WHAT IS A FUNCTION?

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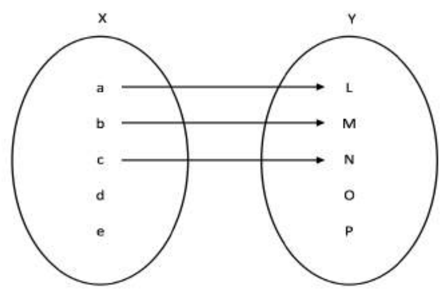
In the mathematical community, two notions of "function" are used: the set-theoretic definition as a univalent set of ordered pairs, and the Bourbaki triple. These definitions entail different interpretations and answers to mathematical questions that even a secondary student might be prompted to answer. However, mathematicians and mathematics educators are often not explicit about which definition they are using. This paper discusses these parallel usages and the related implications for the field of mathematics education.

Keywords: Curriculum, University Mathematics, High School Education

To frame the discussion of this paper, we invite the reader to answer the following questions that a secondary student might be asked about functions.

1. Does the following diagram represent a function?



2. Is the following set of ordered pairs a function? If so, what is its domain?
   $\{(-1,4),(0,7),(2,3),(3,3),(4,-2)\}$
3. Is $y = \sqrt{x}$ a function? If so, what is its domain?
4. Is $g(x) = \ln x$ the inverse function of $f(x) = e^x$?
5. Does $g(x) = \ln x$ have an inverse function? If so, what is it?
6. True or false: A function is invertible if and only if it is injective.

Two Definitions Of "Function"

Function is an important concept in mathematics, and students' understanding of function has been the subject of extensive research in mathematics education (Breidenbach et al., 1992; Leinhardt et al., 1990; Vinner & Dreyfus, 1989). Researchers have noted the mathematical definition of function has evolved: Initially, functions were characterized as explicit rules that assigned numbers to other numbers. Over time, the notion of function became more general— a function could take any type of object as its inputs or outputs (e.g., differentiation can be understood as a function that maps differential real-valued functions to other functions)— and *any* correspondence can be a function, regardless of whether the rule for the function can be explicitly stated or not. Finally, the modern treatment of function is usually provided in set theoretic terms (Kleiner, 1989). Research on students' understanding of function often shows that students hold varied conceptions of function that do not align with the formal theory, even when they can state the formal definition of function (Bardini et

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

al., 2014; Leinhardt et al., 1990; Mirin, 2018; Sfard, 1992; Thompson, 1994; Vinner & Dreyfus, 1989).

It is unsurprising that the meaning of function has evolved. The definitions of many mathematical concepts have become more precise and more abstract over time, and several concepts are now defined in structural or set theoretic terms although they were not originally conceived of in this way (Sfard, 1992). Likewise, students often hold different understandings of the same mathematical concept, with their understandings being internally inconsistent or at variance with the formal theory. We might expect there to be an agreed upon modern definition of function - or, absent a uniform definition, that the different definitions that mathematicians use are logically equivalent. Surprisingly, this is not the case. In fact, the different definitions in use do not even yield the same answers on questions that a secondary student might be asked. In this paper, we describe two commonly used definitions of functions in high school and university mathematics. We show that these definitions actually lead to different answers to the six questions that we posed in the beginning of the paper.

One Definition Of Function: The Bourbaki Triple

One treatment of function is that of Bourbaki (1968), which defines a function as a triple (f, A, B), where A and B are sets and f is a univalent and total subset of $A \times B$. That is, for *all* x in A (*total*), there exists a *unique* y in B (*univalent*) such that (x, y) is a member of f. The set A is called the *domain* of the function, B is the *codomain,* and f is the *graph.* We refer to this as the *Bourbaki Triple* function definition, objects of this type as *Bourbaki Triple functions,* and people who use this definition as *Bourbaki Triple people.*

Another Definition Of Function: The Ordered Pairs

Forster (2003) notes, "some mathematical cultures… [say] a function is an ordered triple of domain, range, and a set of ordered pairs. This notation has the advantage of clarity, but it has not yet won the day" (pp. 10-11). Forster then refers to the alternative definition of a function as *any set of ordered pairs f* that satisfies the following criterion: if (x, y_1) and (x, y_2) are in f, then y_1 and y_2 are equal. In this case, the *domain* of f is the set of all numbers x such that there exists some y where (x,y) is a member of f. There is no unique *codomain;* **a** (rather than **the)** *codomain* is any superset of the range of f. We refer to this as the *Ordered Pairs* function definition, objects of this type as *Ordered Pairs functions,* and people who use this definition as *Ordered Pairs people*.

