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175

Renat Yuldashev

Synthesis of Phase-Locked Loop

Analytical Methods and Simulation

JYVÄSKYLÄN YLIOPISTO

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ABSTRACT

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One of the main approaches for PLL analysis is the numerical simulation of the models on the level of electronic realization. It is a challenging task, because the frequency of oscillators in modern computers and communication systems can reach up to several gigahertz. However, the frequency of the loop bandwidth may be only multiple of kilohertz. This 3 to 6 order of magnitude difference in frequencies makes simulation of PLL very difficult. To overcome these difficulties, a special mathematical model of phase-locked loop, in which only slow time scale of signals phases and frequencies is considered. That, in turn, requires to construct mathematical model of all components of the PLL. One of the main components of the PLL is a phase detector, whose operation depends on the waveforms of considered signals. In this work a general approach for analytical calculation of phase detector characteristic for classical phase-locked loop for various signal waveforms is proposed. Obtained PLL and PLL with a squarer models allow to use phase-frequency model for PLL, thereby significantly reducing time required for numerical simulation.

Keywords: phase-locked loop, phase detector, characteristic, pll with squarer

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PV G.A. Leonov, N.V. Kuznetsov, M.V. Yuldashev, R.V. Yuldashev. Computation of Phase Detector Characteristics in Synchronization Systems. *Doklady Mathematics, Vol. 84, No. 1, pp. 586–590*, 2011.

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PVII N.V. Kuznetsov, G.A. Leonov, P. Neittaanmäki, S.M. Seledzhi, M.V. Yuldashev, R.V. Yuldashev. Nonlinear Analysis of Phase-Locked Loop. *PSYCO 2010 - Periodic Control Systems, IFAC Proceedings Volumes (IFAC-PapersOnline), Vol. 4, No. 1, pp. 34–38*, 2010.

1 INTRODUCTION AND THE STRUCTURE OF THE WORK

1.1 Introduction

Phase-locked loop (PLL) is a circuit invented by a French engineer Anri de Bellescize in the 1930s (Bellescize, 1932). One of the first applications of PLL was in the wireless communication (Wendt and Fredentall, 1943; Richman, 1954). In radio-engineering, PLL-based circuits (e.g., a PLL system with a squarer and Costas loop) were and still are widely used for demodulation, carrier recovery, and frequency synthesis (Costas, 1962; Miyazaki et al., 1991; Fines and Aghvami, 1991; Fiocchi et al., 1999; Bullock, 2000; Kaplan and Hegarty, 2006; Tomkins et al., 2009; Kim et al., 2010; Bullock, 2000; Manassewitsch, 2005). After realization of PLL in integrated circuit, these devices became used in computer architectures and multiprocessor clusters for clock synchronization, frequency synthesis, and others (see e.g. (Kung, 1988; Smith, 1999; Gardner et al., 1993; Simpson, 1944; Lapsley et al., 1997; Buchanan and Wilson, 2001; Xanthopoulos et al., 2001; Wainner and Richmond, 2003; Bindal et al., 2003; Young, 2004; Shu and Sanchez-Sinencio, 2005)). PLLs are also used in optical systems (see e.g. (von Lerber et al., 2009), and gyroscopes (see e.g. (Saukoski et al., 2008)))

The first articles and books on the analysis of PLL were published in the 1960's (see e.g. (Viterbi, 1966; Gardner, 1966; Lindsey, 1972)). Analysis of PLL is an active research area. A large number of articles and patents devoted to the analysis and synthesis PLL published by researchers and engineers in Finland (see e.g. (Kauraniemi and Vuori, 1997; Vankka, 1997; Kauraniemi and Vuori, 1997; Ahola et al., 1998, 1999; Hakkinen and Kostamovaara, 2003; Ahola and Halonen, 2003; Aaltonen et al., 2005; Rapinoja et al., 2006; Saukoski et al., 2008; von Lerber et al., 2009; Rapinoja et al., 2009; Speeti et al., 2009; Tchamov et al., 2009; Rapeli, 1992; Strommer et al., 1992; Niemio, 1992; Jorgensen, 1999; Haapanen and Dekker, 1998; Korhonen, 1994; Latva-Aho, 1996)).

Although a PLL is inherently a nonlinear circuit, in the modern literature, the main approaches to its analysis are based on the use of simplified linear models (Shakhgil'dyan and Lyakhovkin, 1972; Gardner, 1993; Egan, 2000; Best, 2003; Shu and Sanchez-Sinencio, 2005; Tretter, 2007)), the methods of linear analysis (Gardner, 1993; Egan, 2000; Best, 2003), empirical rules, and numerical simulation on the level of electronic realization (see a plenary lecture of D. Abramovitch at the 2002 American Control Conference (Abramovitch, 2002)). However it is known that the application of linearization methods and linear analysis for control systems can lead to untrue results[1].

A rigorous mathematical analysis of PLL models on the level of electronic realization, which are described by a nonlinear non-autonomous system of differential equations, is often a very challenging task (Kudrewicz and Wasowicz, 2007; Suarez and Quere, 2003; Margaris, 2004; Benarjee, 2006; Feely, 2007; Banerjee and Sarkar, 2008; Feely et al., 2012). Therefore, for the analysis of a nonlinear PLL a numerical simulation is often used (see, e.g., (Lai et al., 2005; Best, 2007)). However, in the context of high-frequency signals, complete numerical simulation of PLL-based circuit on the level of electronic realization (in the signal/time space), is a very challenging task. Here, it is necessary to observe simultaneously "very fast time scale of the input signals" and "slow time scale of signal's phases" (Abramovitch, 2008b,a). Therefore, a relatively small discretization step in the simulation procedure does not allow one to consider phase locking processes for the high-frequency signals (up to 10Ghz) in a reasonable time period (see e.g. (Tranter et al., 2010)).

To overcome these difficulties, special nonlinear models of PLL in the phase-frequency space were suggested by Viterbi, Gardner and others (see (Viterbi, 1966; Gardner, 1966; Lindsey, 1972)). But analysis of these models was carried out in the context of sinusoidal and square-wave signals only. This approach requires a mathematical description of the circuit components and a proof of the reliability of the considered model (Leonov et al., 2006; Kudrewicz and Wasowicz, 2007; Kuznetsov et al., 2008; Kuznetsov, 2008; Leonov et al., 2009; Kuznetsov et al., 2009b; Leonov et al., 2010). One of the main components of the classical PLL is a multiplier, used as the phase detector (PD). This element allows one to compare phases of the input signals.

In this work mathematical description of the multiplier operation is presented, and characteristic of the PD for various signal waveforms is presented. The characteristics of a PD obtained in this work are in line with the well-known characteristics for the sinusoidal and square waveforms. All main results are rigorously proved (see proofs of the theorems in (PI; PIV)). PLL models in the phase-frequency domain are justified using the averaging method by Krylov-

1 See chaotic hidden attractors in electronic Chua circuits (Leonov et al., 2010; Kuznetsov et al., 2011bb,a; Bragin et al., 2011; Leonov et al., 2011b, 2012; Leonov and Kuznetsov, 2012; Kuznetsov et al., 2013a; Leonov and Kuznetsov, 2013; Leonov et al., 2011b,a), in drilling systems (Kiseleva et al., 2012, 2014; Leonov G. A., 2013b), in two-dimensional polynomial quadratic systems (Kuznetsov et al., 2013b; Leonov et al., 2011a; Leonov and Kuznetsov, 2010; Kuznetsov and Leonov, 2008; Leonov et al., 2008; Leonov and Kuznetsov, 2007; Kuznetsov, 2008), and in PLL (Leonov and Kuznetsov, 2014). Another examples are Aizerman's and Kalman's conjectures on absolute stability (Leonov et al., 2010b,a; Leonov and Kuznetsov, 2011; Leonov G. A., 2013a; Leonov and Kuznetsov, 2013)

Bogolyubov. Numerical simulation confirms the adequacy of the derived models. The PLL models considered in this work allow to apply well-developed theory of qualitative analysis for dynamic systems (Leonov et al., 1992, 1996a,b; Yakobovich et al., 2004).

Obtained PLL and PLL with a squarer models allow to use phase-frequency model for PLL, thereby significantly reducing time required for numerical simulation. This allows one to determine such important features of the systems involved as pull-in range, pull-out range, and other properties, thereby, significantly reducing time needed for the development of PLL-based circuits and their analysis.

1.2 Structure of the work

The first chapter of this work is devoted to description of the operating principles of the classical PLL. Definition of asymptotic equivalence and conditions describing high-frequency signals are presented. Phase detector characteristic for standard waveforms are presented. Differential equations of the phase-locked loop and phase-locked loop with a squarer are discussed.

In the second chapter simulation of the classical PLL circuit is discussed. Asymptotic equivalence of PLL models for various non-sinusoidal and sinusoidal waveforms is justified numerically. Fast simulation method based on the phase-frequency model of PLL is presented.

Appendix contains definitions and theorems on Fourier series necessary for the proofs of the asymptotic equivalence.

1.3 Included articles and author contribution

At the first stage of the research, the classical phase-locked loop with sinusoidal and square waveform signals was considered. For these waveforms asymptotic equivalence of the phase detector models was proved (see PVII). In this paper co-authors formulated the problems, the author derived the PD characterstic and proved the threorem.

Then, asymptotic equivalence for the general case of the waveforms was proved (see PV). Also, a table of phase detector characteristics for standard signal waveforms and differential equations of the PLL were presented (see PVI). In these papers co-authors formulated the problems, and constructed the table of PD characteristics. The author formulated and proved the theorem.

After that the high-frequency conditions were reformulated, which allowed to remove the assumption of the global stability of the VCO. This required to change the proofs, but the PD characteristics remained unchanged (see PIV). In this paper co-authors formulated the problems, and constructed the table of PD characteristics. The author formulated and proved the theorem.

Efficient simulation method based on these analytical results was considered in (see PIII). In this paper co-authors formulated the problem of effictient simulation of PLL, while author implemented PLL models in Matlab.

