreans, which became the model of secret societies from that day to this. The Pythagoreans persisted as an organization for nearly two centuries. It was a society for the pursuit of knowledge, for the study of mathematics, philosophy, and science. The Pythagoreans discovered the Golden Section. Their symbol was the pentagram or five-pointed star, whose connection with the Golden Section we have already seen, for the pentagram cannot be constructed without the use of the Golden Section. The pentagram is not only a part of the seal of Kappa Mu Epsilon, but it is also the star in the American flag, the star of the P.E.O. society, of the Eastern Stars, etc. Shades of Pythagoras!

The Pythagoreans believed that the essence and explanation of the universe lay in number and form, and that the universe was the incarnation of all wisdom and beauty. The explanation of beauty was to be found in simple ratios. This idea was later greatly strengthened by the Pythagoreans' discovery that the harmony of musical sounds depends upon simple numerical ratios among the vibration frequencies of the notes. Then what would be more natural than to seek the explanation of the beauty of proportion and form in simple ratios? What held for the ear ought also to hold for the eye. The Pythagoreans sought this explanation in the ratio of the Golden Section, and this has been going on nearly ever since. The *Art Digest* published in New York City has had several articles in the last few years on the Golden Section. Witness also the books of Mr. Jay Hambidge, Yale University Press, entitled, *Principles of Dynamic Symmetry, The Parthenon and other Greek Temples and their Dynamic Symmetry,* and *Dynamic Symmetry of the Greek Vase.* The Golden Section is a special case of what Mr. Hambidge calls dynamic symmetry.

According to the Pythagoreans, if divine harmony was to be realized on this imperfect earth then harmonious ratios must exist among earthly things. To find these harmonious ratios we must seek geometric figures making the impression of greatest perfection. These are the regular figures. Regular figures are either plane or solid, and between these two kinds of regular figures there is a strange difference. Of the regular plane figures, the regular polygons, there is an infinite number, while of the regular solid figures, the regular polyhedrons, there are but five, the so-called Platonic bodies. Plutarch tells us that the Pythagoreans believed that these five regular polyhedrons were fundamental forms in the structure of the universe. There were four elements (instead of 92) in the material world; viz., earth, fire, air, and water. The nature of these elements depended on their forms. The smallest constituent particle of earth (its atom) was hexagonal or cubical, the atom of fire was a tetrahedron, that of air an octahedron, and that of water an icosahedron. These polyhedrons are constructed of comparatively simple figures, the square and the equilateral triangle which are easily obtained. The first three were known to the Egyptians. The dodecahedron with its twelve pentagonal faces depending on the Golden Section was much harder to obtain. When discovered it was taken as the symbol of the supernatural, the heavenly domain. As it contains the Golden Section, this ratio was supposed to be the dominating one in the realm of the spirit. And the star pentagram has played an important role in magic and as a talisman ever since. In various European regions it is used as a protection against evil spirits and nightmares. In Goethe's *Faust,* the devil, Mephistopheles, is prevented from escaping from Faust's chamber by a pentagram on the threshold of the door.

We find the Golden Section appearing again with the revival of Platonic philosophy in the early Renaissance. In the latter half of the fifteenth century one of the most important European writers on mathematics was Luca Paciola. Being a monk he was also known as Fra Luca di Borgo. His *Suma* was the first printed work dealing with arithmetic and algebra. It had a wide circulation and much influence. Paciola also wrote *Divina Proportione,* the first work devoted entirely to the Golden Section. Paciola found miraculous attributes in the Golden Section which alone could belong to God. I quote,

"The first is, that this proportion is unique. It is not possible to derive other proportions or variations from it. According to both theological and philosophical doctrine this unity is an attribute of God alone. The second divine property is that of the Holy Trinity. As the Father, Son, and Holy Ghost are one and the same, likewise must one and the same ratio obtain among the three quantities no more and no less. The third attribute is that just as God cannot be defined or made comprehensible to us through words, neither can this ratio be expressed by a rational number but remains always secreted and hidden and is called by mathematicians an irrational. Fourthly, as God cannot change and is the same in all his parts and the same everywhere, so is our proportion always the same and unchanging be it evident in large or small quantities nor can it be understood in any other way. The fifth attribute can with justice be added to the preceding; viz., as God creates divine virtues (the so-called fifth element) and by means of this creates the four other elements, earth, water, air, and fire, and by means of these gives existence to every other thing in nature, so, according to Plato in his *Timaeus*, does our divine proportion give formal existence to Heaven itself, as it gives to Heaven the form of a dodecahedron which cannot be constructed without our proportion."

Such metaphysical significance attributed to the Golden Section will meet with little favor in our modern eyes. It smacks too much of superstition and too little of science. As H. E. Timerding, a recent German author on the Golden Section remarks,

"The Golden Section has again and again enticed men to seek the road into the enchanted land of metaphysics."

The Golden Section in Nature

We shall not, however, be engaged with a metaphysical proposition if we seek to establish the Golden Section as a norm in nature. Let us consider the subject of phyllotaxy, the system of leaf arrangement in plants. It has been found that the seemingly innumerable leaf arrangements can be reduced to comparatively few oft recurring cases. We have the whorled and the spiral arrangements. In the former the leaves in a given plane are equally spaced in a whorl or verticil about the stem or stock. We may think of the spiral arrangement as the result of displacing the leaves of the whorl vertically. The leaves are then no longer arranged in a circle but are equally spaced along a so-called genetic spiral or more properly a cylindrical helix. The spiral or helix winds around the stem of the plant. In this case one obtains a fraction representing the leaf arrangement by wrapping a string about the stem to represent the helix on which the leaves are located. Any leaf is numbered 0. The others then proceed 1, 2, 3, etc., along the string until a leaf is reached which is directly above the leaf numbered zero. Suppose this leaf has the number n. Then n is the denominator of the fraction. The numerator is given by the number of complete revolutions of the string between the zeroth and nth leaves. For example, if the leaf numbered 8 is directly above the zeroth leaf and the string has encircled the stem three times between the zeroth and the eighth leaves then the fraction expressing the leaf arrangement is 3/8. Proceeding in this way we find that the two ranked leaves of all grasses, Indian corn, basswood, and the horizontal branches of the elm and other trees have the fraction one half. One third belongs to all sedges, alder, birch, white hellebore; two fifths (a very common ratio) applies to the willow, rose, drupe (i.e., plum, cherry, apple, apricot, peach, poplar, almond); three eighths to cabbage, asters, hawkweed, holly, plantain; five thirteenths to needles of various conifers, houseleek, and to mulleins; eight twenty-firsts to the scales of spruce and fir cones; and thirteen thirty-fourths belongs to scales of cones of the *pinus larico.*

Consider this sequence of fractions, 1/2, 1/3, 2/5, 3/8, 5/13, 8/21, and 13/34. The law of formation is obvious: Add any two numerators for the next numerator; add any two denominators to form the next denominator. Higher members of the sequence belong to flowers and involucres of the composites, such as the sunflower and the thistle, and in both leaves and cones of the pine family.

If the string is wrapped about the stock in the opposite direction (to left instead of to the right) then in place of the arrangement quotient m/n one obtains the arrangement quotient $(n-m)/n$, or the series 1/2, 2/3, 3/5, 5/8, 8/13, 13/21, etc. The law of formation for this sequence is the same as for the former sequence. This latter sequence we have seen converges to the ratio of the Golden Section. It can easily be shown that the former sequence also converges to the same limit. This regularity of leaf arrangement is sometimes called Ludwig's Law. It is so striking that it was observed as early as 1834, and it led to mystical speculations concerning the so-called "spiral tendency of vegetation." It might be noted also that the great German poet, Goethe, was much fascinated by this subject.

A German writer, Zeising, about the middle of the last century, has perhaps the most successfully and consistently supported the Golden Section as a relationship of universal occurrence in natural forms. He says:

> "The fundamental principle underlying all forms of Nature and Art which approach beauty and perfection (*Totalität*) is to be found in the relationship of the Golden Section and the Golden Section has been from the beginning (*Uranfang*) the highest goal and ideal of all forms and ratios, the cosmic as the individual, the organic as the inorganic, the acoustic as the optical, but which has been realized most completely in the human figure."

Here are a few occurrences of the Golden Section in the ideal human figure:

1. The division of the stature by the waist line is a Golden Section.

2. The point of the middle finger, when the arms hang naturally, divides the height in the Golden Section.

3. The well-known rule that the forehead, nose, and lower portion of the face should be equal can be supplemented by the statement that the lower portion of the face should be divided by the mouth in the Golden Section.

4. The eyebrows should divide the whole height of the head in a Golden Section.

Zeising gives other instances of its occurrence in the human form. The title of his book translated is *New Theory of Proportions in the Human Figure*, (Leipzig, 1854).

The Golden Section in Art

Does the Golden Section give us a division which is pleasing to the eye? H. E. Timerding (*Der Goldene Schnitt*, Berlin, 1937) says that this question is to be answered with an unconditional yes. From time immemorial it has been a rule that pictures with pronounced horizons should be divided in a Golden Section by those horizons. It cannot be said, however, that this rule has been universally used; it has even been deliberately and intentionally ignored by the realists of the last century.

The Golden Rectangle has proportions particularly pleasing to the eye. This was first tested experimentally by psychologist Fechner in 1876. Fechner's procedure was to place ten different rectangles before a person, and to ask that person to choose the rectangle which had for him the most pleasing proportions. Among a large number of persons, the Golden Rectangle was the one most frequently chosen; it seemed to be the norm about which all the choices clustered.

The ratio 5/8 is one of the convergents approximating the ratio of the Golden Section. A beautiful example of the use of this approximate value is found in the division of the height of a certain urn from the Salem cathedral. The urn consists of a base, bowl, cover, and cover ornament. If the bowl is divided into eight parts, the base and the cover each measure five such parts and the cover ornament three parts. Thus, the cover and the bowl are in the ratio of 5/8, while the cover and the cover ornament together bear a ratio of 8/13 to the bowl and base together. The ratio 8/13 is the next convergent in the sequence of convergents approaching the ratio of the Golden Section. The effect of these proportions is exceptionally pleasing.

Wherever the requirements of stability or other reasons do not enter, the modern architect determines the proportions of his buildings from his own artistic or aestetic sense, and is glad of this freedom. The ancient architect sought for fixed rules that would eliminate individual judgment. They were looked upon as incarnations of divine order; to them were ascribed wonder-working powers. The measurements and proportions of the temples of the Egyptians and Babylonians had sacred significances. Division by the Golden Section was such a rule. The Golden ratio was used in the façade of the Parthenon and in façades and floor plans of other Greek temples. A passage from Herodotus leads us to believe that the Great Pyramid of Gizeh, tomb of Cheops, was constructed with the area of each face equal to the altitude. Modern measurements confirm this. If it be true, then the relations between the altitude, slant height, and side of the base can be expressed with the Golden Section.

The literature of the Golden Section is surprisingly extensive. Literally dozens of books, pamphlets, and articles have been written about it. It is said to have other connections which I have not had time to verify, such as a relationship with the periodic table in chemistry and with the distribution of prime numbers. So this paper is far from exhaustive, but constitutes merely an introduction to the subject.

The Geometry of the Pentagon and the Golden Section

By H. v. BARAVALLE

The Geometry of the Pentagon has become almost a foster-child besides other chapters of geometry, as for instance the geometry of the triangles or of the quadrilaterals. Considering terminologies, we find the whole field of trigonometry deriving its name from the geometry of triangles and the "quadrature of areas" (quadratum = square) from the regular representative of the quadrilaterals, all units for measuring areas being also squares.

The characteristic elements of the geometry of the pentagon are neither related to the trigonometric reproduction of forms nor to measuring areas. The regular pentagon, however, and especially the regular stellar pentagon formed by its diagonals, the pentagram, are used today in the flags and emblems of the mightiest nations and had a similar use already two and a half thousand years ago when the pentagram was the emblem of the Pythagorean School. It is the particular appeal of the pentagon to the sense of beauty, and the unique variety of mathematical relationships connected with it which are the characteristics of the geometry of the pentagon. This geometry is therefore particularly fit to simulate mathematical interest and investigations. Outstanding among the mathematical facts connected with the pentagon are the manifold implications of the irrational ratio of The Golden Section.