Comparing and Contrasting Definitions

It's worth emphasizing that a Bourbaki Triple function is a different sort of object than an Ordered Pairs function; a Bourbaki Triple function is an ordered triple, whereas an Ordered Pairs function is a set of ordered pairs. Hence, when a Bourbaki Triple person mentions a function, they are referring to a different type of object than that of an Ordered Pairs person. The way someone understands questions or statements about functions will be related to what type of object they understand a function to be – a triple, or a set of ordered pairs.

Dumas and McCarthy (2015), Bourbaki Triple people, assert the following:

> When you write $f: X \to Y$, you are explicitly naming the intended codomain, and this makes the codomain a crucial part of the definition of the function. You are indicating to the reader that your definition includes more than just the graph of the function. The definition of a function includes three pieces: the domain, the codomain, and the graph. (Dumas & McCarthy, 2015, p. 25)

Dumas and McCarthy are correct *if* one is using the convention of a function as a Bourbaki Triple. In this case, "$f: X \to Y$" names the function (f, X, Y) with domain X, codomain Y, and graph f. However, a look at works by authors who use the Ordered Pair notion of function suggests that adopting such notation does not necessitate endorsing the view that a function is an ordered triple (Devlin & Devlin, 1993; Goldrei, 1998; Halmos, 1960; Stoll, 1979). Using the Ordered Pair interpretation, these authors write "$f: X \to Y$" to mean that f is a function with domain X where $f(x)$ is a member of Y for all x in X (that is, Y is **a** codomain of f). Rather than X and Y being *part* of the function itself, they are *attributes* of the function.

We can see how the differences in the notion of function manifest themselves in interpreting a function definition. Consider the following sentence: "Let $f: N \to Z, f(n) = n + 1$". Under the Ordered Pairs definition, the function is the set f=$\{(n, n + 1): n \in N\}$, and this set (function) has the *property* that N is its domain and Z is **a** codomain. Under the Bourbaki Triple definition, the function actually at hand *is* the entire triple (f, N, Z), where f is still the set $\{(n, n + 1): n \in N\}$, the domain is N, and **the** codomain is Z. The important thing here is that in one interpretation, the function is just the set $\{(n, n + 1): n \in N\}$, while in the other interpretation, the function is the entire triple $(\{(n, n + 1): n \in Z\},\ N, Z)$. Now consider the notation "Let $g: N \to N, g(n) = n + 1$." Under the Ordered Pairs definition, the function g is the same as the function f. Under the Bourbaki Triple definition, the function at hand is the entire triple (g, N, N), which is a different triple than (f, N, Z) (as Z is a different set than N).

Notice that the domain of an Ordered Pairs function is not necessarily stipulated; it is derived as a consequence of the graph itself. If f is a function, x is in the domain of f exactly when there exists a y such that (x, y) is in f. For this reason, the only criterion a set of ordered pairs needs to satisfy is being univalent. It does not make sense to ask whether a relation f is "total" in the abstract; one would need to ask if f was total on a specified set. Similarly, it does not make sense to ask if f is "surjective"; one would need to specify a set (codomain) that f might be surjective upon. The notation "$f: X \to Y$" might be used to stipulate such sets, but this notation is not always used.

A sharp difference between the Bourbaki Triple definition and the Ordered Pairs definition relates to invertibility. With the Ordered Pairs definition, the inverse for a function f, denoted by f^{-1}, is the set $\{(y, x): (x, y)\ \in f\}$. This set f^{-1} is a function if and only if f is one-to-one, i.e., f is invertible as a function if and only if it is injective. With the Bourbaki Triple definition, if (f, A, B) is a function, we consider the inverse of the triple to be (f^{-1}, B, A) where f^{-1} is defined as above. In this case, for (f, A, B) to be invertible as a function, more than injectivity is required; the triple (f^{-1}, B, A) must also be a function. This means that f^{-1}must be total on B, requiring that f be surjective onto B. We will revisit this difference later.