After that, asymptotic equivalence for phase locked loop with a squarer was formulated and proved. Also, new results on frequency doubling for non-sinusoidal waveforms and differential equations for PLL with a squarer were presented. (see PII). In this paper author formulated the theorem and constructed the table of PD characteristics. Co-authors highlighted the problem of frequency doubling and explained the operation and applications of PLL with squarer.

Finally, review of the various analog PLL implementations was considered (see PI). In this book chapter author author formulated problems of nonlinear analysis of various PLL modifications and proved theorems related to PLL and PLL with squarer. Co-authors have written part related to the Costas loops.

The results of this work were presented at a plenary lecture at an international conference Dynamical Systems and Applications (Kiev, Ukrane – 2012), IEEE international Congress on Ultra Modern Telecommunications and Control Systems and Workshops (St.Petersburg, Russia – 2012), at regular lectures at IEEE 4th International Conference on Nonlinear Science and Complexity (Budapest, Hungary – 2012), 9th International Conference on Informatics in Control, Automation and Robotics (Rome, Italy – 2012), IEEE 10-th International Symposium on Signals, Circuits and Systems (Iasi, Romania – 2011), 8th International Conference on Informatics in Control, Automation and Robotics (Noordwijkerhout, The Netherlands – 2011), 4th IFAC Workshop on Periodic Control System (Antalya, Turkey – 2010), International Workshop "Mathematical and Numerical Modeling in Science and Technology" (Jyväskylä, Finland – 2010); at the seminars of the Department of Applied Cybernetics (St. Petersburg State University, Faculty of Mathematics and Mechanics) and at the seminars of the Department of Mathematical Information Technology (University of Jyväskylä, Finland). The main results of this work are also included in (Yuldashev, 2013, 2012).

2 THE MAIN CONTENT

Next, following the papers (PI; PIV; PV; PVI; Yuldashev, 2012, 2013; Kuznetsov et al., 2011a,b; PVII; PII) the main content is presented.

2.1 Asymptotic equivalence of PLL models

Consider the classic PLL with phase detector implemented as a multiplier in signal space shown in Fig. 1

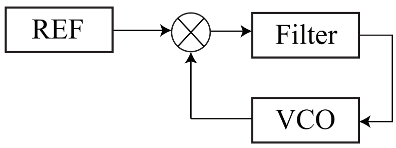


FIGURE 1 Block diagram of classical phase-locked loop

The diagram comprises the following blocks (Boensel, 1967; Best, 2003; Kroupa, 2003): a voltage-controlled oscillator (VCO), a low-pass linear filter, a reference oscillator (REF), and an analog multiplier $\otimes$ used as the phase detector (PD). The phase detector compares the phase of the VCO signal against the phase of the input signal; the output of the PD (error voltage) is proportional to the phase difference between its two inputs. The error voltage is then filtered by the loop filter, whose control output is applied to the VCO by changing its frequency in the direction that reduces the phase difference between the input signal and the VCO.

In the simplest case the waveforms of the VCO and the reference oscillator signals are both sinusoidal (see Fig. 2).

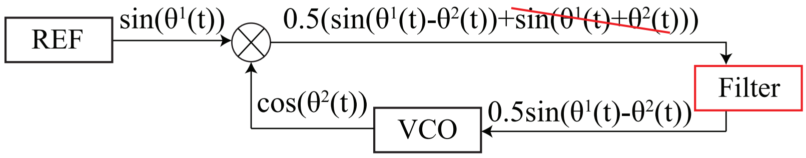


FIGURE 2 Operation of classical phase-locked loop for sinusoidal signals

Then, the output of the multiplier is

$$f^1(t)f^2(t) = 0.5\left(\sin(\theta^1(t) - \theta^2(t)) + \sin(\theta^1(t) + \theta^2(t))\right). \tag{1}$$

The low-pass filter passes low-frequency signal $\sin(\theta^1(t) - \theta^2(t))$ and attenuates high-frequency signal $\sin(\theta^1(t) + \theta^2(t))$. For small phase difference, the low-pass filter output contains only signal approximately equal to difference between the phases of the VCO and reference signal

$$0.5(\sin(\theta^1(t) - \theta^2(t))) \approx 0.5(\theta^1(t) - \theta^2(t)).$$

To carry out a rigorous mathematical analysis of a PLL, consider PLL models in the signal and phase-frequency spaces (Leonov and Seledzhi, 2002; Abramovitch, 2002; Leonov, 2006; Kuznetsov et al., 2009a; PV; PVII). For constructing a nonlinear mathematical model of a classical PLL in the phase-frequency space, it is necessary to calculate characteristic of the PD (Shakhgil'dyan and Lyakhovkin, 1972; Lindsey, 1972; Lindsey and Simon, 1973; Best, 2007). For this it is necessary to consider the block diagram shown in Fig. 3.

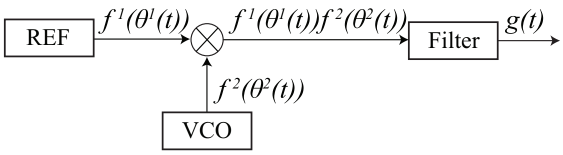


FIGURE 3 Multiplier and filter

Here REF generates high-frequency signal $f^1(t) = f^1(\theta^1(t))$, and VCO is a tunable oscillator which generates high-frequency signal $f^2(t) = f^2(\theta^2(t))$, where $\theta^{1,2}(t)$ are phases. The output of the multiplier $f^1(\theta^1(t))f^2(\theta^2(t))$ enters the linear low pass filter and $g(t)$ is the output of the filter.

Suppose that the waveforms of input signal and VCO $f^{1,2}(\theta)$ are 2π- periodic piecewise differentiable functions, i.e. the functions with a finite number of jump discontinuity points differentiable on their continuity intervals (this is true for the most considered waveforms). Also, let us assume that the phases $\theta^{1,2}(t)$ are smooth, monotonically increasing functions with the derivatives (frequencies) satisfying the inequalities

$$\frac{d\theta^p}{dt}(t) \geq \omega_{min} >> 1, \quad p = 1, 2. \tag{2}$$

This means that frequencies are changing continuously, which is corresponding to classical PLL analysis (Best, 2007; Kroupa, 2003). Here ω_{min} is a fixed positive constant. It should be noted that in the modern electronic devices, generators can reach frequencies up to 10GHz.

Consider description of the model on the Fig. 3 in phase-frequency space.

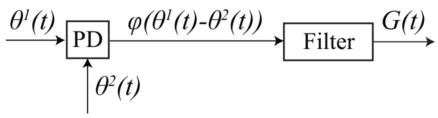


FIGURE 4 Phase detector and filter

Here, the PD is a nonlinear block with the characteristic $\varphi(\theta)$. The output of this block is the function $\varphi\big(\theta^1(t) - \theta^2(t)\big)$ and the inputs are the phases $\theta^{1,2}(t)$. The PD characteristic $\varphi(\theta)$ depends on waveforms $f^{1,2}(\theta)$. Assume that impulse response functions and initial data for the low-pass filters in Fig. 3 and Fig. 4 coincide.

Definition. The block diagrams shown in Fig. 3 and Fig. 4 are called *asymptotically equivalent*, if for a large fixed time interval $[0, T]$ the following relation holds

$$G(t) - g(t) = O(\delta), \quad \delta = \delta(\omega_{min}), \quad \forall t \in [0, T], \tag{3}$$

where $\delta(\omega_{min}) \to 0$ as $\omega_{min} \to \infty$.

The definition of asymptotically equivalent block diagrams allows one to pass from the analysis of the signal space PLL model to the analysis of the phase-frequency space PLL model (Fig. 5)

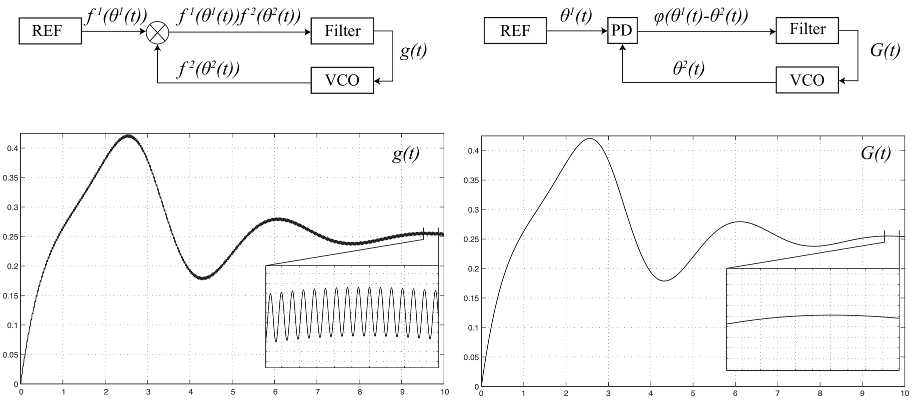


FIGURE 5 Asymptotic equivalence of PLL model in signal space and phase-frequency space PLL model

The equivalence of the block-diagrams shown in Fig. 3 and Fig. 4 was demonstrated by Viterbi and Gardner (Viterbi, 1966; Gardner, 1966), yet only for the sinusoidal signals and without a formal justification. The first mathematically rigorous conditions of high-frequency signals and proof of the asymptotic equivalence of the block-diagrams in the case of the sinusoidal and square-wave signals were presented in (Leonov and Seledzhi, 2005).

In the following section, an analytical method for calculating the characteristic of the multiplier/mixer is proposed (see (PV; PIV)). For various waveforms of high-frequency signals, including non-sinusoidal waveforms (e.g. sawtooth, triangle, etc.) used in practice (Henning, 1981; Fiocchi et al., 1999), new classes of phase-detector characteristics are obtained, and a dynamical model of a PLL is constructed.

