ANOTHER INSTANCE OF THE GOLDEN RIGHT TRIANGLE

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(*Submitted October 1992*)

The golden ratio $\tau = (1+\sqrt{5})/2$, the positive root of $x^2 = x+1$, makes an unexpected appearance in [1], where a certain right triangle turns out to be a "Golden Right Triangle" (GRT), one having sides proportional to $(1, \tau^{1/2}, \tau)$. The author wonders about the existence of other sets of circumstances where the GRT makes an unexpected appearance. In this note, such an occasion arises.

Suppose you are given two line segments of length a and b, then consider the problem in Euclidean geometry of constructing a right-angled triangle with hypotenuse of length equal to the arithmetic mean ($\overline{A}$) of a and b, that is,

$$\overline{A} = \frac{a+b}{2},$$

and one other side equal to their geometric mean ($\overline{G}$), that is

$$\overline{G} = \sqrt{ab}.$$

This problem is readily solved, as indicated in Figure 1.

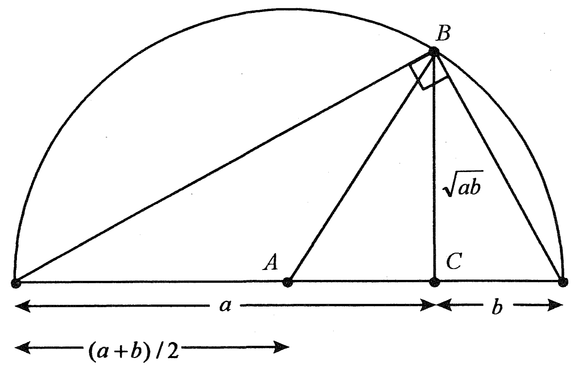

Figure 1

If we now demand that the shortest side AC of $\triangle ABC$ in Figure 1 is the harmonic mean ($\overline{H}$) of a and b, that is, $2/\overline{H} = 1/a + 1/b$, then we obtain the triangle with sides of length indicated in Figure 2. The problem now is to determine a and b so the lengths are proportional to $(\overline{A}, \overline{G}, \overline{H})$.

We can set $b = 1$ without loss of generality, and apply Pythagoras' Theorem to obtain, after some algebra: $a^4 - 18a^2 + 1 = 0$, giving $a^2 = 9 \pm 4\sqrt{5} = \tau^6$ and $1/\tau^6$ for positive and negative signs, respectively.

For the positive sign, and the larger root, we have $a = \tau^3 = 2\tau + 1$. In this case the required triangle has sides of length

$$\left((\tau+1), (2\tau+1)^{1/2}, \frac{(2\tau+1)}{(\tau+1)}\right).$$

This can be written as $(\tau^2, \tau^{3/2}, \tau)$, which is proportional to $(\tau, \tau^{1/2}, 1)$, the GRT above.

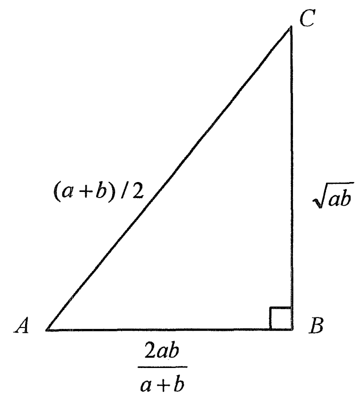

Figure 2

For the negative sign, and smaller root, we have $a = 1/\tau^3$ and $b = 1$. This results in a triangle with sides of length

$$\left(\frac{(\tau+1)}{(2\tau+1)}, \frac{1}{(2\tau+1)^{1/2}}, \frac{1}{(\tau+1)}\right).$$

Again, this is proportional to GRT, the side lengths being reciprocal to those above.

Here, then, is another situation in which the golden ratio makes an unexpected appearance.

REFERENCE

1. Duane W. DeTemple. "The Triangle of Smallest Perimeter Which Circumscribes a Given Semicircle." *The Fibonacci Quarterly* **30.3** (1992):274.

AMS Classification Numbers: 51M04, 11B39

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