How Are Functions Treated In The Literature?

In the mathematics literature, both definitions of function are common. The Bourbaki Triple definition appears in some introductory proof books (e.g., Dumas & McCarthy, 2015) and other domain-specific textbooks, such as Abbott's (2012) textbook on real analysis. On the other hand, the Ordered Pairs definition also appears in some introductory proof textbooks (e.g., Forster, 2003) and in set theory textbooks (e.g., Enderton, 1977; Halmos, 1960; Jech, 2003). Still, other textbooks offer both definitions (e.g. Eccles, 1997). Finally, we will illustrate how others (e.g. Stewart, 2003) are ambiguous.

In the mathematics education literature, we find the Bourbaki Triple definition to be more common. For instance, in their influential review of students' understanding of functions, Leinhardt et al. (1990) refer to the Bourbaki Triple as the modern definition of function that mathematics educators would like their students to master. Other studies have used this definition as their backdrop for what is normatively correct, but they use the *word* "Bourbaki" only, without referring to the internal structure of the triple itself (e.g. Breidenbach et al., 1992; Vinner & Dreyfus, 1989). These authors tend to focus on the arbitrariness of a function as a correspondence rather than as a rule or an equation. Nonetheless, there is variation, and other mathematics educators have adopted the set of ordered pairs definition as their background theory (e.g. Sajka, 2003). Also, it is noteworthy that mathematics educators often describe $g(x) = \ln x$ as the inverse function of $f(x) = e^x$ (e.g. Even, 1990; Mayes, 1994), which seems to suggest the Ordered Pairs definition (discussed below).

In algebra and precalculus texts, we have again found both definitions used. For instance, Hungerford and Shaw (2009) define a function explicitly consisting "of three parts— a set of inputs (called the domain); a rule by which each input determines exactly one output; a set of outputs (called the range)" (p. 155). However, there are also books that define functions as a set of ordered pairs; for instance, Marecek (2017) wrote:

> A relation is any set of ordered pairs (x, y). All the x-values in the ordered pairs together make up the domain. All the y-values in the ordered pairs together make up the range [...] A function is a relation that assigns to each element in its domain exactly one element in the range. (Marecek, 2017, pp. 314-317)

Consistent with the Ordered Pairs definition, Marecek (2017) notes that the domain and range are not specified but derived from the set of ordered pairs, and there is no notion of codomain.

What is especially interesting to us is that some definitions stated in textbooks, and in the education literature, define function in such a way that it is ambiguous as to whether they are using the Bourbaki Triple definition or the Ordered Pairs definition. For instance, consider the way that Stewart (2003) defines functions in his widely used calculus textbook: "A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E" (Stewart, 2003, p.12).

Stewart's (2003) definition contains an ambiguity. We can either (i) view D and E as part of the meaning of function and/or claim they need to be specified in advance, or (ii) view the statement as an existential statement where f is a function if there exist sets D and E that fit the definition. That is, the definition allows for similar interpretation as the notation "$f: D \to E$", under both types of function definition (Bourbaki Triple and Ordered Pairs). In the remainder of Stewart's (2003) text, he appears to treat a function as equivalent to its graph (p. 14) and claims all injective functions are invertible (p. 64). He always treats the range and image of a function equivalently and never concerns himself with codomains. He thus appears to be using the Ordered Pairs definition of function.

Not only is there ambiguity in what an author might mean by "function," but there is also ambiguity in what definition the "Bourbaki approach" to functions implies. Selden and Selden (1992) take a function to be a set of ordered pairs: "the formal *ordered pair definition* of function, first introduced in 1939, is often referred to as the *Bourbaki* approach" (p. 2). Based on the preceding discussion, one might think that a "Bourbaki function" is a Bourbaki Triple. However, the above quote illustrates that this interpretation is not so straightforward.