2.2 The basic assumptions

Following the works (PI; PIV; PV) we consider Fourier series representation of $f^{1,2}(\theta)$

$$\begin{aligned} f^p(\theta) &= \frac{a_0^p}{2} + \sum_{i=1}^{\infty} \big(a_i^p \cos(i\theta) + b_i^p \sin(i\theta)\big), \quad p = 1, 2, \\ a_0^p &= \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) dx, \\ a_i^p &= \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) \cos(ix) dx, \quad b_i^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) \sin(ix) dx, \quad i \in \mathbb{N}. \end{aligned} \tag{4}$$

Needed convergence properties of the Fourier series are described in the appendix (see Appendix, theorems 4, 6). The relationship between the output $\psi(t)$ and the input $\xi(t)$ of the linear low-pass filter is as follows:

$$\psi(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)d\tau, \tag{5}$$

where $\alpha_0(t)$ is an exponentially damped function, linearly dependent on the initial state of the filter at $t = 0$, and $\gamma(t)$ is an impulse response function of the filter. We also assume that $\gamma(t)$ is a differentiable function with a bounded derivative (this is true for the most considered filters (Best, 2007)). From equation (5) we get

$$g(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau) f^1\big(\theta^1(\tau)\big) f^2\big(\theta^2(\tau)\big) d\tau. \tag{6}$$

Let us assume that the frequency error $\frac{d\theta^1}{dt}(\tau) - \frac{d\theta^2}{dt}(\tau)$ is uniformly bounded (see e.g. (Kuznetsov et al., 2008; Leonov, 2008; Leonov et al., 2009, 2010))

$$\Big|\frac{d\theta^1}{dt}(\tau) - \frac{d\theta^2}{dt}(\tau)\Big| \le \Delta\omega, \ \forall \tau \in [0, T], \tag{7}$$

where $\Delta\omega$ is a constant.

We split the time interval $[0, T]$ into the subintervals of length δ

$$\delta = \frac{1}{\sqrt{\omega_{min}}}. \tag{8}$$

Furthermore, we assume that

$$|\frac{d\theta^p}{dt}(\tau) - \frac{d\theta^p}{dt}(t)| \leq \Delta\Omega,\ p = 1, 2, \quad |t - \tau| \leq \delta, \quad \forall \tau, t \in [0, T], \tag{9}$$

where $\Delta\Omega$ does not depend on τ and t. Conditions (7)–(9) imply that the frequencies $\frac{d\theta^p(t)}{dt}$ are almost constant and the signals themselves $f^p(\theta^p(t))$ are rapidly oscillating within small subintervals $[t, t + \delta]$. The boundedness of the derivative of the impulse response function $\gamma(t)$ implies that there exists a constant C, such that

$$|\gamma(\tau) - \gamma(t)| \leq C\delta, \quad |t - \tau| \leq \delta, \quad \forall \tau, t \in [0, T]. \tag{10}$$

2.3 Phase detector characteristics of PLL

The following theorem was formulated for various types of signal waveforms and high-frequency conditions in papers (PI; PIV).

Theorem 1. *Let conditions* (2), (4), (7) – (10) *be satisfied. Then the systems shown in Fig. 3 and Fig. 4 are asymptotically equivalent, where*

$$\varphi(\theta) = \frac{a_0^1 a_0^2}{4} + \frac{1}{2}\sum_{l=1}^{\infty}\Big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + (a_l^1 b_l^2 - b_l^1 a_l^2)\sin(l\theta)\Big). \tag{11}$$

Proof idea. Consider difference

$$\begin{aligned} & g(t) - G(t) = \\ & \alpha_0(t) + \int_0^t \gamma(t-\tau) f^1(\theta^1(\tau)) f^2(\theta^2(\tau)) d\tau - \\ & \alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi(\theta^2(\tau) - \theta^1(\tau)) d\tau = \\ & \int_0^t \gamma(t-\tau)\Big(f^1(\theta^1(\tau)) f^2(\theta^2(\tau)) - \varphi(\theta^2(\tau) - \theta^1(\tau))\Big)\, d\tau. \end{aligned} \tag{12}$$

Then split the interval $[0, T]$ into the subintervals of length δ

$$\int_{k\delta}^{(k+1)\delta} \gamma(t-\tau)\Big(f^1(\theta^1(\tau)) f^2(\theta^2(\tau)) - \varphi(\theta^2(\tau) - \theta^1(\tau))\Big)\, d\tau. \tag{13}$$

For sinusoidal waveforms $f^1(\theta) = \sin(\theta)$, $f^2(\theta) = \cos(\theta)$ the PD characteristic is $\varphi(\theta) = 0.5\sin(\theta)$. Therefore we have

$$\int\limits_{k\delta}^{(k+1)\delta} \gamma(t-\tau)\left(\sin\left(\theta^1(\tau)\right)\cos\left(\theta^2(\tau)\right) - 0.5\sin(\theta^2(\tau) - \theta^1(\tau))\right)d\tau = \\ 0.5\int\limits_{k\delta}^{(k+1)\delta} \gamma(t-\tau)\sin(\theta^1(\tau) + \theta^2(\tau))d\tau. \tag{14}$$

Since $\sin(\theta^1(\tau) + \theta^2(\tau))$ is high-frequency signal (see conditions (2), (9)), from (10) we have

$$\int\limits_{k\delta}^{(k+1)\delta} \gamma(t-\tau)\sin(\theta^1(\tau) + \theta^2(\tau))d\tau = O(\delta^2). \tag{15}$$

Finding the sum of these integrals, we obtain the desired estimate.

For non-sinusoidal waveforms it is possible to use the property of uniform convergence for the Fourier series on the intervals of continuity (see Appendix, (6)). The neighborhood of the points of discontinuity may be chosen sufficiently small that they do not affect the estimates. This allows you to proceed with the evaluation of the following integrals of the individual harmonics

$$\int\limits_{k\delta}^{(k+1)\delta} \gamma(t-\tau)\sin(i\theta^1(\tau) \pm j\theta^2(\tau))d\tau, \quad \int\limits_{k\delta}^{(k+1)\delta} \gamma(t-\tau)\cos(i\theta^1(\tau) \pm j\theta^2(\tau))d\tau \tag{16}$$

for various combinations of integer numbers i and j (see PIV). ■

Remark 1. *For the most considered waveforms (e.g. sinusoidal, squarewave, sawtooth, polyharmonic), infinite series* (11) *can be truncated up to the first* $\sqrt{\omega_{min}}$ *terms.*

A remainder $R_{[\frac{1}{\delta}]}$ of series (11) can be estimated as

$$|R_{[\frac{1}{\delta}]}(x)| \le \sum_{l=[\frac{1}{\delta}]+1}^{\infty} (|a_l^1 a_l^2 + b_l^1 b_l^2| + |a_l^1 b_l^2 - b_l^1 a_l^2|) \le \\ \le \sum_{l=[\frac{1}{\delta}]+1}^{\infty} (|a_l^1 a_l^2| + |b_l^1 b_l^2| + |a_l^1 b_l^2| + |b_l^1 a_l^2|).$$

Since $a_l^{1,2} = O(\frac{1}{l})$ and $b_l^{1,2} = O(\frac{1}{l})$ (see, e.g., (Zygmund, 1968; Walker, 2003)) then

$$|R_{[\frac{1}{\delta}]}(x)| \le O\left(\sum_{l=[\frac{1}{\delta}]+1}^{\infty} \frac{1}{l^2}\right), \tag{17}$$

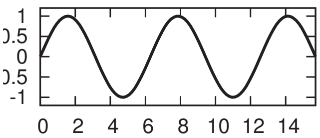
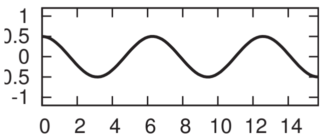
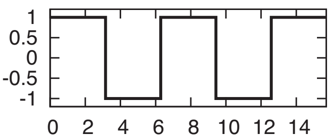
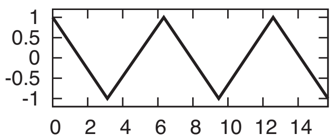
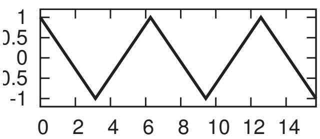
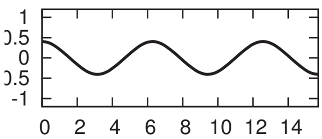
the sum is bounded by corresponding integral of function $\frac{1}{x^2}$. Then

$$|R_{[\frac{1}{\delta}]}(x)| \leq O(\delta), \tag{18}$$

which can be easily prooved by integration.

Theorem above allows one to calculate the phase detector characteristic for standard signal waveforms shown in the table below. Under the assumptions made, function $\varphi(\theta)$ is continuous, because the Fourier series $\varphi(\theta)$ converges uniformly. Convergence rate for the coefficients of $\varphi(\theta)$ is given by theorem 5 in appendix. To the left, the waveforms $f^{1,2}(\theta)$ of the input signals are shown; depicted to the right are the corresponding PD characteristics $\varphi(\theta)$.