The first figure shows a regular pentagon, and inscribed in it the pentagram formed by its diagonals. The central area of the pentagram forms against a regular pentagon in reverse position. In this pentagon another pentagram has been inscribed. The total diagram of Figure 1 contains three horizontal lines, among them the base of the pentagon. Due to symmetry there is a group of three parallel

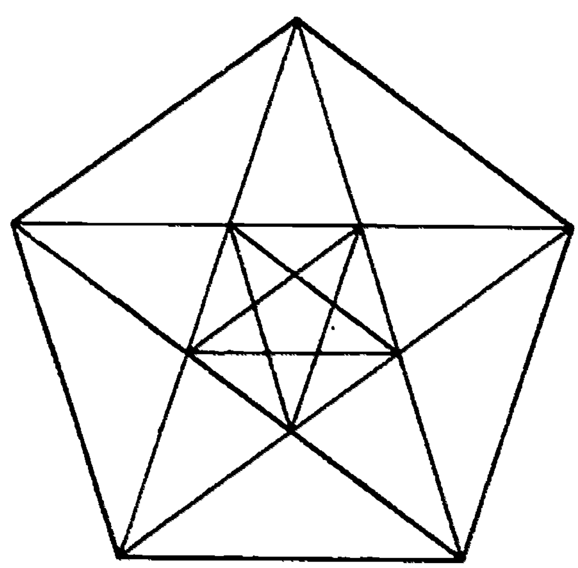

Fig. 1. Pentagon with inscribed pentagrams.

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lines coordinated in the same way to every one of the five sides of the pentagon. These parallel lines form between them two types of rhombi, smaller and larger ones. One of the smaller and one of the larger rhombi is marked in Figure 2 and Figure 3. A diagonal divides a rhombus into two congruent isosceles triangles. By folding and bending over the marked rhombus in Figure 2 along its horizontal diagonal we shall always reach exactly the opposite vertex of the central area. By cutting a pentagram out of paper, then bending over its outer parts and holding the paper before a light will make the inner pentagram appear in the central area. Folding the marked rhombus of Figure 3 in the same way along its horizontal diagonal will bring on both ends of this diagonal two angles to coincidence into which the diagonals divide the interior angles of a regular pentagon. Consequently, a pentagram trisects the interior angles of a circumscribed pentagon. If one of the partial angles is denoted ϕ the angles of the large rhombus of Figure 3 are 2ϕ; 3ϕ; 2ϕ; 3ϕ and those of the small rhombus in Figure 2: ϕ; 4ϕ;

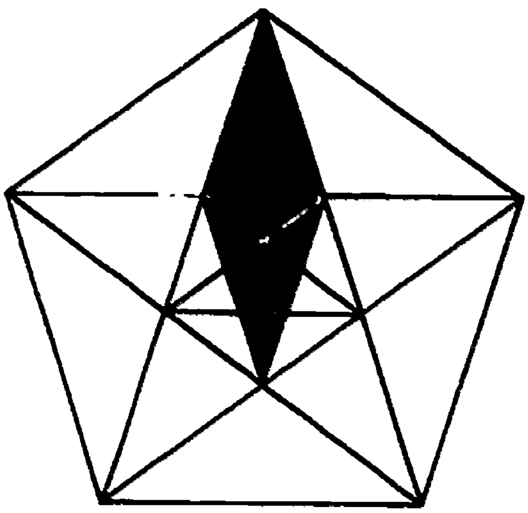

Fig. 2. Smaller rhombus contained in the pentagon-pentagram diagram.

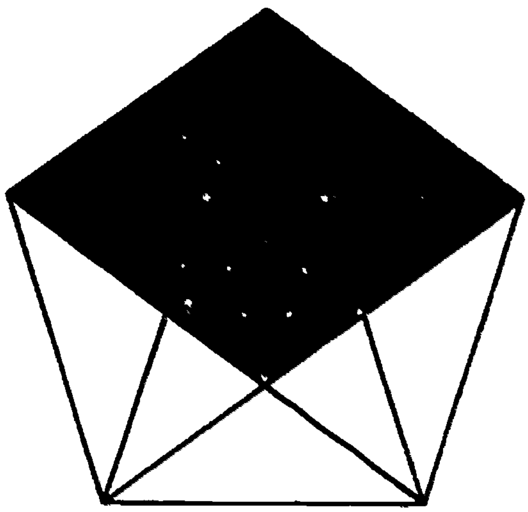

Fig. 3. Larger rhombus contained in the pentagon-pentagram diagram.

ϕ; 4ϕ. The sum of the angles of any of the two rhombi being 10ϕ ($\phi = 360°/10 = 36°$). All angles which come in the diagrams of the Figures 1-3 are of the sizes: 36°; 72°; 108°; 144°; 180°; 216°; 252°; 288°; 324°; and 360°; thus forming an arithmetic progression with a difference of 36°.

The line segments in Figure 1, including both the partial segments between points of intersection and also their sums, are of six different sizes. The largest are the diagonals of the large pentagon. Counting them as of size No. 1 and then continuing with numbering until we come to size No. 6 with the sides of the inmost pentagram, the various sizes appear in the following quantities:

Line segments of size No. 1 come in the diagram	5 times:
Line segments of size No. 2 come in the diagram	15 times:
Line segments of size No. 3 come in the diagram	15 times:
Line segments of size No. 4 come in the diagram	15 times:
Line segments of size No. 5 come in the diagram	10 times:
Line segments of size No. 6 come in the diagram	5 times:
Total amount of line segments	65

In Figure 4, three isosceles triangles which are contained in the pentagram are marked through shading. The sides of the largest one are of the sizes 1; 1; 2. The sides of the middle sized triangle are: 2; 2; 3 and those of the smallest triangle: 3; 3; 4. In the complete diagram of Figure 1 further triangles of still the same form are contained which are smaller and have sides of the sizes 5 and 6. The similarity of all these triangles establishes the following equations of the ratios of the line segments:

$$\frac{\text{segm 1}}{\text{segm 2}} = \frac{\text{segm 2}}{\text{segm 3}} = \frac{\text{segm 3}}{\text{segm 4}} = \frac{\text{segm 4}}{\text{segm 5}} = \frac{\text{segm 5}}{\text{segm 6}}.$$

Therefore, the six sizes of line segments are members of a geometric progression. Whereas the angles in the pentagon-diagram make up an arithmetic progression, the line segments form a geometric progression. Denoting x as the ratio of this geometric progression and "a" for the length of a line segment of size 1, we have:

segment size No. 1 = a
segment size No. 2 = ax
segment size No. 3 = ax^2
segment size No. 4 = ax^3
segment size No. n = ax^{n-1}

The value of x can be found through the fact that one line segment of size 3 and one of size 2 make up a pentagram side of size 1. Therefore $ax^2 + ax = a$ or $x^2 + x = 1$. Solving the quadratic for x we get $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 1}$. The positive root is

$$-\frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{\sqrt{5} - 1}{2} = 0.61803398875 \cdots$$

which is the number of the Golden Section: G.

The expression $G = -\frac{1}{2} + \sqrt{\frac{1}{4} + 1}$ suggests the construction of a right triangle with the legs of one-half and of one unit. From its hypotenuse $\sqrt{\frac{1}{4} + 1}$ we subtract one-half unit and obtain the length of G units. Starting

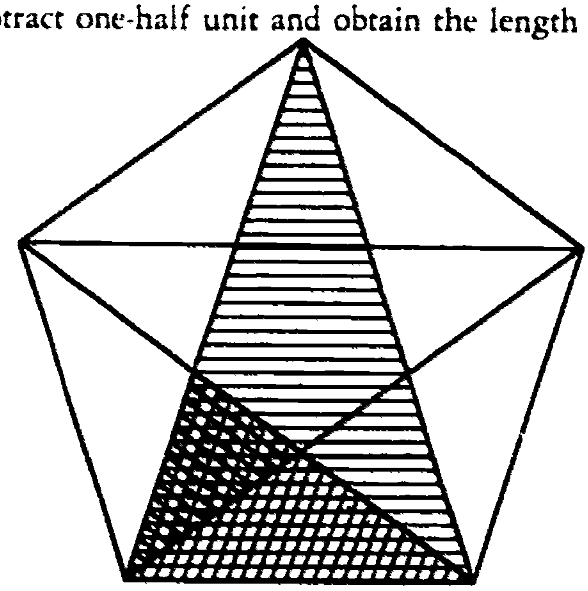

Fig. 4. Similar triangles in a pentagram.

the construction with any given line segment, one obtains the original length multiplied by the factor G. All the 65 line segments of the diagram in Figure 1 can thus be obtained from the large pentagram side by repeated application of the described construction.

Other lengths connected with the pentagram, for instance, the relative altitudes of its vertices can also be expressed through G. This can be done by applying the theory of complex roots of an equation and of the complex-number plane. The geometry of a regular n sided polygon reappears in the nth roots of unity. For the pentagon, we use a 5th root of unity corresponding to the equation $x^5 - 1 = 0$. One of the roots being 1, we get through synthetic division:

$$\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ & 1 & 1 & 1 & 1 & 1 & \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & \end{array}$$

and obtain the quartic equation $x^4 + x^3 + x^2 + x + 1 = 0$. Applying to it the methods of reciprocal equations, we first divide by x^2 and get:

$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0.$$

Then we regroup:

$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0.$$

Substituting y for $x + (1/x)$ and therefore y^2 for $x^2 + 2 + (1/x^2)$ or $y^2 - 2$ for $x^2 + (1/x^2)$ the equation takes on the form $y^2 + y = 1$ which is again the characteristic equation which has G as its positive root. The two roots

$$y = \frac{-1 \pm \sqrt{5}}{2}$$

can be expressed through G as

$$\frac{-1 + \sqrt{5}}{2} = G \text{ and } \frac{-1 - \sqrt{5}}{2} = -(G + 1).$$

The values for x are obtained by solving the equations:

$$x + \frac{1}{x} = \frac{-1 + \sqrt{5}}{2} = G \quad \text{and } x + \frac{1}{x} = \frac{-1 - \sqrt{5}}{2} = -(1 + G)$$

By multiplying the first equation with x we get: $x^2 - Gx = -1$. Its roots are:

$$x = \frac{G}{2} \pm \sqrt{\frac{G^2}{4} - 1}.$$

By multiplying the second equation with x we get: $x^2 + (1 + G)x = -1$ and the roots are

$$x = -\frac{1 + G}{2} \pm \sqrt{\frac{(1 + G)^2}{4} - 1}.$$

As $G < 1$ both $G^2/4$ and $(1+G)^2/4$ are smaller than 1 and consequently all the four roots are complex. The five roots of the original equation $x^5 - 1 = 0$ are:

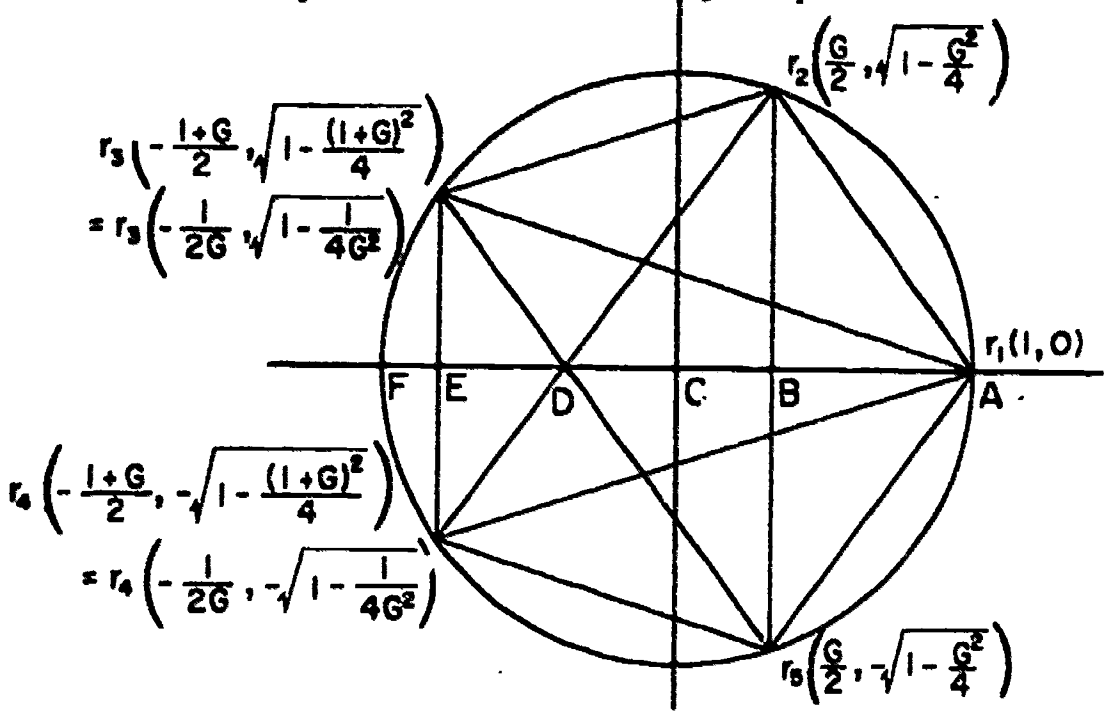

Fig. 5. The ratio G in the roots of the equation: $x^5 - 1 = 0$.