Our main point thus far is as follows: There are two different definitions of function that are not logically equivalent: (i) both definitions are used in advanced mathematics, secondary mathematics, and the writing of mathematicians; and (ii) even when an author defines functions in their text, it can sometimes still be ambiguous as to which definition they are using. In the next section, we discuss how the different definitions entail different interpretations of mathematical questions that a secondary student might encounter.

Revisiting Questions

We address each of the six previously posed questions using the most straightforward interpretation of the two definitions of function above:

1. Consider the diagram in the first question at the start of this paper. A Bourbaki Triple person might interpret the question to be asking if the triple $(\{(a, L), (b, M), (c, N)\}, X, Y)$ is a

function. The answer is "no"; there are members of X that are not assigned a value in Y (the structure is not total). We believe this is regarded as the normatively desired response. However, an Ordered Pairs person might interpret the question to be asking if the set $\{(a, L), (b, M), (c, N)\}$ is a function. Of course the answer is "yes", because it is a univalent set of ordered pairs (assuming *a, b,* and *c* are distinct).

2. This question was taken from Redden (2012), p. 249, Example 2. According to the Ordered Pairs definition, $\{(-1,4), (0,7), (2,3), (3,3), (4,-2)\}$ is a function because it is univalent. Its domain is $\{-1, 0, 2, 3, 4\}$. We believe this is regarded by most as the normatively desired response (as in Redden, 2012; Even, 1990). According to the literal Bourbaki definition, we would say this set is *not* a function, for the simple reason that functions are triples and not ordered pairs; that is, no domain and codomain are stipulated here. If we were less literal, this question would be undefined; we would need to know what domain and codomain were stipulated. It is possible to interpret the question to be asking if there *exists* a domain *D* and codomain *E* such that (f, D, E) is a function, where f is the given set of ordered pairs above. However, we would not expect this interpretation from a secondary student.
   A Bourbaki Triple person can adopt conventions so that their answer to this question agrees with that of an Ordered Pairs person. If a domain is not specified, the domain of a real-valued relation is stipulated to be the largest set of real numbers to which the relation can apply, and if the codomain is not specified, it is assumed to be *R* (in the context of secondary algebra, calculus, or real analysis).
3. According to the set of ordered pairs definition, $y = \sqrt{x}$ is a function. That is, we interpret the function at hand to be the set $\{(x, y) \in R{\times}R: y = \sqrt{x}\}$. As this set is univalent, it is a function. We believe this is regarded by most as the normatively desired response. However, interpreting the question using the Bourbaki Triple definition is less straightforward. What object are we asking is a function? If we are stipulating that the domain and codomain are both R, which is arguably a convention in the context of a calculus or real analysis course, then our question is "is $(\{(x, y) \in R{\times}R: y = \sqrt{x}\}, R, R)$ a function?" and the answer is a clear "no", since this triple is not total on R. On the other hand, we can interpret the domain to be $[0, \infty)$, the largest domain on which it can be defined. In this case, the answer is "yes", as the triple $(\{(x, y) \in R{\times}R: y = \sqrt{x}\}, [0, \infty), R)$ is a function.
4. Using the Ordered Pairs definition, $g(x) = \ln x$ is the inverse of $f(x) = e^x$. That is, the set $\{(x, y) \in R{\times}R: y = e^x\}$ has the inverse $\{(x, y) \in R{\times}R: y = \ln x\}$, which is a function. We believe this is regarded as the normatively desired response; see, for instance, Stewart (2003, p.67), who defines logarithmic functions as the inverses to exponential functions. However, a Bourbaki Triple person who assumed the convention that the codomain of a real-valued function is *R* unless otherwise stated would assume that the question is asking if $(\{(x, y) \in R{\times}R: y = e^x\}, R, R)$ has an inverse function, and such a structure is not invertible as a function because it is not surjective onto R.
5. For similar reasons as in the previous example, an Ordered Pairs person would straightforwardly believe that $g(x) = \ln x$ has an inverse function (it is injective) and its inverse is $f(x) = e^x$, which, as we noted above, we believe is the normatively desired response.
   The Bourbaki Triple person would have a less straightforward answer. If they accept that $g(x) = \ln x$ is a function, then they would interpret it as a function from R^+ to R to ensure that it were total. Now, since *g* is bijective between R^+ to R, *g* must have an inverse. In this case, they would be interpreting "$g(x) = ln\, x$" to name the triple $(\{(x, y) \in R{\times}R: y = ln\, x\}, R^+, R)$. However, the inverse is not $f(x) = e^x$ because they would interpret "$f(x) = e^x$" to name the triple $(\{(x, y) \in R{\times}R: y = e^x\}, R, R)$, and this function's