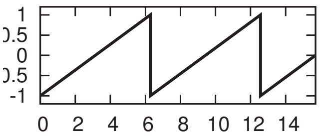
TABLE 1 Phase detector characteristics

Signals waveforms	PD characteristic
$f^{1,2}(\theta) = \sin(\theta)$ 	$\varphi(\theta) = \frac{1}{2}\cos(\theta)$ 
$f^{1,2}(\theta) = \text{sign}(\sin(\theta)) =$ $\frac{4}{\pi}\sum\limits_{n=1}^{\infty}\frac{\sin((2k-1)t)}{2k-1}$ 	$\varphi(\theta)=\frac{8}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big) =$ $\begin{cases} \frac{2}{\pi}\theta+1, & \theta\in[-\pi,0], \\ 1-\frac{2}{\pi}\theta, & \theta\in[0,\pi]. \end{cases}$ 
$f^{1,2}(\theta)=\frac{8}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big) =$ $\begin{cases} \frac{2}{\pi}\theta+1, & \theta\in[-\pi,0], \\ 1-\frac{2}{\pi}\theta, & \theta\in[0,\pi]. \end{cases}$ 	$\varphi(\theta) = \frac{32}{\pi^4}\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\big((2n-1)\theta\big)$ 
$f^{1,2}(\theta) = -\frac{2}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{n}\sin(n\theta) =$ $\frac{2}{\pi}\theta-1, \theta\in[0,2\pi]$	$\varphi(\theta) = \frac{2}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{1}{n^2}\cos(n\theta)$

(continues)

Table 1 (continues)

<table>
<tr><th>Signal waveforms</th><th>PD characteristic</th></tr>
<tr><td>



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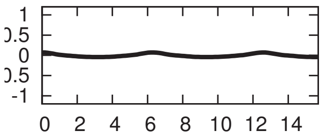

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</table>

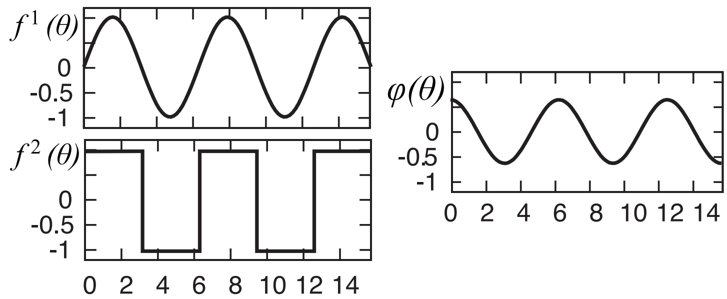
TABLE 2 PD characteristics of PLL for different waveforms

<table>
<tr><td>

$f^1(\theta) = \sin(\theta),$

$f^2(\theta) = \operatorname{sign}\sin(\theta) = \frac{4}{\pi}\sum\limits_{n=1}^{\infty}\frac{\sin((2k-1)t)}{2k-1},$

$\varphi(\theta) = \frac{2}{\pi}\cos(\theta)$



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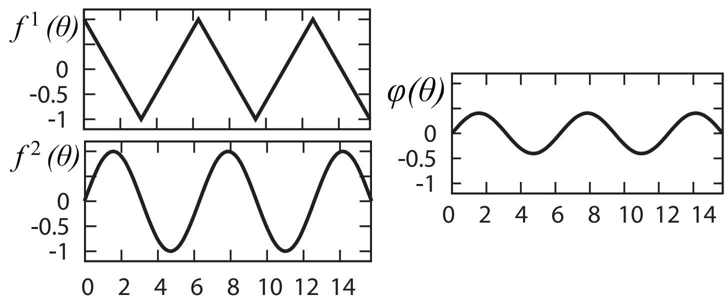
<tr><td>

$f^1(\theta) = \frac{8}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right) = \begin{cases} \frac{2}{\pi}\theta + 1, & \theta \in [-\pi, 0], \\ 1 - \frac{2}{\pi}\theta, & \theta \in [0, \pi]. \end{cases}$

$f^2(\theta) = \sin(\theta),$

$\varphi(\theta) = \frac{4}{\pi^2}\sin(\theta)$



</td></tr>
<tr><td>

$f^1(\theta) = -\frac{2}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{n}\sin\left(n\theta\right) = \frac{2}{\pi}\theta - 1, \theta \in [0, 2\pi],$

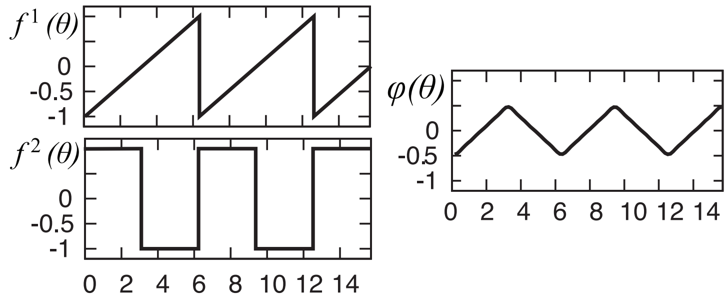
$f^2(\theta) = \operatorname{sign}\sin(\theta) = \frac{4}{\pi}\sum\limits_{n=1}^{\infty}\frac{\sin((2k-1)t)}{2k-1},$

$\varphi(\theta) = -\frac{4}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{\cos((2n-1)\theta)}{(2n-1)^2} = \begin{cases} \frac{1}{2\pi}\theta, & \theta \in [0, \pi], \\ 1 - \frac{1}{2\pi}\theta, & \theta \in [\pi, 2\pi]. \end{cases}$

</td></tr>
</table>

(continues)

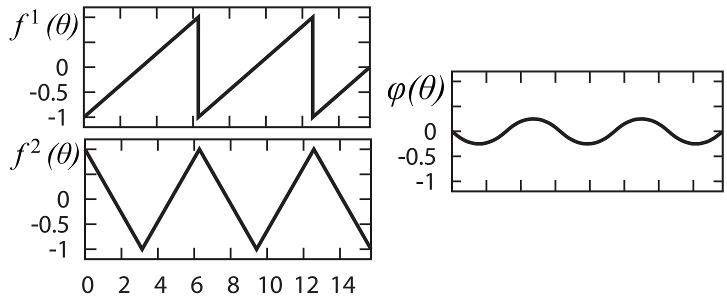
Table 2 (continues)



$f^1(\theta) = -\frac{2}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{n}\sin\left(n\theta\right) = \frac{2}{\pi}\theta - 1, \theta \in [0, 2\pi]$

$f^2(\theta) = \frac{8}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right) = \begin{cases} \frac{2}{\pi}\theta + 1, & \theta \in [-\pi, 0], \\ 1 - \frac{2}{\pi}\theta, & \theta \in [0, \pi]. \end{cases}$

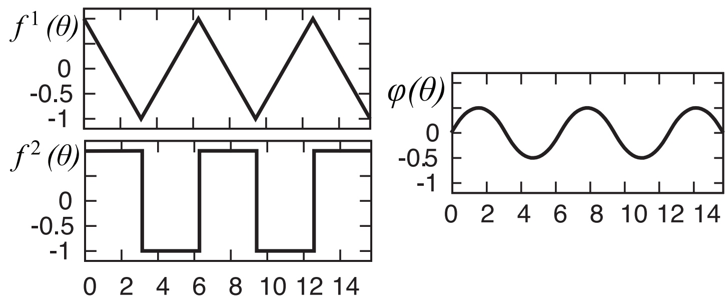
$\varphi(\theta) = -\frac{8}{\pi^3}\sum\limits_{n=1}^{\infty}\frac{\sin((2n-1)\theta)}{(2n-1)^3}$



$f^1(\theta) = \frac{8}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right) = \begin{cases} \frac{2}{\pi}\theta + 1, & \theta \in [-\pi, 0], \\ 1 - \frac{2}{\pi}\theta, & \theta \in [0, \pi]. \end{cases}$

$f^2(\theta) = \operatorname{sign}\sin(\theta) = \frac{4}{\pi}\sum\limits_{n=1}^{\infty}\frac{\sin((2k-1)t)}{2k-1},$

$\varphi(\theta) = -\frac{16}{\pi^3}\sum\limits_{n=1}^{\infty}\frac{\sin((2n-1)\theta)}{(2n-1)^3}$



2.4 Phase detector characteristics of PLL with squarer

Consider now the block diagram of a classical PLL system with a squarer (see, e.g., (Hershey et al., 2002; Goradia et al., 1990; PII)) in signal space shown in Fig. 6.

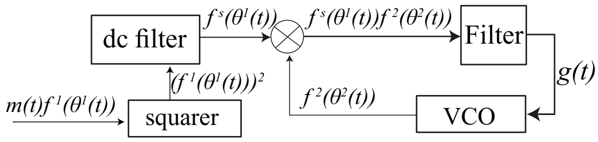


FIGURE 6 PLL with squarer in signal space

This circuit (sometimes called "squaring loop") is often used for the carrier recovery during the data transfer (see, e.g., (Lindsey and Simon, 1973)). The circuit consists of the following blocks: a multiplier, a linear low-pass filter, a voltage-controlled oscillator, and a squarer. The transmitted data is denoted by $m(t) = \pm 1$, and the carrier is denoted by $f^1(t) = f^1(\theta^1(t))$ The VCO generates the signal $f^2(t) = f^2(\theta^2(t))$. The squarer multiplies the input signal $m(t)f^1(\theta^1(t))$ by itself and the subsequent filter erases the DC offset (Best, 2007)

$$f^s(\theta^1(t)) = m(t)f^1(\theta^1(t))m(t)f^1(\theta^1(t)) = f^1(\theta^1(t))f^1(\theta^1(t)). \qquad (19)$$

Consequently, the data signal $m(t)$ does not affect the frequency of the VCO.