$$r_1 = 1$$

$$r_2 = \frac{G}{2} + \sqrt{1 - \frac{G^2}{4}} \cdot i$$

$$r_3 = -\frac{1+G}{2} + \sqrt{1 - \frac{(1+G)^2}{4}} \cdot i$$

$$r_4 = -\frac{1+G}{2} - \sqrt{1 - \frac{(1+G)^2}{4}} \cdot i$$

$$r_5 = \frac{G}{2} - \sqrt{1 - \frac{G^2}{4}} \cdot i$$

The expressions for r_3 and r_4 can be simplified through the relation $1 + G = 1/G$ which derives itself from the fundamental equation $x^2 + x = 1$ through dividing it by x: $x + 1 = 1/x$. As G is its positive root we have $G + 1 = 1/G$. Therefore:

$$r_3 = -\frac{1}{2G} + \sqrt{1 - \frac{1}{4G^2}} \cdot i;$$

$$r_4 = -\frac{1}{2G} - \sqrt{1 - \frac{1}{4G^2}} \cdot i.$$

The Figure 5 shows the location of the five roots on the complex number plane. They lie on the circle with the radius of one unit. The abscissae of the five points are the real parts of the roots:

$$1;\ \frac{G}{2};\quad -\frac{1+G}{2} = -\frac{1}{2G};\quad -\frac{1+G}{2} = -\frac{1}{2G};\ \frac{G}{2}$$

and the ordinates the imaginary parts:

$$0;\quad \sqrt{1-\frac{G^2}{4}};$$

$$\sqrt{1-\frac{(1+G)^2}{4}} = \sqrt{1-\frac{1}{4G^2}};$$

$$-\sqrt{1-\frac{(1+G)^2}{4}} = -\sqrt{1-\frac{1}{4G^2}};$$

$$-\sqrt{1-\frac{G^2}{4}}.$$

From the abscissae we obtain the ratios of the line segments in a pentagram along its axis of symmetry to the radius of the circumscribed circle. They are all simple linear functions of G.

$$AB = 1 - \frac{G}{2}$$

$$BC\ .\ \frac{G}{2}$$

$$CD = BD - BC = AB - BC = 1 - \frac{G}{2} - \frac{G}{2} = 1 - G$$

$$DE = CE - CD = \frac{1+G}{2} - (1 - G) = \frac{3G}{2} - \frac{1}{2}$$

$$EF = 1 - \frac{1+G}{2} = \frac{1}{2} - \frac{G}{2} = \frac{1-G}{2}.$$

Also the radius of the circle which is inscribed in the pentagon is a simple linear function of G. The radius equals

$$EC = \frac{1+G}{2} = \frac{1}{2G}.$$

Thus the ratios of the segments of the pentagram to the pentagon-sides s being expressed through G, and the ratios of its segments along an axis of symmetry to the radius of the circumscribed circle R being also expressed by G, all that remains is to tie the two groups together. This will be achieved through finding the ratio between s and R. The answer is contained in the ordinate for r_1 in Figure 5. R being the radius of the circumscribed circle, half the side of the pentagon is

$$\frac{s}{2} = \sqrt{1 - \frac{1}{4G^2}} \cdot R \text{ or } \frac{s}{R} = 2\sqrt{1 - \frac{1}{4G^2}}$$

which again expresses itself through G.

The number G is also the ratio of areas which are formed between the pentagon and the pentagrams. The sequence of areas which is marked in the five diagrams of Figure 6 constitutes a geometric progression with the ratio G. The

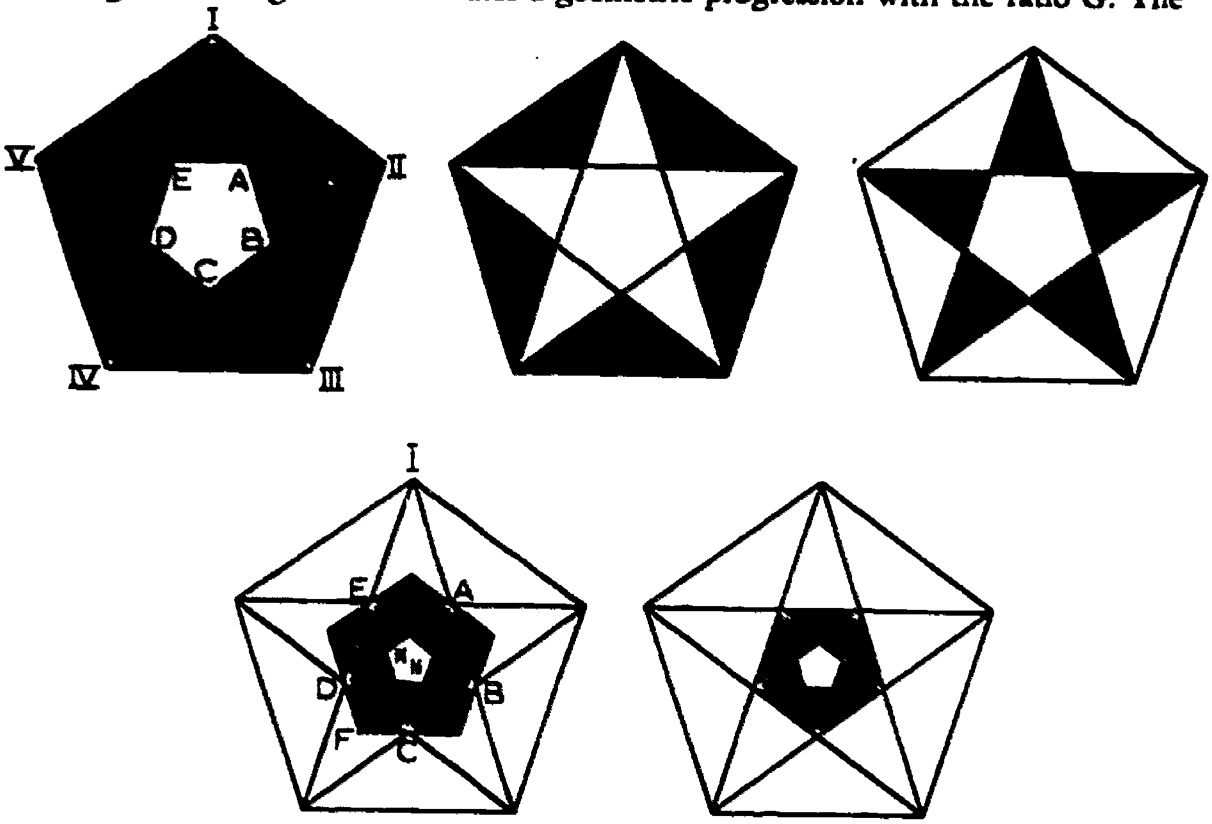

Fig. 6. Areas forming a geometric progression with the ratio G.

ring-shaped area marked in the first diagram (upper row left is composed of five times the area of $\triangle$I IIE. Taking IIE as the base and diminishing it by multiplying with G while keeping the altitude of the triangle unchanged, we get $\triangle$IIAI which is one of the five marked triangles of the second diagram (upper row middle). In comparison to $\triangle$IIAI the base of $\triangle AE$I is again reduced by G while its altitude remains the same. Five times the triangle AEI equals the marked triangles of the third diagram. In order to take the next step to the fourth diagram (lower row left) we consider again $\triangle AE$I which is congruent to $\triangle AEC$. Taking AC as its base and reducing it by the ratio G to CH without changing the altitude we get $\triangle HCE$ which is congruent to $\triangle CED$. By subtracting from the triangle *CED* the triangle *EDK* and adding instead the congruent triangle *CDF* we obtain the quadrilateral *CKDF* which taken five times makes up the marked area of the fourth diagram. This area, therefore, represents the third diagram's area

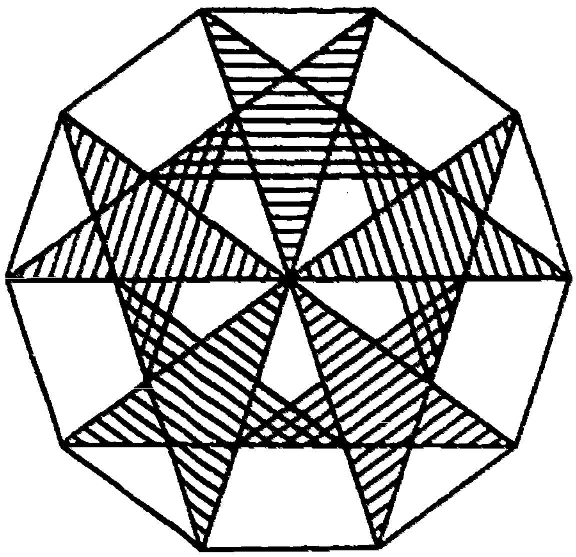

Fig. 7. The ratio G in the regular decagon.

reduced by *G*. Finally, we take up once more the triangle *CED* which equals one-fifth of the marked area of the fourth diagram. Reducing its base *CE* by the ratio *G* without changing the altitude, we obtain the triangle *CKD* which taken five times makes up the marked ring-shaped area of the fifth diagram. Denoting the total marked area of the first diagram with A the marked areas of the successive diagrams form the geometric progression: A; AG; AG^2; AG^3; AG^4. The last ring-shaped area occupies the same place within the inner pentagram as the first ring-shaped area in the outer one. The ratio between the two rings is therefore G^4 which checks with a previously found result that corresponding sides of the two pentagons have the ratio G^2. The white area left over in the middle of the last diagram is also G^4 times the white area in the middle of the first diagram.

The part the ratio *G* plays in a pentagram also carries over into the domain of the regular decagon: *G* is the ratio of the side of a regular decagon to the radius of its circumscribed circle. Figure 7 shows a regular decagon. Its vertices are joined with the center and thus the angle of 360° around the center is divided into ten equal angles of 36°. Ten pentagrams can be placed around the center to fit in these spaces. Every second of them is drawn in Figure 7 and marked through shading. The sides of these pentagrams equal the radius of the circle circumscribed about the decagon, and the sides of the pentagons drawn around these pentagrams equal the decagon side. The ratio between these two sizes is *G*. The usual construction of the side of a regular decagon to be inscribed in a given circle is an application of *G*.

In solid geometry *G* reappears in the geometry of the pentagondodecahedron and of the icosahedron which contain pentagons as their faces or as plane sections.

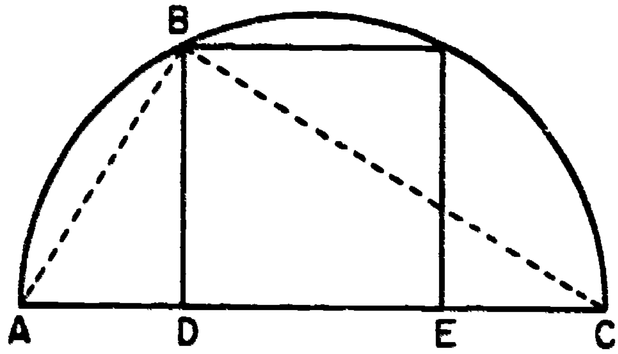

Fig. 8. The ratio of G in a square inscribed in a semi-circle.

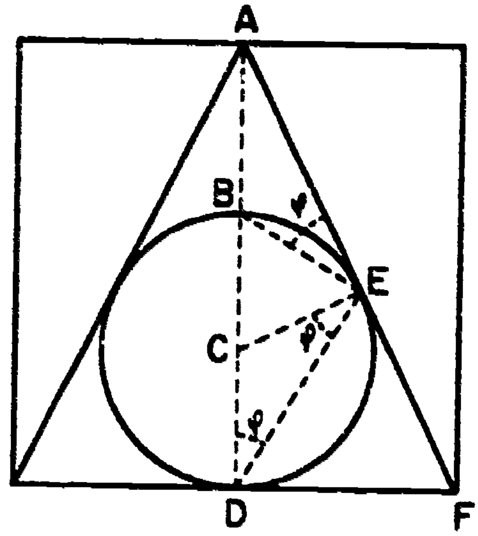

Fig. 9. The ratio G in a circle inscribed in an isosceles triangle which is in turn inscribed in a square.

The ratio G appears furthermore in geometric figures which are not connected with pentagons or decagons. One of them is a square which is inscribed in a semi-circle (Figure 8). Whereas the three line segments of a pentagram side have the smaller one in the middle and the larger ones to the sides, we have the reverse sequence in Figure 8. Nevertheless, the ratio between the two sizes of segments is again G. To prove it we use the similar triangles BCD and ABD. Denoting the ratio of the shorter to the longer legs as x we have:

$$\frac{BD}{CD} = x; \quad \frac{AD}{BD} = x$$

and therefore $BD = CD \cdot x$; $AD = BD \cdot x = CD \cdot x^2$. As $AD + BD = AD + DE = AE$ and $AE = CD$ we have: $CDx^2 + CDx = CD$ or $x^2 + x = 1$, the positive root being G. This result can also be interpreted for solid geometry, dealing with an equilateral cylinder inscribed in a hemisphere.