codomain (R) is not the domain of $(\{(x, y) \in R \times R: y = ln\, x\}, R^+, R)$. The inverse would be the function $h: R \to R^+$ defined by $h(x) = e^x$, which is the triple $(\{(x, y) \in R \times R: y = e^x\}, R, R^+)$. There is nothing contradictory about this; the Bourbaki Triple definition implies that functions with the same graphs but different codomains are different functions. We only observe that we ordinarily would not expect a student to distinguish between the differing codomains to receive credit for identifying the inverse of $g(x) = ln\, x$.

6. As we noted in the previous section, the Ordered Pairs person agrees that a function is invertible whenever it is injective. The Bourbaki Triple person disagrees, claiming a function needs to be surjective as well. It appears there is no consensus in textbooks or amongst mathematics educators for whether injective functions are necessarily invertible. Some, like Abbott (2012, p. 155) and Stewart (2003, p. 64) assert that all injective functions are invertible. Others, like Mattuck (1999) and Friedberg et al. (1989), claim that injectivity and surjectivity are both necessary.

Two Approaches For Coping With Difference In Mathematics Education

Functions and their inverses are fundamental concepts. Naturally, mathematics educators would like students to develop productive and normatively correct understandings of functions and inverses. However, there are two different ways of defining functions that lead to divergent answers to basic questions from secondary mathematics. For instance, one would hope that there is a straightforward consensus answer as to whether $f(x) = e^x$ is invertible, at least in the context of secondary mathematics, but this is not the case. How, then, is an educator to teach students or evaluate the quality of a student's understanding of functions and their inverses?

There are at least two positions that an educator may adopt: *dogmatism* or *contextualism*. With dogmatism, we can insist that one of the two definitions is *the right definition,* argue that textbook writers and other researchers should use this definition, and regard those who do not act in accordance with this definition as being mathematically sloppy or incorrect. For instance, a Bourbaki Triple dogmatist might insist on using the Bourbaki Triple definition; the Bourbaki Triple dogmatist might call for textbooks to clarify Questions like 2 and 3 in the beginning of the paper to be mathematically accurate. This would involve rewrites such as "is there a set *D* (domain) and a set *E* (codomain) such that $(\{(-1,4), (0,7), (2,3), (3,3), (4, -2)\}, D, E)$ is a function?" If $f(x) = \sqrt{x}$ defines a graph of a function, what could its domain be?"). The Bourbaki Triple dogmatist acknowledges that some textbooks and even some mathematics educators use the ordered pairs definition of function, but that does not mean mathematics educators *should* be flexible with their definition of function. Indeed, it would be unwise policy to draw conclusions about the nature of mathematics based on errors that textbook writers and mathematics educators sometimes make. Similar dogmatism could be recommended by an advocate for the Ordered Pairs definition. We admit that a dogmatist approach has several advantages. For one, it would unify the differential treatment of functions in textbooks and mathematics education literature. Further, it would provide clear normative guidelines for how functions and inverses should be discussed and how students' understanding of these functions should be evaluated.

The alternative approach, *contextualism,* is to declare that there is no universal definition of function, but rather that the definition of function depends on context. Consider the following passage from Dumas and McCarthy's (2015) text in which they justify adopting the Bourbaki Triple definition:

> If you identified the function with its graph, then every function would have many possible codomains (take any superset of the original codomain). Set theorists think of functions this way, and if functions are considered as sets, extensionality requires that functions with the

> same graph are identical. However, this convention would make a discussion of surjections clumsy, so we shall not adopt it. (p. 25)

We highlight three ideas here. First, Dumas and McCarthy acknowledge that there was more than one definition of function that they could have used in their textbook. In particular, they do not say set theorists are wrong for defining a function as a univalent set of ordered pairs. Second, they view the decision on which definition to adopt as their choice. Third, they do not view their choice as arbitrary; they provide a reason justifying their choice; they adopted the Bourbaki triple definition because it made it easier to discuss and reason about surjective functions, which was one of their goals in the textbook.