Consider the block-diagram in Fig. 7

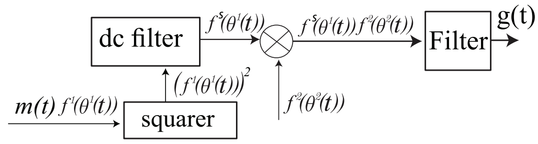


FIGURE 7 Classical PLL with squarer

From equation (5) we get

$$g(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau) f^s\big(\theta^1(\tau)\big) f^2\big(\theta^2(\tau)\big) d\tau. \qquad (20)$$

Theorem 2. *Let conditions* (2), (4), (7) – (10) *be satisfied. Then the systems in Fig. 7 and Fig. 4 are asymptotically equivalent, where*

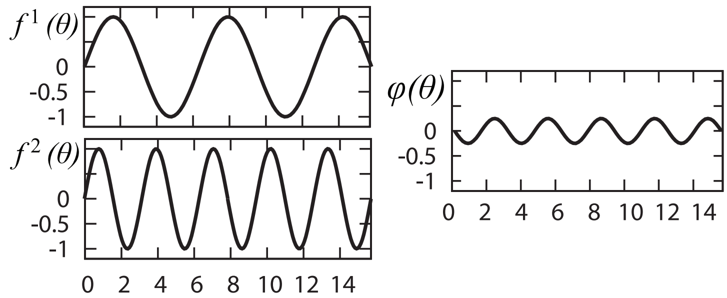
$$\begin{aligned}
\varphi(\theta) &= \tfrac{1}{2}\sum_{l=1}^{\infty}\Big((A_l^1 a_l^2 + B_l^1 b_l^2)\cos(l\theta) + (A_l^1 b_l^2 - B_l^1 a_l^2)\sin(l\theta)\Big),\\
A_l^1 &= \tfrac{1}{2}\sum_{m=1}^{\infty}\big(a_m^1(a_{m+l}^1 + a_{m-l}^1) + b_m^1(b_{m+k} + b_{m-k}^1)\big),\\
B_l^1 &= \tfrac{1}{2}\sum_{m=1}^{\infty}\big(a_m^1(b_{m+l}^1 - b_{m-l}^1) - b_m^1(a_{m+k} + a_{m-k}^1)\big).
\end{aligned} \qquad (21)$$

Proof idea. The theorem above is the direct corollary of the Theorem 1. Since all properties describing high-frequency oscillations hold, we can use the signal $f^s(\theta^1(s))$ as the input signal $f^1(\theta)$ from Theorem 1. For the full proof, see paper PII.

Phase detector characteristic of PLL with squarer for standard signal waveforms.

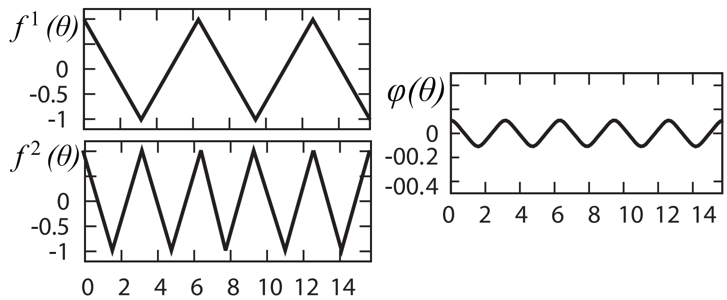
TABLE 3 PD characteristics of PLL with squarer for different waveforms

$f^1(\theta) = \sin(\theta)$, $f^2(\theta) = \sin(2\theta)$, $\varphi(\theta) = -\frac{1}{4}\sin(2\theta)$

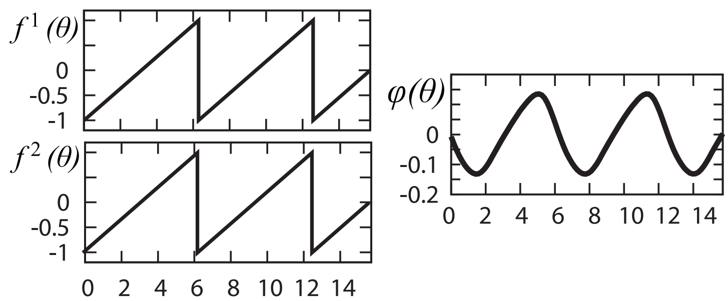


$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big)$,

$f^2(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((4n-2)\theta\big)$, $\varphi(\theta) = \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{4}{(4n-2)^4}\cos\big((4n-2)\theta\big)$



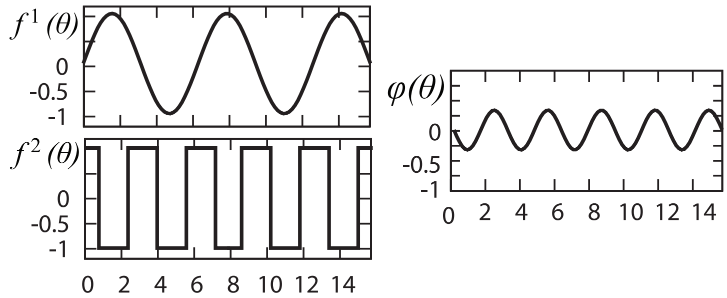
$f^{1,2}(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\big(n\theta\big)$, $\varphi(\theta) = -\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\big(n\theta\big)$



$f^1(\theta) = \sin(\theta)$, $f^2(\theta) = \operatorname{sign}\sin(2\theta)$, $\varphi(\theta) = -\frac{1}{\pi}\sin(2\theta)$

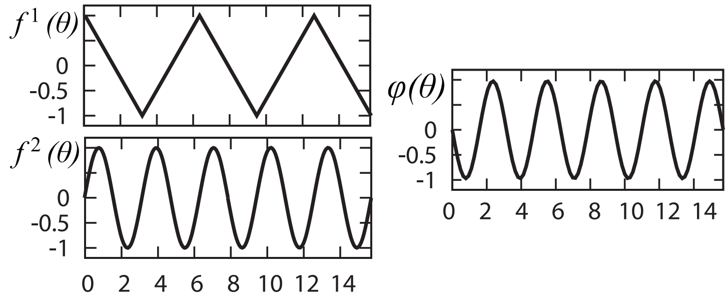
(continues)

Table 3 (continues)



$f^1(\theta) = \frac{8}{\pi^2} \sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\big((2n-1)\theta\big)$

$f^2(\theta) = \sin(2\theta)$, $\varphi(\theta) = -\frac{8}{\pi^2} \sin(2\theta)$



2.5 Differential equations of PLL

Let us derive system of ordinary differential equations for a PLL in signal space and in the phase-frequency space (see papers (PIV; PV; PVI; PI)).

First, consider signal space PLL model (on the level of electronic realization, see Fig. 8)

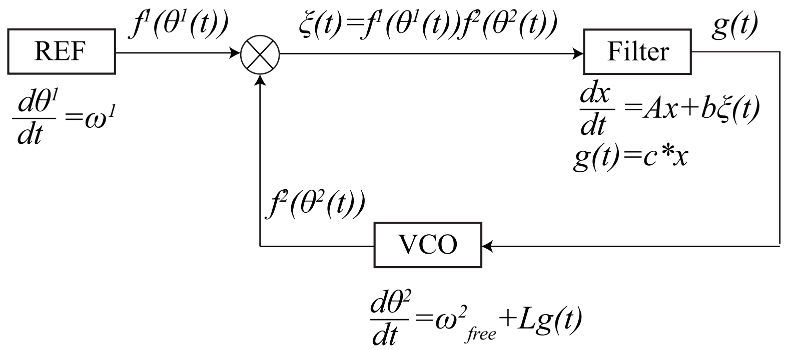


FIGURE 8 Differential equation of PLL, signal space model

Suppose that the frequency of the input (reference) signal is constant $\frac{d\theta^1}{dt} = \omega^1$. The model of a voltage-controlled oscillator is usually assumed to be a linear one (Viterbi, 1966; Stensby, 1997; Kroupa, 2003; Best, 2007):

$$\frac{d\theta^2}{dt} = \omega^2_{free} + Lg(t), \tag{22}$$

where ω^2_{free} is a free-running frequency of VCO and L is the input gain. It is also possible to use nonlinear models of tunable generator (see, e.g., (Suarez and Quere, 2003)). Thus, the output of the multiplier is $\xi(t) = f^1(\theta^1(t))f^2(\theta^2(t))$.

From a mathematical point of view, a linear low-pass filter can be described by the following system of ordinary differential equations

$$\frac{dx}{dt} = Ax + b\xi(t), \ g(t) = c^* x. \tag{23}$$

Here scalar functions $\xi(t)$ and $g(t)$ are the input and the output of the filter correspondingly, n-dimensional vector x denotes the state of the system, n-by-n constant matrix A is a transition matrix, b and c are constant n-dimensional vectors, and * denotes conjugate transpose operation. A solution of this system takes the form (6), where

$$\gamma(t-\tau) = c^* e^{A(t-\tau)} b, \quad \alpha_0(t) = c^* e^{At} x_0. \tag{24}$$

Let us denote

$$\theta(t) = \theta^2(t) - \theta^1(t). \tag{25}$$

Then differential equations of PLL in signal space take the form

$$\begin{aligned} \frac{dx}{dt} &= Ax + bf^1(\omega^1 t) f^2(\theta + \omega^1 t), \\ \frac{d\theta}{dt} &= \omega^2_{free} - \omega^1 + Lc^* x. \end{aligned} \tag{26}$$

System (26) is nonautonomous, thereby, being difficult for investigation (Margaris, 2004; Kudrewicz and Wasowicz, 2007). Theorems 1 and 2, and the averaging method (Krylov and Bogolubov, 1947; Mitropolsky and Bogolubov, 1961) allow one to study more simple, autonomous systems of differential equations.

In order to apply the averaging method, it is necessary to convert the system (26) into the following form

$$\frac{dy}{d\tau} = \varepsilon F(y, \tau), \quad y(0) = y_0, \tag{27}$$

where F is T-periodic on τ and Lipschitz continuous. Let D be an bounded open set, containing x_0, choose ε_0 such that $0 < \varepsilon \leq \varepsilon_0$.