Another appearance of G occurs in a circle inscribed in an isosceles triangle which in turn is inscribed in a square (Figure 9) or, intepreted by solid geometry, in a sphere inscribed in a cone which in turn is inscribed in an equilateral cylinder or in a cube. The three angles in Figure 9 marked as ϕ are equal to one another (one pair are angles on the base of an isosceles triangle and another pair perpendicular angles) therefore; $\triangle ABE \sim \triangle AED$ (the triangles have one angle in common and contain another pair of equal angles ϕ). Therefore

$$\frac{AB}{AE} = \frac{AE}{AD}.$$

Denoting these ratios as x we have $AE = AD \cdot x$ and $AB = AE \cdot x = AD \cdot x^2$. Then $\triangle ADF \sim AEC$ (the triangles have both right triangles and have their angle at A is common). In the large triangle ADF the ratio of the larger to the smaller leg is 2; therefore the corresponding ratio in the smaller triangle is also 2 and AE must equal twice the radius of the circle. Therefore, $AE = BD$. Substituting AE which is $AD \cdot x$ for BD and $AB = AD \cdot x^2$ for AB into the equation $AB + BD = AD$ we get $ADx^2 + ADx = AD$ which is again the fundamental equation $x^2 + x = 1$ with the positive root $x = G$.

G is also the ratio of the smaller leg to the hypotenuse of a right triangle the sides of which form a geometric progression. (The right triangle the sides of which form an arithmetic progression is the Egyptian triangle with the sides 3; 4; 5). Denoting the hypotenuse of a right triangle whose sides form a geometric progression as "a" the larger leg is $a \cdot x$ and the smaller leg $a \cdot x^2$. From the theorem of Pythagoras we get $a^2 = (ax)^2 + (ax^2)^2$ or $x^4 + x^2 = 1$ which gives for x^2 the positive root G. Therefore the smaller leg of the right triangle $a \cdot x^2$ equals $a \cdot G$.

Arithmetically the number G shows also outstanding qualities. First, it has the same infinite sequence of decimals as its reciprocal value: $G = 0.61803398875 \cdots$. $1/G = 1.61803398875 \cdots$. It is the only positive number which forms its reciprocal value by adding 1. This results from the equation $x^2 + x = 1$ by dividing it by x: $x + 1 = (1/x)$. Then G can be expressed as the limit of a continued fraction written only by figures 1.

$$G = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}$$

By computing this continued fraction step by step, we get the following fractions:

$$1;\ 1/2;\ 2/3;\ 3/5;\ 5/8;\ 8/13;\ 13/21;\ 21/34;\ 34/55;\ 55/89;\ 89/144;\ \cdots.$$

The numerators can be obtained by adding the numerators of the two preceeding fractions, and the same holds good for the denominators. Both the numerators and the denominators form a Series of Fibonacci:

1; 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; $\cdots$. in which each term is the sum of the two preceding ones. G is the limit of the ratios of two successive terms in the Series of Fibonacci. This Series starts with two terms of 1. If instead any other numbers are chosen (excluding zero) which can be integers or fractions and the same procedure is applied to them G ll still appear as the limit of the ratios of two successive terms. This is shown in : following example in which arbitrarily the numbers 5 and 24 have been chosen:

$= 5$	$5 \div 24 = 0.2083 \cdots$
$= 24$	$24 \div 29 = 0.8276 \cdots$
$5 + 24 = 29$	$29 \div 53 = 0.5472 \cdots$
$24 + 29 = 53$	$53 \div 82 = 0.6463 \cdots$
$29 + 53 = 82$	$82 \div 135 = 0.6074 \cdots$
$53 + 82 = 135$	$135 \div 217 = 0.6221 \cdots$
$82 + 135 = 217$	$217 \div 352 = 0.6165 \cdots$
$135 + 217 = 352$	$352 \div 569 = 0.6187 \cdots$
$217 + 352 = 569$	$569 \div 921 = 0.6178 \cdots$
$352 + 569 = 921$	$921 \div 1490 = 0.6181 \cdots$
$569 + 921 = 1490$	$1490 \div 2411 = 0.6180 \cdots$
$921 + 1490 = 2411$	$2411 \div 3901 = 0.6180 \cdots$
$1490 + 2411 = 3901$	. . . .

In our case the first four decimals of G are obtained at the 11th division.

G can also be expressed as a limit of square roots in which 1 is again the only figure used:

$$G = \cfrac{1}{\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}}}$$

(the proofs which follow the theory of limits are here omitted).

In the history of mathematics references to the number G lead back to oldest geometric records. There is a passage in Herodotus in which he relates that the Egyptian priests had told him that the proportions of the Great Pyramid at Gizeh were so chosen that the area of a square whose sides is the height of of the Great Pyramid equals the area of a face triangle. Writing $2b$ for the side of the base of the Great Pyramid (see Figure 10) and "a" for the altitude of a face triangle and "b" for the height of the Pyramid, Herodotus relation is expressed in the following equation: $b^2 = (2b \cdot a)/2 = a \cdot b$. As "$a$" is the hypotenuse of a right triangle with the legs "b" and "b" we can apply the theorem of Pythagoras and get: $a^2 = b^2 + h^2$ or $h^2 = a^2 - b^2$. Equating the expressions for b^2 in the two equations we obtain $a^2 - b^2 = ab$ or $b^2 + ab = a^2$. Dividing the equation by a^2 we have $(b^2/a^2) + (b/a) = 1$. Substituting x for the ratio b/a we are back at the equation $x^2 + x = 1$ which has G as its positive root. Therefore, G is the ratio of half the side of the base square of the Great Pyramid to the altitude of the face triangle. Checking with the actual measurements taken at the Great Pyramid, we have:

$b = 148.2\,m$ (reconstructed height of undamaged apex)

$b = 116.4\,m$

which makes

$a = \sqrt{148.2^2 + 116.4^2} = 188.4$

and gives the ratio $b/a = 0.6178 \cdots$.

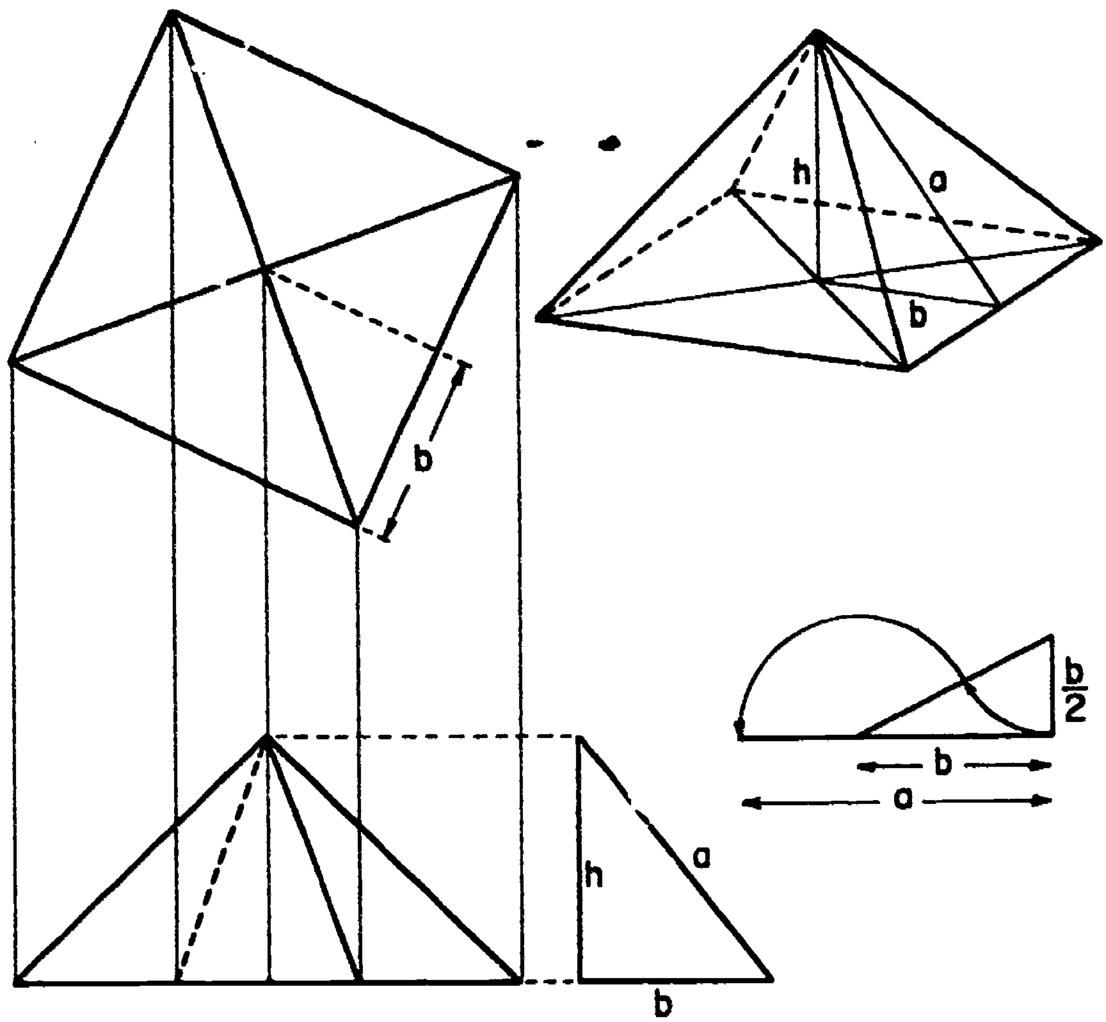

Fig. 10. Construction of the form of the Great Pyramid.

Comparing with $G = 0.6180\cdots$ the difference is $0.0002\cdots$.

A further consequence of the statement of Herodotus is the fact that G also appears as the ratio of the base to the lateral area of the Great Pyramid. The sum of the areas of the four face triangles of the great Pyramid is $4 \cdot (2b \cdot a)/2 = 4ab$. The area of the base is $(2b)^2 = 4b^2$. The ratio of the area is therefore,

$$\frac{4b^2}{4ab} = \frac{b}{a} = G.$$

The ratio G can be used to construct the form of the Great Pyramid. In Figure 10 first the ground plan of the Pyramid has been drawn. It is a square with its diagonals. Then the elevation is drawn with the positions of the base vertices determined through vertical lines dropped down from the corresponding points of the ground plan. What remains to be drawn is the height of the Pyramid. The altitude of a lateral face is $(1/G) \cdot b$ repeating the construction for G as described before using b as the base, one obtains $b \cdot G$. Adding $b \cdot G$ to b furnishes $b(G+1) = b1/G = a$. Using "a" as the hypotenuse and b as one leg of a right triangle, the length of the second leg is the height of the Pyramid h. Thus the elevation can be completed. The third projection (upper right in Figure 10) has been obtained from the ground plan and elevation through the methods of descriptive geometry, (described in THE MATHEMATICS TEACHER, April 1946).

In the sixteenth century (1509) Paciolo di Borgo wrote his treatise "De Divina Proportione" (Of the Divine Proportion) on the ratio *G*. Kepler refers to it as "sectio divina" (divine section) and Leonardo da Vinci as "sectio aurea" (the golden section) which is a term still in use for it. In an extensive literature on The Golden Section, numerous facts have been collected which show its appearance in forms of nature and art. Hambidge based on it his aesthetic research on "Dynamic Symmetry." Kepler, whose sense of proportional relations led him to his three astronomical laws which are the starting point of modern astronomy, speaks of the properties of *G* in his "Mysterium Cosmographicum de Admirabile Proportione Orbium Celestium" as of those of one of the two "great treasures" of geometry, the second being the Theorem of Pythagoras.

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Meet Mr. Tau

JOHN SATTERLY

1. In calculations made during the study of pentagons, pentagrams and the two regular solids, the icosahedron and (pentagonal) dodecahedron, a certain number continually occurs. It has been named "Tau" and is designated by τ. Its properties are many and curious, as will be seen below.

2. "Tau" is equal to

(1) $2 \cos 36°$ or $1/(2 \sin 18°)$

(2) $(\sqrt{5}+1)/2$

(3) The positive root of the equation $x^2-x-1=0$

(4) The value of

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots \text{ad infinitum}}}}$$

or as this is written for brevity

$$1+\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\cdots \text{ad infinitum}$$

(5) $1.618033989\cdots$

(6) The ratio, when n is very large, of the $(n+1)$th term to the nth term in the celebrated Fibonacci series

$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55\cdots$$

where each number after the second is the sum of the two immediately preceding numbers.