Similarly, Joel David Hamkins, a mathematician and philosopher from Oxford University, justifies why set theorists like himself prefer to think of functions as ordered pairs for mathematically practical reasons. For instance, Hamkins explains why it is difficult to speak of sequences of ordinals in set theoretic terms using the Bourbaki Triple definition. Responding to a challenge that the concept of function is "imprecise", Hamkins responds:

> Many words lack meaning out of context, while becoming precise in a context. Why should you expect that there is a meaning for this word [function] outside of any context? [...] The function concept *has* been made absolutely precise. In fact, it has been made fully precise twice, in two different ways. Each group prefers to use their own precise definition, for sound reasons. (Hamkins, 2010)

The contextual position, for which we advocate, synthesizes the comments above. The contextualist acknowledges that for mathematicians, everything else being equal, it would be best for the same concept to be defined in the same way across all mathematical contexts and communities. This adds clarity and facilitates communication between different mathematical communities. However, mathematicians consider other factors to consider when choosing a concept's definition. In particular, mathematicians desire that their definitions should facilitate their communication, problem posing, and problem solving. Because the needs with respect to function vary by mathematical community, it is not surprising that different mathematical communities would define the function concept in different ways. The value of uniformity in these cases is not necessarily worth more than the value of utility.

In the case of *function* in secondary mathematics, functions are usually used for purposes of modeling and equation solving. In these cases, the totality of a function tends not to matter. The functions $f(x) = 1/(x^2 + 1)$ and $g(x) = 1/x^2$ are, for the most part, interpreted and acted upon in the same way, even though the former is total while the latter is not. It would be detrimental to the theory to exclude partial functions on R like $g(x) = 1/x^2$ or $h(x) = tan\,x$, and it would be cumbersome to constantly stipulate domains. For these reasons, we think it is prudent for textbooks to ignore the totality of functions in this context, except in the cases where the lack of totality matters. Similarly, inverse functions are generally used to assist in equation solving, graphing, and differentiation. In most cases, the use of inverses does not depend on whether the inverse is total on the codomain of the original function. It would be detrimental to the theory to eliminate inverse functions for functions that are not surjective, and it would be cumbersome to complicate the reasoning by introducing and changing codomains. Textbooks are justified in treating injective functions as always having inverses in this context. However, in other contexts in which non-total functions are not the object of consideration or would needlessly complicate the theory (e.g., group homomorphisms), it makes sense to adopt the Bourbaki Triple definition. Likewise, in contexts in which surjectivity or function composition play a central role, it might make sense to adopt the Bourbaki Triple definition.

We conclude by offering some recommendations to the mathematics education community for their investigations of functions and inverses:

- Be aware that there are two definitions of functions in mathematical practice and that these definitions entail different answers to questions that a secondary student might be asked.
- In research reports, state which conception you have in mind and justify your choice. How did your conception of function allow you to achieve your pedagogical or research goals? If your definition of function was not germane to the study (e.g., you were only focusing on univalence and issues of totality and inverses did not arise), explain that too.
- Avoid being a dogmatist when evaluating research papers. Just because a scholar used a different definition of function than you would prefer does not mean that they are mathematically incorrect. On the contrary, regardless of whether they adopted the Bourbaki Triple definition or the Ordered Pairs definition, they are in good company with many prestigious mathematicians.
- Avoid being a dogmatist when evaluating students. Regardless of whether a student asserted that "every injective function is invertible" or its negation, it would be a mistake to evaluate this comment as mathematically incorrect. It would be more appropriate to look at the reasonings and understandings that the student used to justify their assertion.
- In instruction, it is misleading to assert that mathematical definitions are always universal. There are multiple definitions for mathematical concepts that are not logically equivalent, even in secondary mathematics. Other examples besides functions include natural numbers (does this set include 0?) and trapezoids (is a parallelogram a trapezoid?). Rather than speak in absolutes, we suggest acknowledging that some mathematical concepts are defined in different ways, and to highlight the benefit of the definitional choice in the particular classroom context in question.

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