Theorem 3. *Consider the following system*

$$\frac{dx}{dt} = \varepsilon F(t, x). \tag{28}$$

Assume that the right-hand side $F(t,x)$ is uniformly bounded and integrals

$$\int_0^t \int_c^x F(t,x)dxdt, \quad 0 \le t < \infty, \quad x \in D \tag{29}$$

are smooth for any fixed $c \in D$; the following limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(t,x)dt = \bar{F}(x) \tag{30}$$

exists (uniformly with respect to $x \in D$); uniformly with respect to x, r_m

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{F_\Delta(t,x_1,\dots,x_i+r_m,\dots,x_n) - F_\Delta(t,x)}{r_m} dt = 0, \\ x + r_m \in D, \tag{31}$$

where r_m is a decreasing sequence ($r_m \to 0$ while $m \to \infty$), and $F_\Delta = F(t,x) - \bar{F}(x)$; $\bar{F}(x)$ are k-Lipschitz functions; solution $z(t)$ of the averaged equation

$$\frac{dz}{dt} = \varepsilon \bar{F}(z),$$

for any t from $0 \le t < \infty$ belongs to the domain D together with its ρ-neighborhood, and solution of the (28) *with initial conditions $x(0) = \xi(0)$ is unique.*

Then for any $\eta > 0$, $\quad T > 0$ there is an $\varepsilon_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$ the solution $x(t)$ of (28) *satisfies*

$$|x(t) - \xi(t)| < \eta, \quad t \in [0, \frac{T}{\varepsilon}] \tag{32}$$

Now let us apply this theorem to the equations of PLL in signal space

$$\begin{aligned} \frac{dx}{dt} &= Ax + bf^1(\omega^1 t) f^2(\theta + \omega^1 t), \\ \frac{d\theta}{dt} &= \omega^2_{free} - \omega^1 + Lc^* x. \end{aligned} \tag{33}$$

where $\theta = \theta^2 - \theta^1$ is the phase difference between the VCO and the input signal. Since frequencies are assumed to be large (see (2),(7)), it is reasonable to introduce the following notations

$$\begin{aligned} &\tau = \omega^1 t, \quad \varepsilon = \frac{1}{\omega^1}, \quad y(\tau) = \begin{pmatrix} x(\frac{\tau}{\omega^1}) \\ \theta(\frac{\tau}{\omega^1}) \end{pmatrix}, \\ &F(y,\tau) = \begin{pmatrix} Ax(\frac{\tau}{\omega^1}) + bf^1(\tau) f^2(\theta(\frac{\tau}{\omega^1}) + \tau) \\ \omega^2_{free} - \omega^1 + Lc^* x(\frac{\tau}{\omega^1}) \end{pmatrix} \end{aligned} \tag{34}$$

Since only piecewise-differentiable waveforms are considered, condition (29) is satisfied. From theorem 1 we get (30)

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_{t_0}^{t_0+T} f^1(\tau)f^2(\theta(\frac{\tau}{\omega^1})+\tau)d\tau=\varphi(\theta). \tag{35}$$

Therefore, averaged equations for PLL are

$$\begin{aligned}\frac{dx}{dt}&=Ax+b\varphi(\theta),\\ \frac{d\theta}{dt}&=\omega^2_{free}-\omega^1+Lc^*x.\end{aligned} \tag{36}$$

Since

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T\Big(f^1(\omega^1T)f^2(\theta+\omega^1t+r_m)-\varphi(\theta+r_m)-\\ &f^1(\omega^1T)f^2(\theta+\omega^1t)+\varphi(\theta)\Big)dt=0\end{aligned} \tag{37}$$

for any $r_m \to 0$ where r_0 is a constant condition (31) is also met. So, (32) can be used to estimate the difference between solutions of differential equations in signal space and in phase space.

The system of differential equations for a PLL with a squarer is the same as the equations for the classic PLL (see PII).

2.5.1 Conclusions

In this section derivation of the PLL equations and their justification are considered. Differential equations of PLL derived in this section were well-known to engineers for the case of sinusoidal signals without formal justification (Viterbi, 1966; Gardner, 1966). G.A.Leonov introduced a high-frequency conditions (see e.g. (Leonov and Seledzhi, 2005; Leonov and Seledghi, 2005; Leonov, 2006)), which allows to justify PD characteristics for sinusoidal and square waveform signals. To justify PLL equations in phase-frequency space, the averaging method was also applied for the PLL equations, but only for sinusoidal signals (see e.g. (Margaris, 2004; Kudrewicz and Wasowicz, 2007)). In this work the PLL models for new classes of signals were obtained using the conditions describing high-frequency oscillations and the transition from signals space model to phase-frequency space model was justified.

2.6 Numerical simulation of PLL

One of the approaches for PLL analysis is the numerical simulation of the signal space models (see a plenary lecture of D. Abramovitch at the 2002 American Control Conference (Abramovitch, 2002)). However, according to (Abramovitch, 2008b,a), numerical simulation of a PLL in the signal space is a very slow process. Therefore, usually two time scales are used for simulation of the PLL. The first is "the very fast time scale of the input signals to the phase detector (namely the reference or data signal)". The second is "the relatively slow time scale of the signal's phase; often called modulation domain". Frequency of oscillators in modern computers and communication systems can reach up to several gigahertz, while the frequency of the loop bandwidth may be only multiple of kilohertz. This 3 to 6 order of magnitude difference in frequencies makes simulation of PLL very difficult. The discretization step should be small enough to clearly observe input signals and phase detector dynamics, which makes very difficult to observe the dynamics of the feedback loop. Simulation time increases, as frequencies of input signals increases. Typical software based on these principles are SimPLL, ADISimPLL, various models implemented in Matlab Simulink, and others.

One way to overcome this problem is simulation of the models in the frequency-phase space only (see e.g. (Tranter et al., 2010)). However, this requires the construction and validation of rigorous mathematical models that describe each of the elements of the circuit. Phase detector is one of the PLL elements which is a challenge to build a mathematical model for. In the case of classical PLL, such models have been known only for sinusoidal and square waveform signals.

In the previous chapter the PD models for various waveforms (including non-sinusoidal) was presented. The following is a numerical justification of the adequacy of the models presented in the first chapter for classical PLL in phase-freqeuncy space. The simulation results confirm the analytical results and show the possibility to speed up the simulation by using models of PLL in phase-frequency space.

Model implementation

Consider implementation of the signal space model and phase-frequency space model for classical PLL in Matlab Simulink (see 9), which allows to compare filters outputs.

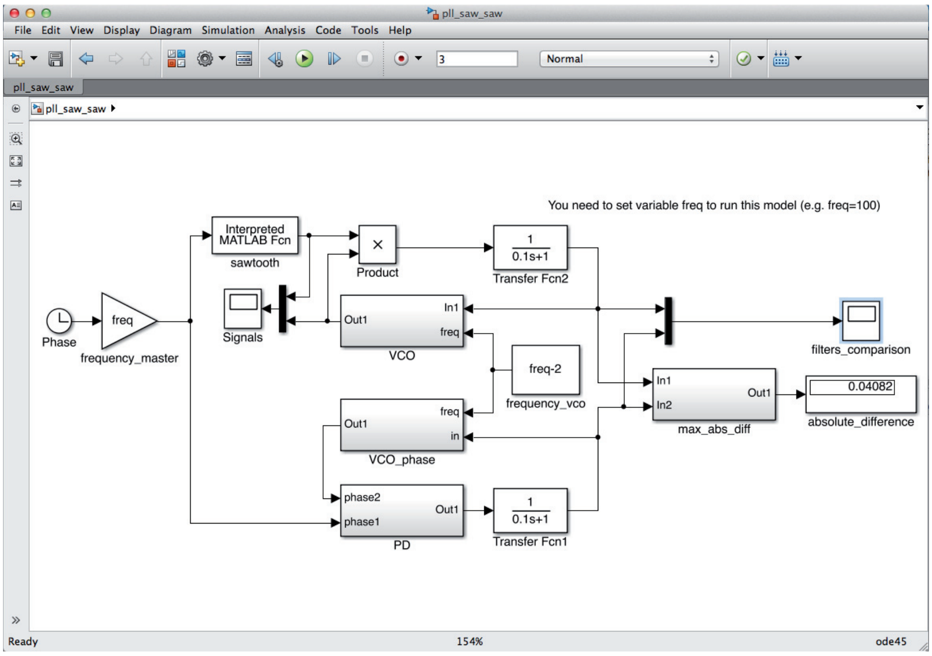


FIGURE 9 Comparison of PLL models in signal space and phase-frequency space, Matlab Simulink

Here the output of the "frequency_master" block is reference signal's phase $\omega^1 t$; the outpus of the block "max_abs_diff" is equal to the maximum of the absolute value of the difference of the filter outputs $g(t) - G(t)$; blocks "Transfer Fcn1" and "Transfer Fcn2" implement linear low-pass filters; block "Product" implements multiplier; block "VCO" implements voltage-controlled oscillator in signal space; block "VCO_phase" implements voltage-controlled oscillator in phase-frequency space; block "PD" implements phase detector characteristic. The frequency of the reference signal is set by a variable "freq". The free-running frequency of the controlled oscillator is set by a block "frequency_vco". The outputs of the filters are displayed by the block called "filters_comparison". The waveform of the reference signal is implemented by the block "Interpreted MATLAB Fcn", called "waveform". The reference and VCO signals can be displayed by the block "Signals".

Since the model of a tunable generator is usually assumed to be a linear one, VCO is implemented as follows

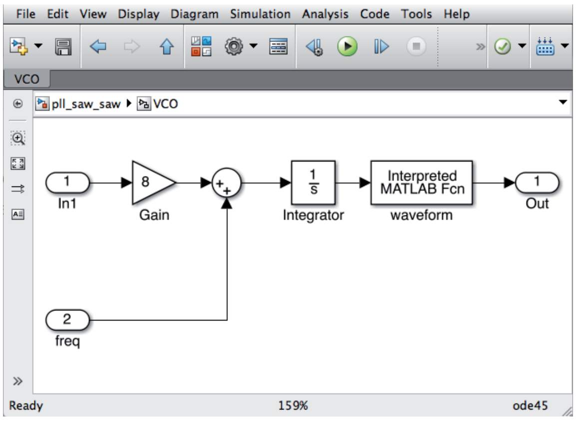


FIGURE 10 VCO implementation in time space, Matlab Simulink

Here blocks "In" and "Out" implement the input and the output of the VCO respectively; block "Gain" implements VCO input gain of the PLL; block "freq" implements a free-running running frequency of VCO; block "Integrator" integrates it's input; block "waveform" implements the waveform of the VCO signal $f^2(\theta)$.