(7) The ratio of the larger segment of a straight line to the smaller segment when this line has been divided "In Extreme and Mean Ratio" an operation sometimes called "The Divine Section" or "The Golden Section." Thus if a line of length $x+y$ is divided into two parts x and y so that $(x+y)/x=x/y=\tau$ say, then $(\tau+1)/\tau=\tau$ and $\tau^2-\tau-1=0$ which is the same as (3) above.

3. *Notes*

On 1. Most Trigonometries prove

$$\cos 36° = ¼\,(\sqrt{5}+1).$$

As

$$\sqrt{5} = 2.236080 \therefore \cos 36° = 0.8090170 = \tau/2$$

On 3. If $x^2-x-1=0$ the usual method of solution of a quadratic equation gives

$$x=\frac{+1\pm\sqrt{1+4}}{2}.$$

Therefore

$$\tau=\tfrac{1}{2}(\sqrt{5}+1).$$

We may notice here that since $\tau^2-\tau-1=0$ other equations follow:

(1) $\tau^2=\tau+1$, (2) $1=\tau^2-\tau$ and dividing by τ we get $1/\tau=\tau-1$

(3) $$2-\tau=(1-[\tau-1])=1-\frac{1}{\tau}=\frac{\tau-1}{\tau}=\frac{1}{\tau^2}$$

On 4. Derivation of the Continued fraction expression. We have $\tau^2=\tau+1$. Divide by τ, we get

$$\tau=1+\frac{1}{\tau}$$

Substitute for τ in the right hand term and we get

$$\tau=1+\cfrac{1}{1+\cfrac{1}{\tau}}$$

and so on.

On 6. The ratios of any two consecutive numbers in the Fibonacci series are successive convergents to τ. Thus we have

$$\frac{2}{1},\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{13}{8},\ \frac{21}{13}\ldots$$

These are alternately $>$ and $<\tau$.

The curious may like to know that

10000/9899 = *1.010203050813213 4559*

but the resemblance to the Fibonacci numbers appears to go no further.

On 7. Rectangles whose sides are in the ratio of τ to 1 are said to be of the shape most pleasing to the eye; test this on the rectangles which you see around you.

We shall see later how τ enters into the shape of and the calcuations with regard to the Pentagon and its associated plane and solid figures.

4. *Mr. Tau and Algebra.* Table I shows Integral Powers of τ. To read easily start at τ^0 and work down using the all powerful expression $\tau^2=\tau+1$. Then work up from τ through its inverse powers.

TABLE I

. . . .	
$\tau^{-4}=5-3\tau$	$=(7-3\sqrt{5})/2=0.1459$
$\tau^{-3}=2\tau-3$	$=\sqrt{5}-2 \quad =0.2361\ldots$
$\tau^{-2}=2-\tau$	$=(3-\sqrt{5})/2 =0.3820\ldots$
$\tau^{-1}=\tau-1$	$=(\sqrt{5}-1)/2 =0.6180\ldots$
$\tau^{0}=1$	$=1 \quad =1.0000$
$\tau^{1}=\tau$	$=(\sqrt{5}+1)/2 =1.6180\ldots$
$\tau^{2}=\tau+1$	$=(3+\sqrt{5})/2 =2.6180\ldots$
$\tau^{3}=2\tau+1$	$=2+\sqrt{5} \quad =4.2361$
$\tau^{4}=3\tau+2$	$=(7+3\sqrt{5})/2=6.8541$
$\tau^{5}=5\tau+3$	. . . .
$\tau^{6}=8\tau+5$	
. . . .	

Each power of τ is the sum of the two immediate expressions above. Note the occurrence of the Fibonacci numbers.

TABLE II
Products of τ and (τ + Integers)

$$\tau^2 = \tau\tau = \tau+1$$

Also $\tau^3 = \tau(\tau+1)=2\tau+1$
$$\tau(\tau+2)=3\tau+1$$
$$\tau(\tau+3)=4\tau+1$$
. . . .

Also $\tau^4 = (\tau+1)(\tau+1)=3\tau+2$
$$(\tau+1)(\tau+2)=4\tau+3$$
$$(\tau+1)(\tau+3)=5\tau+4$$
. . . .

Also $(\tau+2)(\tau+3)=6\tau+7$
$$(\tau+2)(\tau+4)=7\tau+9$$
$$(\tau+2)(\tau+5)=8\tau+11$$
. . . .

Also $\tau(\tau+1)(\tau+2)=7\tau+4$
$$(\tau+1)(\tau+2)(\tau+3)=19\tau+13$$
. . . .

6. In the act of working out problems on Pentagons, etc., involving expressions containing τ many simplifications may be made if certain identities containing τ are recognized, remembered and used. Table III gives a list.

Table III
Identities Involving τ

$$\tau + 2 = \tau\sqrt{5}. \quad 2\tau - 1 = \sqrt{5}. \quad 2\tau + 1 = \tau^3$$
$$2 - \tau = (\tau - 1)/\tau = (\tau - 1)^2 = 1/\tau^2 = 1/(\tau + 1)$$
$$4\tau + 3 = (\tau + 1)(\tau + 2) = \tau^3\sqrt{5}$$
$$1/(4\tau + 3) = (2 - \tau)/(\tau + 2) = 0.1056$$
$$(3 - \tau)(\tau + 2) = 5$$
$$(\tau + 2)/(3 - \tau) = \tau^2$$
$$\sqrt{(4\tau + 3)(3 - \tau)} = \sqrt{5 + 5\tau} = \tau\sqrt{5} = \tau + 2$$
$$\sqrt{4\tau + 3} = \tau^{3/2}5^{1/4} = 3.0777$$
$$\sqrt{(4\tau + 3)/(\tau + 2)} = \tau$$
$$(4\tau + 3)/(3 - \tau) = \tau^4 = 3\tau + 2$$
$$3 - \tau = \sqrt{5}/\tau = (\tau + 2)/(\tau + 1) = (\tau^2 + 1)/\tau^2 = 1.3820$$
$$3 - \tau = 4 - \tau^2 = (2 + \tau)(2 - \tau)$$
$$(3 + \tau)/(3 - \tau) = [(\tau + 2)(\tau + 3)]/5 = (6\tau + 7)/5 = 3.3416$$
$$\sqrt{3 - \tau} = (5^{1/4})/(\tau^{1/2}) = 1.1756$$
$$\tau/\sqrt{3 - \tau} = (\tau^{3/2})/(5^{1/4}) = 1.3764$$
$$\tau\sqrt{3 - \tau} = 5^{1/4}\tau^{1/2} = 1.9021$$
$$(1 - 1/3)\,\tau^3)/(2 - \tau) = 1/3$$

7. Answers to problems involving τ often look quite different if obtained by different methods or routes. To guard against this always get rid if possible of any τ-term in the denominator by using a relation in Table III, also reduce the numerator to terms in the first power of τ and numerics by using the relations in Table I. Thus

$$(28\tau + 11)/(\tau - 1) = (28\tau + 11)\,\tau = 28\tau^2 + 11\tau = 39\tau + 28$$
$$(10\tau + 9)/(3 - \tau) = [(10\tau + 9)(\tau + 2)]/5 = (10\tau^2 + 29\tau + 18)/5 = (39\tau + 28)/5$$
$$(46\tau + 30)/(\tau + 2) = [(46\tau + 30)(3 - \tau)]/5 = (62\tau + 44)/5$$
$$\frac{1}{\tau^4}\,\frac{46\tau + 30}{\tau + 2} = \frac{62\tau + 44}{5}(5 - 3\tau) = \frac{1}{5}(34 - 8\tau)$$
$$\tau^4(17 - 4\tau) = 31\tau + 22$$

8. *Mr. Tau and Trigonometry*

The textbooks of Trigonometry give the values of the trigonometrical functions of 36° and 18° in terms of $\sqrt{5}$. Table IV lists these and gives also the values in terms of τ.

3

TABLE IV

The function	In terms of $\sqrt{5}$	In terms of τ	Numerical value
sin 18°	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{1}{2\tau}$	0.3090
cos 18°	$\frac{1}{4}\sqrt{10+2\sqrt{5}}$	$\frac{1}{2}\sqrt{\tau+2}$	0.9511
tan 18°	$\frac{\sin 18°}{\cos 18°}$	$\frac{\tau-1}{\sqrt{\tau+2}}$	0.3249
sin 36°	$\frac{1}{4}\sqrt{10-2\sqrt{5}}$	$\frac{\sqrt{3-\tau}}{2}$	0.5878
cos 36°	$\frac{1}{4}(\sqrt{5}+1)$	$\frac{\tau}{2}$	0.8090
tan 36°	$\frac{\sin 36°}{\cos 36°}$	$\frac{\sqrt{3-\tau}}{\tau}$	0.7265

9. *Tau in Geometry*

(1) The following (Fig. 1a) shows a simple geometrical construction for the "Divine Section" of a line AB. Make $BC = \frac{1}{2}AB$, $CD = CB$, $AE = AD$. Take BC as unity, then $AC = \sqrt{5}$, $AD = \sqrt{5}-1 = AE$, and $EB = 3-\sqrt{5}$. Therefore

$$AB/AE = 2/(\sqrt{5}-1) = \tfrac{1}{2}(\sqrt{5}+1) = \tau$$

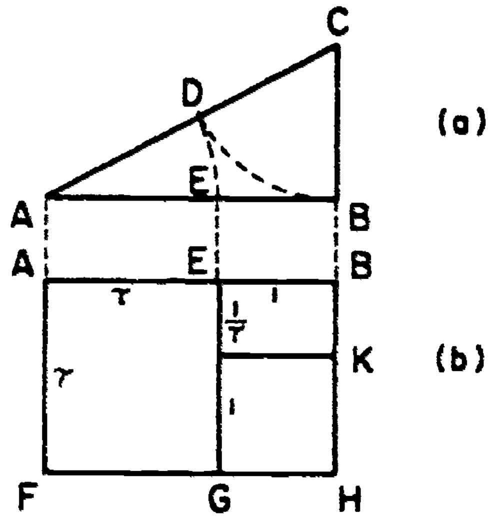


Fig. 1a, b

and $AE/EB = (\sqrt{5}-1)/(3-\sqrt{5}) = \frac{1}{2}(\sqrt{5}+1) = \tau$
and $AB/AE = AE/EB$.

(2) *The Aesthetic Rectangle.* $ABHF$ (Fig. 1b) is such a rectangle of sides $AB = \tau+1$, $AF = \tau$.

$$AB/AF = (\tau+1)/\tau = \tau^2/\tau = \tau/1.$$

E is the "divine section" of AB and $AE/EB = \tau/1$. Take the square on AE away from the rectangle. It leaves the rectangle EH of sides $BH = \tau$ and $BE = 1$.

Therefore rectangle EH is similar to rectangle AH. Take away from rectangle EH the square of GH, and there is left the rectangle EK of sides $BE = 1$, $BK = \tau - 1 = 1/\tau$, therefore rectangle EK is similar to the previous rectangles. The ratios of the respective areas are simple.

Rectangle $AH = (\tau + 1)\tau = \tau^3$
Square $AG = \tau \times \tau = \tau^2$
Rectangle $EH = 1 \times \tau = \tau$
Square $GK = 1 \times 1 = 1$
Rectangle $EK = 1 \times (\tau - 1) = 1/\tau$

The process of subdivision may be continued indefinitely, also the converse process of addition. Defining a *gnomon* as that figure which added to a given figure makes a similar figure we see that the squares are gnomons to their respective rectangles. Fig. 2 shows the rectangles on a larger scale with the process of subdivision more advanced. If the original rectangle is not true to shape the smaller figures go far astray. To keep the diagram correct draw the straight lines BF, EH. These are at right angles to each other and intersect at O the limit of the

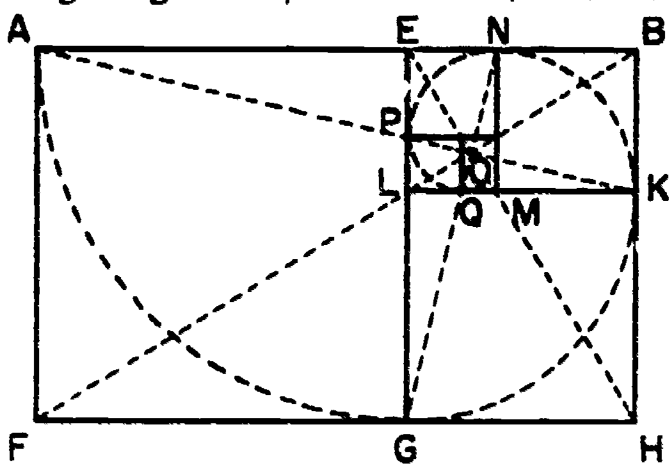


Fig. 2

ingrowing squares and rectangles. The intersection of BF and EG gives L the starting point of the line LK, the intersection of EH with LK gives M the starting point of MN and so on. If we take an origin at F, axes of x and y along FH, FA the coordinates of O are $\tau^3/\sqrt{5}, \tau^2/\sqrt{5}$. Draw also $OA, OG, OK \cdots OG$ is $\perp$ OA and to OK, in fact the region around O is divided into 8 equal angles of 45° each. Calculation shows that

$$OA = \sqrt{\frac{2}{5}(4\tau + 3)},\quad OG = \sqrt{\frac{2}{5}(\tau + 2)},\quad OK = \sqrt{\frac{2}{5}(3 - \tau)}$$

so that $OA/OG = \tau = OG/OK = \cdots$ Therefore an *equiangular spiral* may may be drawn through $AGKNPQ \cdots$ having its pole at O. The spiral looks as if it were tangential to AF at A, FH at G, HB at K, etc., but it is not quite so.

(3) *The Isosceles Triangle of Vertical Angle* 36° (Fig. 3). Let DAB be such a triangle of base $= a$. Then $DB/\tfrac{1}{2}AB = 1/\sin 18° \therefore DB = a\tau$. Draw

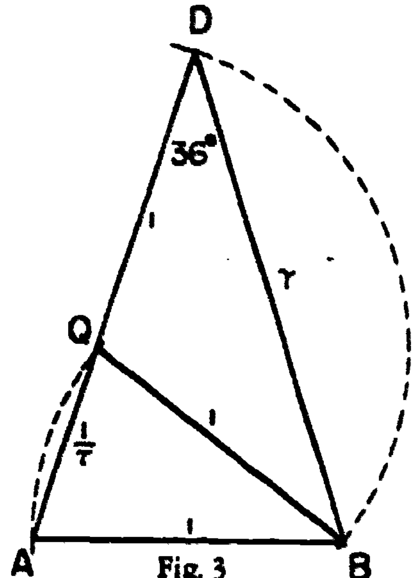


Fig. 3
(Side *AB* is taken as unity)

BQ bisecting ∠ DBA. Then $BQ = AB = a = DQ$, and $AQ = a\tau - a = a(\tau - 1) = a/\tau$. This is the "three-isosceles" triangle, and in area

$$\triangle ABQ : \triangle QBD : \triangle ABD :: 1 : \tau : \tau^2.$$

The $\triangle BQD$ is a gnomon to $\triangle ABQ$ for added to $\triangle ABQ$ it makes a similar $\triangle ADB$.

(4) *The Pentagon. ABCDE* (Fig. 4) is a regular pentagon of side $AB = a$. O is the center of the circumcircle and DOP is $\perp AB$. Then it can be shown that

$$OP = \frac{1}{2}\, a/\tan 36^\circ = \frac{1}{2}\, a \frac{\tau}{\sqrt{3-\tau}} = 0.6822a$$

$$OA = \frac{1}{2} a/\sin 36^\circ = a/(\sqrt{3-\tau}) = 0.8506a$$

$$DA = \frac{1}{2} a/\sin 18^\circ = \tau a = 1.6180a$$

$$DP = a\sqrt{\tau^2 - \frac{1}{4}} = \frac{a}{2}\sqrt{4\tau + 3} = 1.5388a$$

$$\text{Area of pentagon} = \frac{5}{4}\, a^2 \frac{\tau}{\sqrt{3-\tau}} = 1.7204a^2.$$

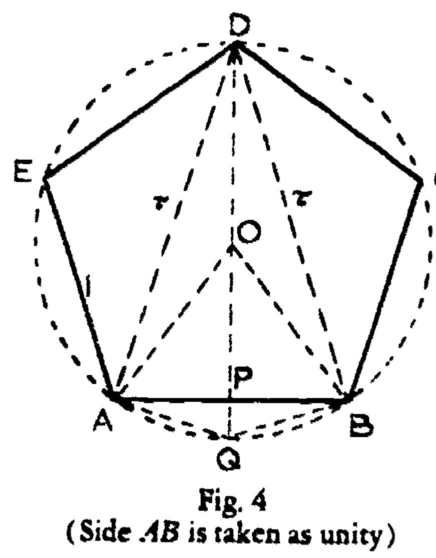


Fig. 4
(Side *AB* is taken as unity)

Also DA, DB divide the area of the pentagon into three areas $\triangle DEA$, $\triangle DAB$, $\triangle DBC$ whose ratios are

$$\frac{1}{\tau}:1:\frac{1}{\tau}$$

and

$$\frac{\text{Area of Pentagon}}{\text{Area of }\triangle DAB}=\sqrt{5}=2\tau-1$$

NOTE. If DP is produced to cut the circle in Q, AQ is the side of a regular decagon fitting in the circle and $AQ = 2R\sin 18^\circ = R/\tau$ where $R = OA$. Substituting for R we find that the side of the decagon

$$= a/(\tau\sqrt{3-\tau}) = a/(\sqrt{\tau+2}) = 0.5257a.$$

(5) *The Pentagram* (a so-called mystic figure). The construction is obvious (see Fig. 5). Let $AB = a$. Each star triangle is a mirror image of a triangle in the pentagon, e.g. $\triangle AFB$ is a mirror image in AB of $\triangle ADB$. Slant height of star triangle, e.g. $AF = a\tau$.

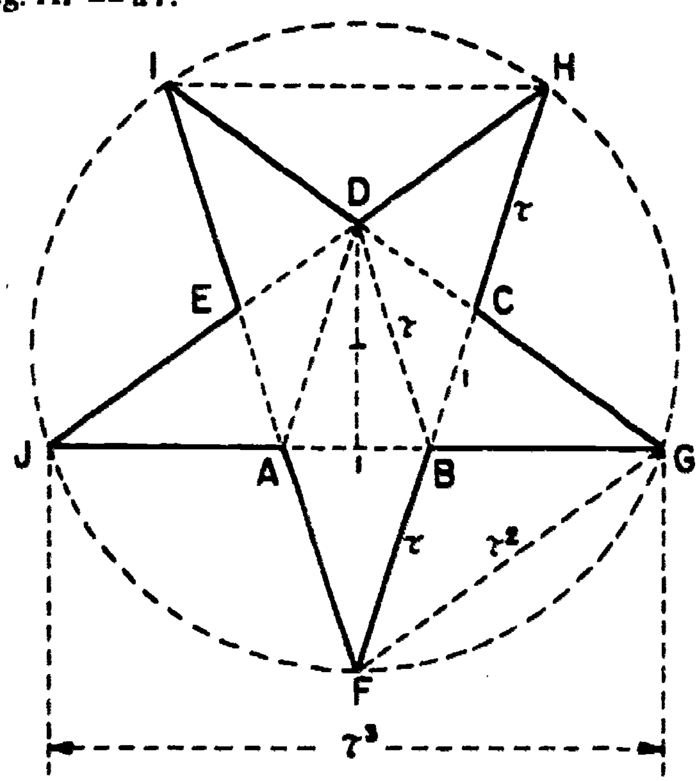


Fig. 5
(The side AB is taken as unity)

Distance between vertices of two adjacent star triangles, e.g. $FG = a\tau^2$
Distance from one vertex of a star triangle to the next but one, e.g.

$$FH = (2\tau + 1)\,a = a\tau^3$$

Area of star triangle e.g.

$$\Delta AFB = \frac{a^2}{4}\sqrt{4\tau + 3} = 0.7694a^2$$

$$\frac{\text{Area of star } \Delta}{\text{Area of } \Delta OAB} = \left(\frac{a^2}{4}\sqrt{4\tau + 3}\right)\bigg/\left(\frac{a^2}{4}\,\frac{\tau}{\sqrt{3-\tau}}\right)$$

$$= \sqrt{(4\tau + 3)(3-\tau)}/\tau = \tau\sqrt{5}/\tau = \sqrt{5}.$$

$\therefore$ (Area of Pentagram)/Area of Pentagon) $= \sqrt{5} + 1 = 2\tau$.

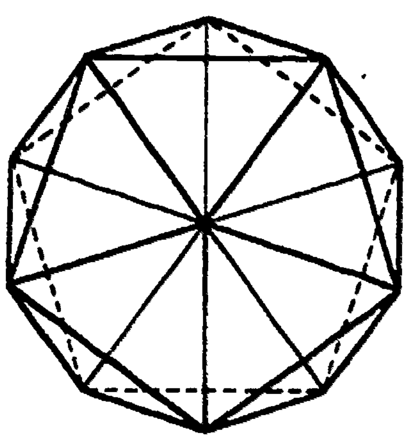
(6) *The Icosahedron.* The surface of the regular icosahedron (Fig. 6a) consists of twenty equal equilateral triangles. If edge $= a$

Radius of circumsphere $= \frac{1}{2}\, 5^{1/4}\tau^{1/2}\, a = 0.951a$

Radius of sphere through mid-points of edges $= \frac{1}{2}\tau a \doteq 0.8090a$

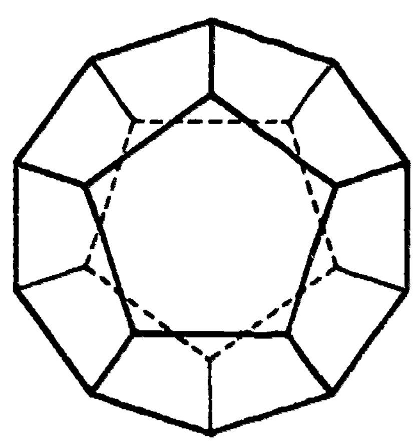
Radius of inscribed sphere $= \dfrac{1}{2\sqrt{3}}\,\tau^2 a = 0.755a$

The Volume $= \frac{5}{6}\tau^2 a^3 = 2.1817a^3$



(a)

Fig. 6a



(b)

Fig. 6b

(7) *The Dodecahedron.* The surface of the regular (pentagonal) dodecahedron (Fig. 6b) consists of twelve equal regular pentagons.

The radii corresponding to the descriptions above are

$$\frac{\sqrt{3}}{2}\tau a = 1.401a \qquad \tfrac{1}{2}\tau^2 a = 1.309a, \qquad \tfrac{1}{2}5^{-1/4}\tau^{5/2}a = 1.114a$$

respectively.

$$\text{The Volume} = \tfrac{1}{2}5^{1/2}\tau^4 a^3 = 7.663a^3$$

The proofs of the above results are not beyond a student's capabilities but they are long.

10. *Mr. Tau's Relations.* We have seen that if we start with the numbers 0, 1, 1 and add (the last) two at a time we get the Fibonacci numbers and the ratio of two consecutive numbers converges to $1.6180\cdots$, i.e. to τ, and τ is one root of the equation $x^2-x-1=0$. If we take *any* two numbers at the start and add two at a time the ratio of two consecutive numbers also converges upon $1.618\cdots$. For example starting with 1.5 we get

$$1\ 5\ 6\ 11\ 17\ 28\ 45\ 73\ 118\cdots$$

A Fibonaccian will show that this series is the sum of

$$1\times(1\ 1\ 2\ 3\ 5\ 8\ 13\ 21-)$$

and

$$4\times(0\ 1\ 1\ 2\ 3\ 5\ \ 8\ 13-)$$

and this is true (with a different numeric from 4) for all such series so that we still are linked to the Fibonacci series. If we start with any three numbers and add three numbers at a time we get a series having a convergent ratio of $1.839286\cdots$ and this is a root of $x^3-x^2-x-1=0$. Similarly starting with four numbers and five numbers as shown in the table below (Table V).

TABLE V
Fibonacci and Post-Fibonacci Series

Start with	Add	Convergent Ratio	A Root of the Equation
Any two numbers	Two at a time	$\tau=1.618\ldots$	$x^2-x-1=0$
Any three numbers	Three at a time	$1.839\ldots$	$x^3-x^2-x-1=0$
Any four numbers	Four at a time	$1.927\ldots$	$x^4-x^3-{}^2x-x-1=0$
Any five numbers	Five at a time	1.965	$x^5-x^4-x^3-x^2-x-1=0$
Any no. of numbers	All at a time	2.000	. . . .
Any two numbers	$3\times$last$+2\times$last but one	3.56	$x^2-3x-2=0$
Any two numbers	$3\times$last$-2\times$last but one	2.00	$x^2-3x+2=0$
Any two numbers	$2\times$last$+3\times$last but one	3.00	$x^2-2x-3=0$

Variants are shown in the lower part of the table but we must stop here. Useful reference books are Sir D'Arcy W. Thompson's "Growth and Form" (Cambridge, University Press and Macmillan Company, New York) 2nd edition, 1942, Chs. XI, XIV and H. M. Cundy and A. P. Rollett, "Mathematical Models" (Clarendon Press, Oxford), 1942, Ch. II, but other references are available e.g. H. S. M. Coxeter, "The Golden Section, Phyllotaxis and Wythoff's Game" (*Scripta Mathematica*, New York) Vol. XIX, Nos. 2–3, 1953.