Implementation of VCO in the phase-frequency space differs from the implementation of VCO in the signal space. Here "waveform" block is absent. So the output of the "VCO_phase" block is phase $\theta^2(t)$.

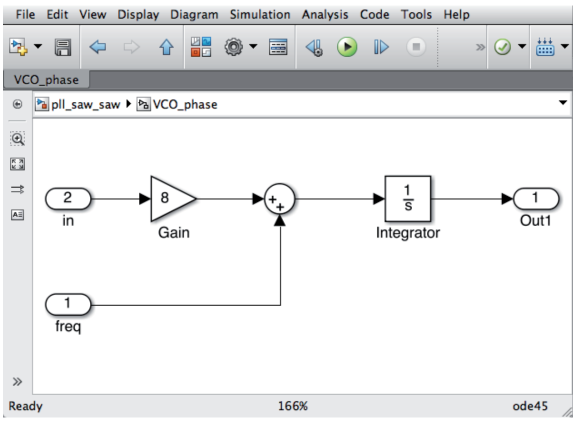


FIGURE 11 VCO implementation in phase-frequency space, Matlab Simulink

Phase detector is implemented as a function (PD characteristic) of the difference of input phases

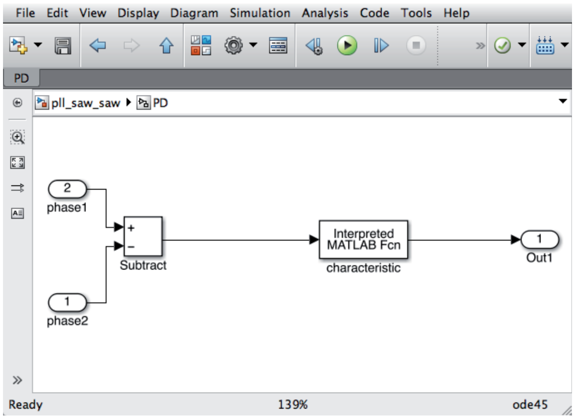


FIGURE 12 Phase detector implementation in phase-frquency space, Matlab Simulink

The block "max_abs_diff" is implemeted as follows

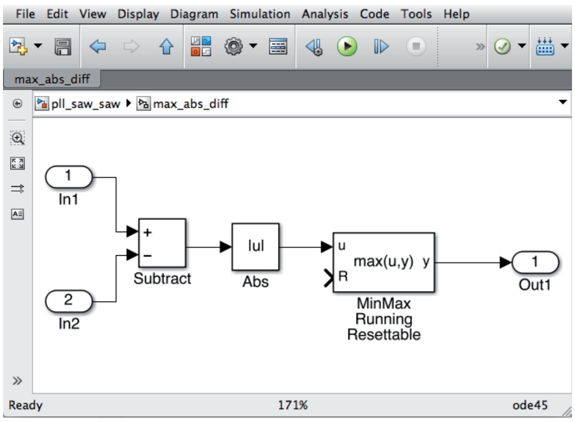


FIGURE 13 "max_abs_diff" block implementation, Matlab Simulink

Simulation results.

The following table contains maximum of absolute errors for various signal waveforms and frequencies. Here filter transfer function is $\frac{1}{0.1s+1}$, loop gain is $L = 8$, simulation time is $T = 3$ seconds, discretization step is $\frac{1}{freq*20}$.

waveforms \ $\omega_{ref}, \omega_{free}$ (rad/s)	100, 98	1000, 998	10000, 9998
sin, sin	0.0256	0.00255	0.000256
triangle, triangle	0.0203	0.00209	0.000210
triangle, square	0.0725	0.00779	0.000835
sawtooth, sawtooth	0.0455	0.00527	0.000945
square, square	0.0814	0.00984	0.001611
sawtooth, square	0.0772	0.00925	0.001121
triangle, sawtooth	0.0611	0.00666	0.000934
dirac, sin	0.00262	0.0003865	0.00004922

Here “dirac” denotes approximation of the Dirac’s Comb is approximate by “triangle” waveform with “eps=0.1”.

```
function y = dirac_triangle( x,eps )
y = zeros(1,length(x));
for i=1:length(x)
    u = rem(x(i)+pi,2*pi)-pi;
    if abs(u) < eps/2
        y(i)=sawtooth(u*pi/eps+pi,0.5)/eps;
    end;
end;
end
```

Despite the fact that the theoretical part of this work does not allows to consider Dirac’s comb, here (11) was used to formally obtain the PD characteristic

$$0.0795608945286980229205653358 80383\sin(\theta).$$

Simulation results show, that the observed convergence rate of the absolute difference $|g(t) - G(t)|$ is almost linear (i.e. $|g(t) - G(t)| = O(\frac{1}{\omega_{min}})$) for sinusoidal and triangle waveforms. However, for discontinuous waveforms (square, sawtooth) the convergence rate is less than linear, but more than square root (i.e. $|g(t) - G(t)| = O(\frac{1}{\omega_{min}})$). These results show the possibility to improve the estimate of the convergence rate for curtain signal waveforms (i.e. smooth and Lipschitz waveforms). This is a question for the further study.

Typical simulation results are presented on Fig. 14–Fig. 17

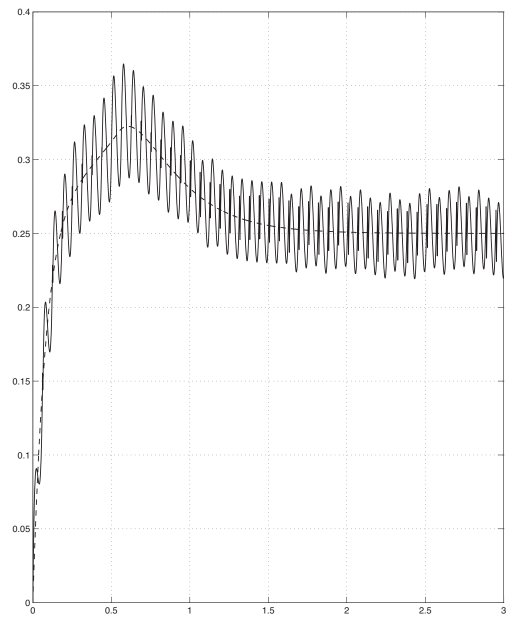


FIGURE 14 Sawtooth,filter $\frac{1}{0.1s+1}$, vco gain 8, frequencies 100 rad/s and 98 rad/s

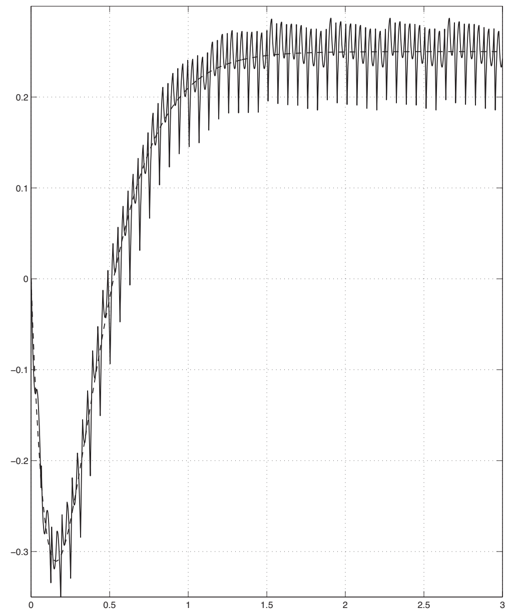


FIGURE 15 Sawtooth+square,filter $\frac{1}{0.1s+1}$, vco gain 8, frequencies 100 rad/s and 98 rad/s

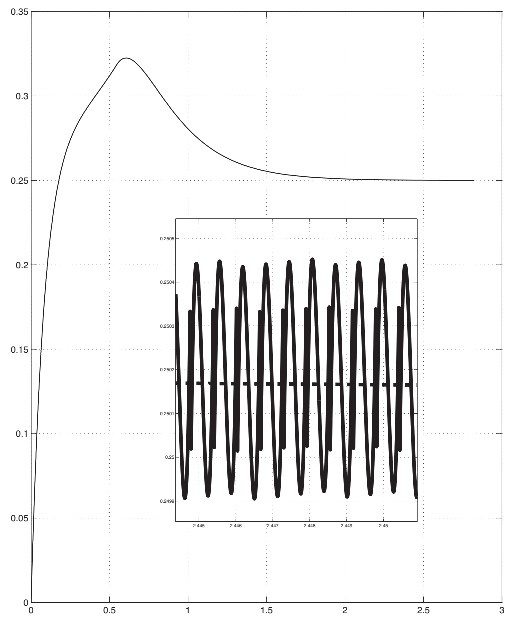


FIGURE 16 Sawtooth,filter $\frac{1}{0.1s+1}$, vco gain 8, frequencies 10000 rad/s and 9998 rad/s

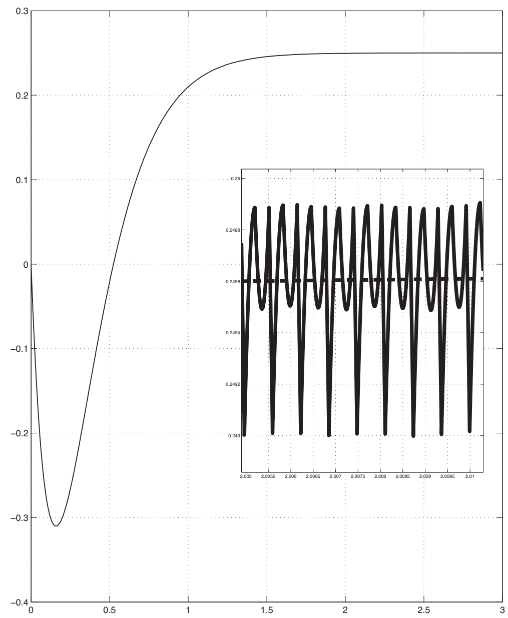


FIGURE 17 Sawtooth+square,filter $\frac{1}{0.1s+1}$, vco gain 8, frequencies 10000 rad/s and 9998 rad/s,

Simulation shows, that filter outputs for calssical PLL in phase-frequency space and time space are close. However, the filter output in the signal space model contains small amplitude parasitic high-frequency oscillations, which interfere with efficient simulation.