"Good bye, Mr. Tau, we have enjoyed our meeting."

The Golden Section, Phyllotaxis, and Wythoff's Game

By H. S. M. COXETER

Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.

—. KEPLER (1571-1630)

1. The odd-sounding phrase "division of a line into extreme and mean ratio" was used by Euclid to signify division of a line segment into two unequal parts such that the ratio of the whole to the larger part is equal to the ratio of the larger to the smaller. Calling each ratio τ (after $\tau o\mu\eta$, "the section"), we see that this requires

$$\tau^{-1}+\tau^{-2}=1,$$

so that τ is the positive root of the equation

$$x^2-x-1=0,$$

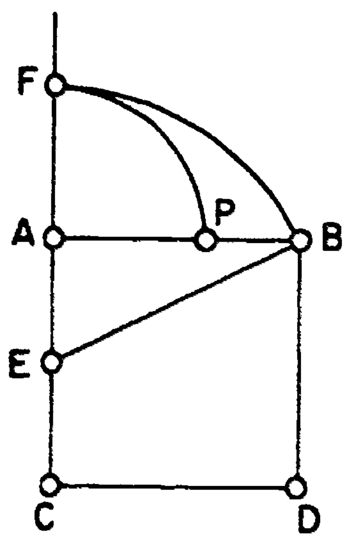


Fig. 1

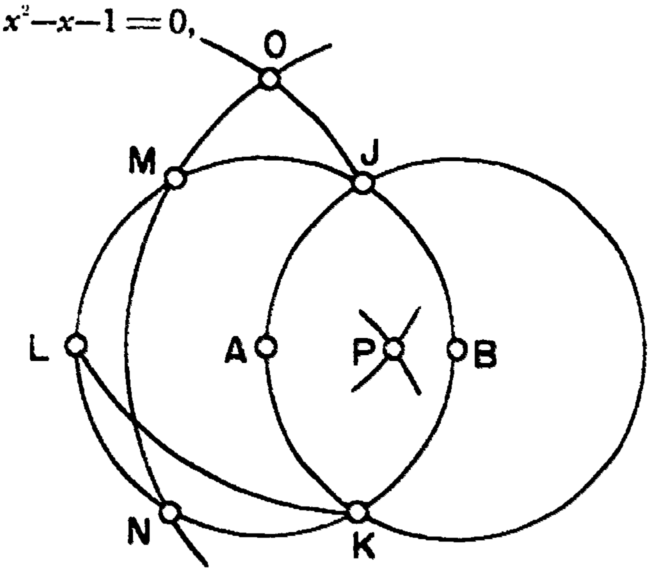


Fig. 2

viz.,

$$\tau=\tfrac{1}{2}(\sqrt{5}+1)=1.618033989\ldots,$$

whence

$$\tau^{-1}=\tfrac{1}{2}(\sqrt{5}-1), \quad \tau^{-2}=\tfrac{1}{2}(3-\sqrt{5}).$$

The classical construction (Euclid II, 11) is as follows. To divide a given segment AB in extreme and mean ratio (Fig. 1)), let E be the mid-point of the side AC of the square $ABDC$; take F on CA produced, so that $EF=EB$; take P on AB, so that $AP=AF$. Then P is the dividing point (such that $AB\times PB=AP^2$).

Nils Pipping [14] has recently devised a new construction, in the spirit of Mascheroni and Mohr [11, 12], who proved that every ruler-and-compasses construction can be duplicated with the compasses alone. Pipping's division of the given segment AB requires just seven circles, of three different radii, as in Fig. 2. The circle $A(AB)$ (with center A and radius AB) meets the equal circle $B(AB)$ in two points J and K. Then $J(JK)$ determines L, $B(JK)$ determines M and N, $L(JK)$ determines O, and finally the two circles $M(AO)$ and $N(\quad)$ intersect in a point P which divides AB in extreme and mean ratio (as can easily be verified by several applications of Pythagoras's Theorem).

It is interesting to compare this with M roni's third solution to the problem of locating the *mid-point* of a given segment [11, Problem 66], which likewise requires seven circles.

The division into extreme and mean ratio, later known as the *golden section*, was used by Euclid (IV, 10) "to construct an isosceles triangle having each of the angles at the base double of the remaining one" and (IV, 11) "in a given circle to inscribe an equilateral and equiangular pentagon." The figure that he obtained is essentially a regular pentagon with its inscribed star pentagon or *pentagram*. This can be displayed by tying a simple knot in a long strip of paper and carefully pressing it flat. In modern notation, the connection between τ and the pentagon is expressed by the formula

$$\tau = 2 \cos \frac{\pi}{5} \cdot$$

Euclid's construction for the pentagon is one of the thirteen properties of τ described by Fra Luca Pacioli in his book, *Divina proportione* [13] which was illustrated by his friend Leonardo da Vinci. Successive chapters are entitled: The First Considerable Effect; The Second Essential Effect; The Third Singular Effect; The Fourth Ineffable Effect; The Fifth Admirable Effect; The Sixth Inexpressible Effect, and so on. "The Seventh Inestimable Effect" is that a regular decagon of side 1 has circumradius τ. (We can thus inscribe a pentagon in a given circle by first inscribing a decagon and then picking out alternate vertices.) "The Ninth Most Excellent Effect" is that two crossing diagonals of a regular pentagon divide one another in extreme and mean ratio. "The Twelfth Incomparable Effect" and "The Thirteenth Most Distinguished Effect" are constructions for the icosahedron and the dodecahedron. The next chapter tells "how, for the sake of our salvation, this list of effects must end" (because there were just thirteen at table at the Last Supper).

The faces surrounding a corner of the icosahedron belong to a pyramid whose base is a regular pentagon. Any two opposite edges belong to a rectangle whose longer sides are diagonals of such pentagons. Since the diagonal of a pentagon is τ times its side, this rectangle is a *golden* rectangle, whose sides are in the ratio $\tau:1$. In fact, the twelve vertices of the icosahedron (Fig. 3) are the

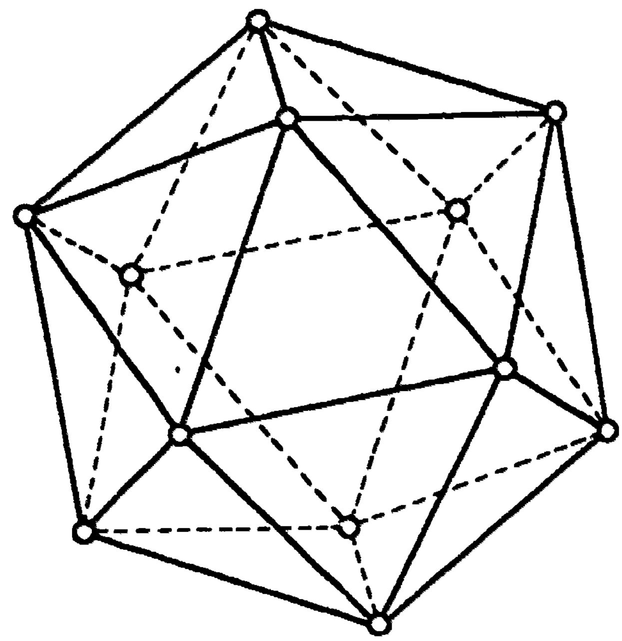


Fig. 3

twelve vertices of three golden rectangles in perpendicular planes (Fig. 4). Thus [16] the vertices of an icosahedron of edge 2 can be represented by the coordinates

$$(0, \pm 1, \pm\tau), \qquad (\pm\tau, 0, \pm 1), \qquad (\pm 1, \pm\tau, 0).$$

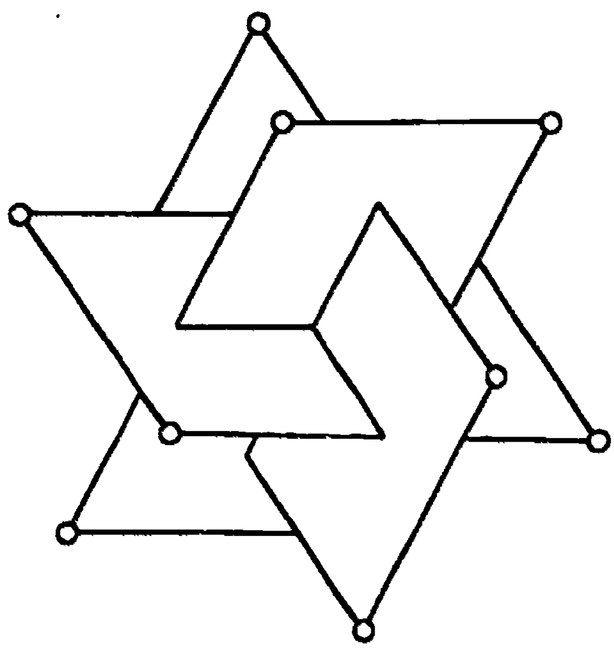


Fig. 4

The identity

$$\tau = 1 + 1/\tau$$

shows that the golden rectangle can be dissected into two pieces: a square and a smaller golden rectangle. Given the square $ABDC$, we can construct the side CF of the rectangle by Euclid's method (Fig. 1). From the smaller rectangle $ABGF$ (Fig. 5) we can cut off another square, leaving a still smaller rectangle, and continue the process indefinitely. Quadrants of circles, inscribed in the successive squares, form a composite spiral of rather agreeable appearance. More interestingly, the end points $D, A, H, I, \ldots$ of the quadrants lie on a true logarithmic spiral whose pole is the point of intersection $CG \cdot BF$.

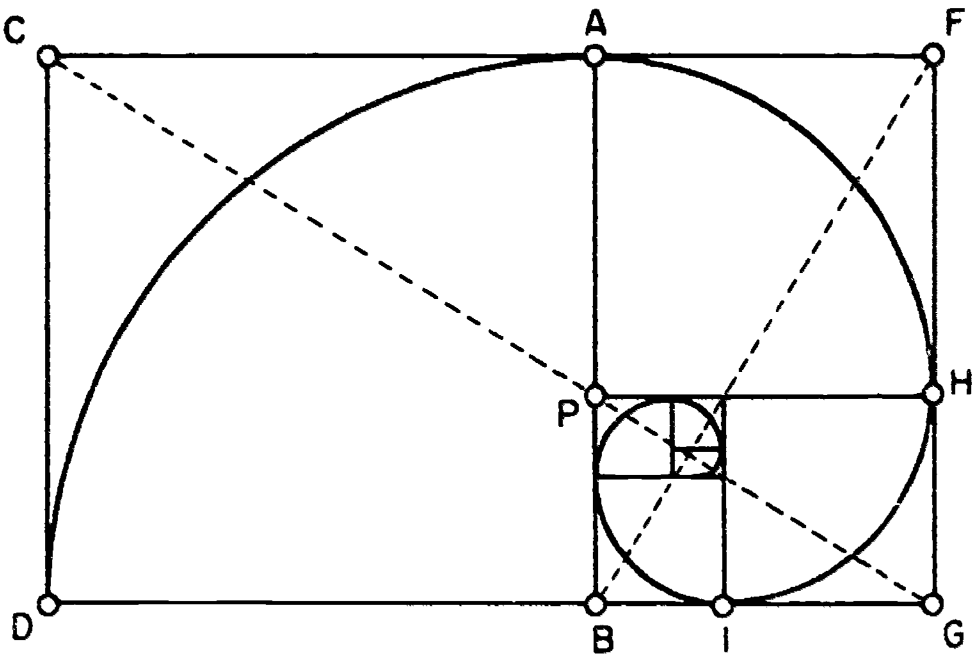


Fig. 5

It was pointed out by Cundy and Rollett [5] that this spiral cuts each of the lines $CF, FG, GB, BP, \ldots$ twice, instead of touching them like the circular quadrants. In fact, its angle ϕ (between tangent and radius vector) satisfies the equation $\pi \cot \phi = 2 \log \tau$, so that $\phi = 72^\circ\, 58'$. But our eyes can scarcely distinguish it from the logarithmic spiral of angle $74^\circ\, 39'$ (satisfying $\pi \cot \phi = \frac{2}{3} \log \tan \phi$) which, being its own evolute, has the same contact properties as the composite spiral of Fig. 5. Of course, the rectangle is no longer golden; in fact, the ratio of its sides is not

$$\tau = 1.6180\ldots$$

but

$$\tan^{1/3}\phi = 1.5387\ldots$$

We leave it to the psychologists to decide which of these two rectangular shapes is the more aesthetically satisfying [6].

In 1202, Leonardo of Pisa, nicknamed Fibonacci (not "son of an ass," as has been suggested, but rather "son of good nature" or "prosperity"), came across his celebrated sequence of integers in connection with the breeding of rabbits [1, 9]. He assumed that rabbits live forever, and that every month each pair begets a new pair which becomes productive at the age of two months. In the first month the experiment begins with a newborn pair of rabbits. In the second month, there is still just one pair. In the third month there are two; in the fourth, three; in the fifth, five; and so on. Let f_n denote the number of pairs of rabbits in the nth month. The first few values may be tabulated as follows:

n:	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
f_n:	1	1	2	3	5	8	13	21	34	55	89	144	$\cdots$

Four centuries later, Girard [7] noticed that each of these numbers (after the second) is equal to the sum of the preceding two:

$$f_1 = f_2 = 1, \quad f_{n+2} = f_{n+1} + f_n \qquad (n \geq 1).$$

Another hundred years passed before Simson [17] observed that f_{n+1}/f_n is the nth convergent to the continued fraction

$$1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots.$$

To see that this converges to τ, he merely had to express the relation $\tau = 1 + 1/\tau$ in the form

$$\tau = 1 + \frac{1}{\tau} = 1 + \frac{1}{1+}\,\frac{1}{\tau} = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\tau} = \cdots.$$

Simson also obtained the identity

$$f_{n-1}f_{n+1} - f^2_n = (-1)^n,$$

which yields the following puzzle-dissection [15]. A rectangle $f_{n+1} \times f_{n-1}$ is cut into four pieces which can apparently be reassembled to form a square of side f_n (Fig. 6). The figure should be drawn on squared paper, so that the audience

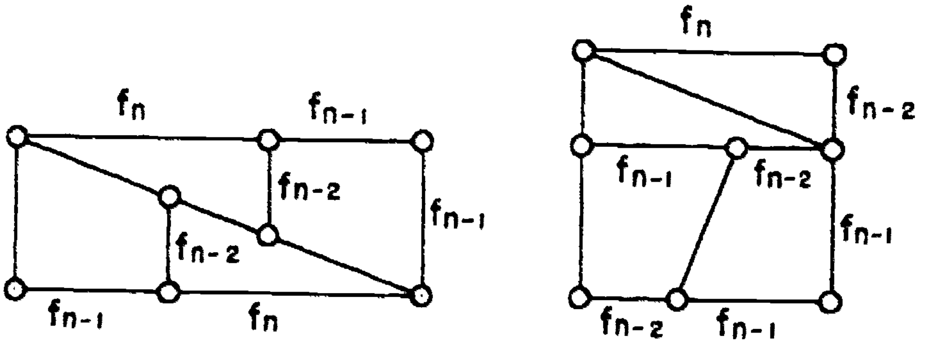


Fig. 6

can "see" that there is no cheating. The value $n = 6$ is sufficient in practice, but of course the error is still less detectable when $n = 7$.

Lagrange [8] noticed that the residues of the Fibonacci numbers, for any given modulus, are periodic; e.g., their final digits (in the denary scale) repeat after a cycle of sixty:

1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, . . . , 7, 2, 9, 1, 0.

In 1876, Lucas obtained the identities

$$f_{2n+1} = f^2_{n+1} + f^2_n, \qquad f_{3n} = f^3_{n+1} + f^3_n - f^3_{n-1},$$
$$1 + 1 + 2 + 3 + \cdots + f_n = f_{n+2} - 1.$$

More interestingly [10], he discovered the explicit formula in terms of binomial coefficients:

$$f_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots,$$

which can be established by observing that

$$\begin{aligned} 1 + t + 2t^2 + 3t^3 + \cdots f_{n+1}\, t^n + \cdots &= (1 - t - t^2)^{-1} \\ &= 1 + (t + t^2) + (t + t^2)^2 + (t + t^2)^3 + \cdots . \end{aligned}$$

Setting $t = 0.01$, we obtain the decimal

$$\frac{10000}{9899} = 1.0102030508132134559 \cdots$$

(which is spoilt by the necessary "carrying" after the nineteenth significant digit).

Lucas also observed that the recursion formula

$$f_{n+2} = f_{n+1} + f_n$$

is satisfied by any linear combination of the nth powers of the roots of the equation

$$x^2 = x + 1,$$

whence, in virtue of the initial conditions $f_0 = 0, f_1 = 1,$

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\ &= 5^{-1/2}\,\{\tau^n - (-\tau)^{-n}\}. \end{aligned}$$

It follows that

$$\tau^n = \tfrac{1}{2}(f_n \sqrt{5} + g_n),$$

where

$$g_n = f_{n-1} + f_{n+1}.$$

2. The Fibonacci numbers arise naturally in the botanical phenomenon called *phyllotaxis* [19]. In some trees, such as the elm and lime, the leaves along a twig appear alternately on two opposite sides, and we speak of "1/2 phyllotaxis." In others, such as the beech and hazel, the passage from one leaf to the next, involves a screw-twist through one-third of a turn, and we speak of "1/3 phyllotaxis." Similarly, the oak and cherry exhibit 2/5 phyllotaxis; the poplar and pear, 3/8; the willow and almond, 5/13; and so on. We recognize the fractions as being quotients of alternate Fibonacci numbers. But *consecutive*

Fibonacci numbers could be used just as well; e.g., a clockwise rotation through 5/8 of a turn is equivalent to a counterclockwise rotation through 3/8.

Another manifestation of phyllotaxis is the arrangement of the florets of a sunflower, or of the scales of a pine cone, in spiral or helical *whorls*. We observe that the numbers of right-handed and left-handed whorls are two consecutive Fibonacci numbers, viz., 2 and 3 (or vice versa) for the balsam cone, 3 and 5 for the hemlock cone, 5 and 8 for the pine cone, 8 and 13 for the pineapple (the clearest instance of all) and higher numbers for sunflowers of various degrees of cultivation. Church [4] gives photographs of a (34, 55) sunflower and of a giant (55, 89) sunflower. The Russians are said to have succeeded in cultivating a super-giant (89, 144).

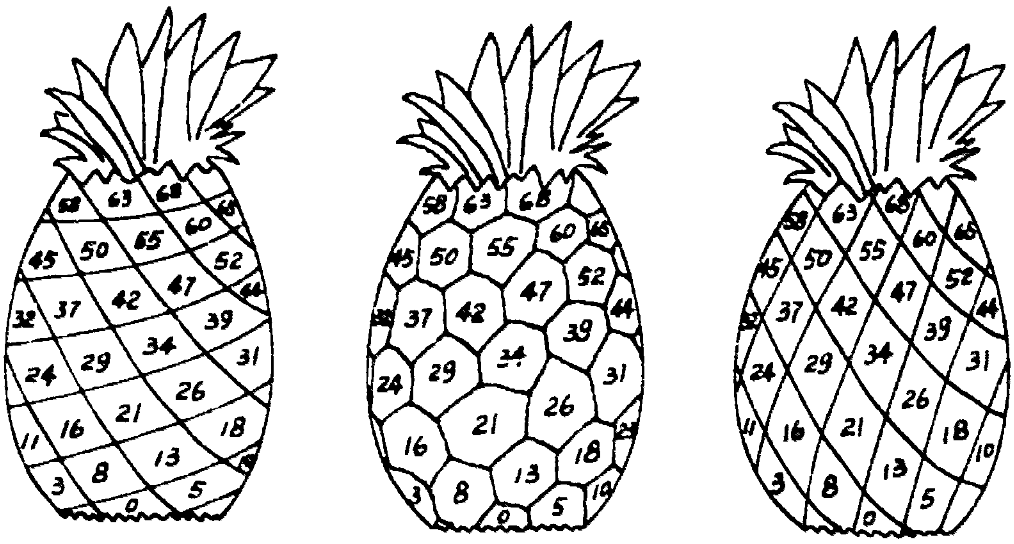


Fig. 7

The fact that the numbers of whorls can be increased by intensive cultivation suggests an evolutionary explanation for the phenomenon. We can imagine that a simple (1, 1) plant evolved into a (1, 2) plant, then into a (2, 3) plant, and so on. The transition can be explained by observing that the florets are not really quadrangular but hexagonal, so that each belongs not only to two kinds of whorl but to a third as well. A slight distortion suffices to make the third kind supersede one of the others. In Fig. 7, a pineapple has been sketched between two hypothetical variants; a simpler fruit, exhibiting (5, 8) phyllotaxis; out of which the pineapple could have evolved, and a super-pineapple, exhibiting unquestionable (8, 13) phyllotaxis, which might be produced by intensive cultivation. The scales of the pineapple have been numbered systematically with the multiples of 5 and 8 in the directions in which 5 or 8 whorls occur. The remaining numbers then follow by "vector addition," e.g., we have the multiples of $5 + 8 = 13$ in the intermediate direction, in which there are 13 whorls. Thus the numbers in any whorl form an arithmetical progression. The same kind of numbering could be applied to the florets of a sunflower.

Such an explanation for phyllotaxis was first given by Tait [18]. According to Dr. A. M. Turing (who is preparing a new monograph on this subject), the continuous advance from one pair of parastichy numbers to another, such as

(5, 8) to (8, 13), takes place during the growth of a single plant, and may or may not be combined with an evolutionary development.

3. Another application of the golden section is to the theory of *Wythoff's game* [20]. Like the well-known Nim (2), this is a game for two players, playing alternately. Two heaps of counters are placed on a table, the number in each heap being arbitrary. A player either removes from one of the heaps an arbitrary number of counters or removes from both heaps an equal number (e.g., heaps of 1 and 2 can be reduced to 0 and 2, or 1 and 1, or 1 and 0, or 0 and 1). A player wins by taking the last counter or counters.

An experienced player, player against a novice, can nearly always win by remembering which pair of numbers are "safe combinations": safe for him to leave on the table with the knowledge that, if he does not make any mistake later on, he is sure to win. (If both players know the safe combinations, the outcome depends on whether the initial heaps form a safe or unsafe combination.)

The safe combinations

$$(1,2), (3,5), (4,7), (6,10), (8,13), (9,15), (11,18), \ldots$$

can be written down successively by the following rule. At each stage, the smaller number is the smallest natural number not already used, and the larger is chosen so that the difference of the numbers in the *n*th pair is *n*. Thus every natural number appears exactly once as a member of a pair, and exactly once as a difference. It follows that, if player A leaves a safe combination, B cannot help changing it into an unsafe combination (unsafe for B). It is slightly harder to see that any such unsafe combination left by B can be rendered safe by A. Suppose B leaves the pair (p, q) $(p \leq q)$ which is not one of the safe combinations. If $p = q$, A wins immediately. If not, let (p, p') or (p', p) be the safe combination to which p belongs. If $p' < q$, A reduces the q heap to p'. If $q < p'$ (so that $p < q < p'$ and $q - p < p' - p$), he reduces both heaps by equal amounts, so as to leave the safe combination whose difference is $q - p$.

Thus A can win, no matter what B does, unless A is confronted with a safe combination before his first move (in which case he will remove one counter and trust B to make a mistake).

It is easier to write down a lot of safe combinations than to discover a general formula. Such a formula was given by Wythoff's "out of a hat"; but a more natural approach is provided by the following theorem of Beatty [3]:

If $x^{-1} + y^{-1} = 1$, *where x and y are positive irrational numbers, then the sequences*

$$[x], [2x], [3x], \cdots, \qquad [y], [2y], [3y], \cdots$$

together include every positive integer just once.

(Here $[x]$ means the integral part of x.)

The following proof was devised jointly by J. Hyslop in Glasgow and A. Ostrowski in Göttingen.

For a given integer N, the numbers of members less than N of the sequences

$$x, 2x, 3x, \ldots \quad \text{and} \quad y, 2y, 3y, \ldots$$

are, respectively, $[N/x]$ and (N/y). Since $x^{-1}+y^{-1}=1$, where x and y are irrational, N/x and N/y are two irrational numbers whose sum is the integer N. Hence their fractional parts must add up to exactly 1, and

$$[N/x]+(N/y)=N-1.$$

This is the number of members less than N of the two sequences together. By taking $N = 1, 2, 3, \ldots$ in turn, we deduce that the multiples of x and y are "evenly" distributed among the natural numbers: one between 1 and 2, one between 2 and 3, and so on. Hence their integral parts, $[nx]$ and $[ny]$, are the natural numbers themselves.

This is one of the two requirements for the safe combinations in Wythoff's game. The other, that the difference shall be n, is secured by taking

$$y=x+1.$$

Since $x^{-1}+y^{-1}=1$, it follows that

$$x^2-x-1=0,$$

whence $x=\tau$, $y=\tau^2$, and the nth safe combination is

$$[n\tau], \qquad [n\tau^2].$$

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FOR FURTHER READING AND STUDY

There is an embarassment of riches when we look to the literature of The Golden Section, including the related topics of Fibonacci numbers, continued fractions, the geometry of the pentagon, dynamic symmetry, phyllotaxy, and other interesting sidelights. The bibliography below is but a very small part of this storehouse of ideas.

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