Nowadays, oscillators can reach frequencies up to 10Ghz. Therefore, the discretization step has to be chosen sufficiently small for a clear observation of the dynamics of the phase detector. Depending on application, the time period required for the synchronization of oscillators can vary from several seconds to several minutes. This makes the full numerical simulation of a PLL in the signal space almost impossible for the high-frequency signals. According to the engineers of the ST-Ericsson firm, the full simulation of a PLL can take up two weeks. On the other hand, numerical simulation in the phase-frequency space allows one to consider only low-frequency signals, something that allows one to increase a discretization step and decrease time needed for numeriacal simulation. This approach for simulation of PLL is presented in patent (Kuznetsov et al., 2011) and patent application FI20130124.

3 CONCLUSIONS

In this work mathematical description of the classical phase-locked loop operation is considered, and characteristic of the phase detector for various signal waveforms is presented. The characteristics of a phase detector obtained in this work are in line with the well-known characteristics for the sinusoidal and square waveforms. All main results are rigorously proved. PLL models in the phase-frequency domain are justified using the averaging method by Krylov-Bogolyubov. Numerical simulation confirms the adequacy of the derived models and shows the possibility to improve theoretical results. Particularly, convergence rate for the smooth and Lipschitz waveforms is observed to be linear, while the theoretical results predict a square root convergence rate.

Obtained PLL and PLL with a squarer models allow one to use phase-frequency model for PLL, thereby significantly reducing time required for numerical simulation. This allows one to determine such important features of the systems involved as pull-in range, pull-out range, and other properties, thereby, significantly reducing time needed for the development of PLL-based circuits and their analysis.

YHTEENVETO (FINNISH SUMMARY)

Elektronisten piirien numeerinen simulointi on yleinen lähestymistapa Vaihelukittujen piirien (phase locked loop) ominaisuuksien analysointiin. Se ei kuitenkaan sellaisenaan sovellu tietokoneissa ja tietoliikennejärjestelmissä tarvittavien piirien simulointiin, joissa signaaligeneraattorit toimivat jopa useiden gigahertsien taajudella, kun vaihelukittujen piirien sisäinen toiminta on korkeintaan kilohertsien taajuusalueella.

Useiden (3-6) kertaluokkien ero taajuuksissa edellyttäisi hyvin raskasta laskentaa vaihelukittujen piirien toiminnan simuloimiseksi suoraan fysikaalisen mallin pohjalta. Tässä työssä tutkitaan taajuustasossa formuloituja matalataajuisia malleja vaihelukituille piireille. Näiden mallien avulla on mahdollista tarkastella vaihelukittujen piirien dynamiikkaa erillään korkeataajuisten signaalien yksityiskohtaisesta simuloinnista. Taajustason mallin muodostaminen edellyttää kaikkien piirien, erityisesti ns vaiheilmaisimen, mallintamista taajuustasossa.

Tässä työssä johdetaan yleinen lähestymistapa vaiheilmaisimen mallintamiseen taajuustasossa eri tyyppisille referenssisignaaleille. Työssä yleistetään aiemmin vain harmoonisille signaaleille tunnetut mallit vaiheilmaisimelle ja todistetaan näiden asymptoottinen tarkkuus referenssisignaalien taajuuden funktiona.

Työssä johdetut mallit mahdollistavat vaihelukittujen piirien ominaisuuden tehokkaan simuloinnin ja analysoinnin ja helpottavat näin oleellisesti piirien suunnittelua.

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APPENDIX 1 FOURIER SERIES

Following reviews (Zygmund, 1968; Walker, 2003), definitions and theorems needed to prove theorems on the asymptotic equivalence (see chapter 1) of the PLL models.

Consider f — Periodic function, i.e. $f(x+P) = f(x), \forall x$. If the function has period 2π, then its Fourier series is

$$\frac{1}{2}a_0 + \sum_0^\infty a_n \cos(nx) + b_n \sin(nx) \tag{38}$$

with Fourier coefficients a_n ,b_n and defined by the integrals

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \end{aligned} \tag{39}$$

(see e.g. (Zygmund, 1968)).

It is also possible to express Fourier series in an algebraically simpler form involving complex exponentials

$$\begin{aligned} \cos(\theta) &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \\ \sin(\theta) &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \end{aligned} \tag{40}$$

From these equations, it follows that (38) can be rewritten as

$$\frac{1}{2}a_0 + \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{41}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx, \quad c_0 = \frac{1}{2}a_0. \tag{42}$$

It is sometimes useful to use the following notation for partial sums of Fourier series

$$\begin{aligned} S_n(x) &= \frac{1}{2}a_0 + \sum_{\nu=1}^{\nu} a_\nu \cos(\nu x) + b_\nu \cos(\nu x) \\ \tilde{S}_n(x) &= \sum_{\nu=1}^{n} a_\nu \cos(\nu x) - b_\nu \cos(\nu x) \end{aligned} \tag{43}$$

Using (39) we have

$$\begin{aligned} S_n(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\{\frac{1}{2} + \sum_{\nu=1}^{n} \cos(\nu(t-x))\}dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)dt, \\ \tilde{S}_n(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\{\sum_{\nu=1}^{n} \sin(\nu(t-x))\}dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\tilde{D}_n(t-x)dt, \end{aligned} \tag{44}$$

where

$$D_n(v) = \frac{1}{2} + \sum_{\nu=1} n\cos(\nu v) = \frac{\sin(n+\frac{1}{2})v}{2\sin(\frac{1}{2}v)},$$
$$\tilde{D}_n(v) = \sum_{\nu=1} n\sin(\nu v) = \frac{\cos\frac{1}{2}v - \cos(n+\frac{1}{2})v}{2\sin(\frac{1}{2}v)}. \tag{45}$$

The polynomials $D_n(v)$ and $\tilde{D}_n(v)$ are called Direchlet's kernel and Direchlet's conjugate kernel respectively. The formulae for for S_n and $\tilde{S}_n$ may also be written

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)du, \quad \tilde{S}_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\tilde{D}_n(u)du, \tag{46}$$

It is also useful to take the last term of S_n or $\tilde{S}_n$ with a factor $\frac{1}{2}$. The new expressions will be called the modified partial sums

$$S^*_n(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{n-1}(a_\nu \cos \nu x + b_\nu \sin \nu x) + \frac{1}{2}(a_n \cos nx + b_n \sin nx) = \frac{1}{2}\left(S_n(x) + S_{n-1}(x)\right), \tag{47}$$

and S^*_n is defined similarly.

The three types of convergence that we shall describe here are pointwise, uniform, and norm convergence. All convergence theorems are concerned with how the partial sums

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx} \tag{48}$$

converge to $f(x)$.

A function is said to be Lipschitz at a point x_0 if, for some positive constant A there is some positive constant δ, such that

$$|f(x) - f(x_0)| \leq A|x - x_0|, \forall |x - x_0| \leq \delta. \tag{49}$$

APPENDIX 1.1 Pointwise convergence.

The question of pointwise convergence concerns whether

$$\lim_{N\to\infty} S_N(x_0) = f(x_0) \tag{50}$$

holds for each fixed x-value x_0. If $\lim_{N\to\infty} S_N(x_0)$ does equal $f(x_0)$, then we say that the Fourier series for f converges $f(x_0)$ to at x_0.

Theorem 4 (see e.g. (Walker, 2003))**.** *Suppose f has period 2π, that $\int_{-\pi}^{\pi} |f(x)|dx$ is finite, and that f is Lipschitz at x_0. Then the Fourier series for f converges to $f(x_0)$ at x_0.*

Proof. Without loss of generality, assume that $f(x_0) = 0$. Define the function $g(x) = \frac{f(x)}{e^{ix}-e^{ix_0}}$. This function is 2π–periodic. Furthermore, $\int_{-\pi}^{\pi} |g(x)|dx$ is finite, because $\frac{f(x)}{e^{ix}-e^{ix_0}}$ is bounded for x near x_0. In fact, for such x,

$$\left|\frac{f(x)}{e^{ix}-e^{ix_0}}\right| = \left|\frac{f(x)-f(x_0)}{e^{ix}-e^{ix_0}}\right| \leq A\left|\frac{x-x_0}{e^{ix}-e^{ix_0}}\right| \tag{51}$$

and $\frac{x-x_0}{e^{ix}-e^{ix_0}}$ is bounded in magnitude, because it tends to $-ie^{-ix_0}$. Denote the n^{th} Fourier coefficient of $g(x)$ by d_n. Then we have $c_n = d_{n-1} - d_n e^{ix_0}$ because $f(x) = g(x)(e^{ix} - e^{ix_0})$. Then the partial sum S_N is

$$S_N(x_0) = \sum_{n=-N}^{n=N} c_n e^{inx} = d_{-N-1}e^{-iNx_0} - d_N e^{i(N+1)x_0}. \tag{52}$$

Since $d_n \to 0$ as $|n| \to \infty$, by the Riemann-Lebesgue lemma, we conclude that $S_N(x_0) \to 0$. ■

Theorem A1 implies that the Fourier series $S_N(x)$ for the picewise-differentiable waveforms (e.g. square-wave) converges to $f(x)$ at all of its points of continuity.

APPENDIX 1.2 Fourier coefficients for functions with bounded variation.

We say that a function f has a bounded variation on a closed interval $[a, b]$ if there exists a positive constant B such that, for all finite sets of a points $a = x_0 < x_1 < \cdots < x_n = b$ the inequality

$$\sum_{\nu=1}^{n} |f(x_\nu - f(x_{\nu-1}))| \leq B \tag{53}$$

is satisfied. Jordan proved that a function has bounded variation if and only if it can be expressed as the difference of two non-decreasing functions.

Theorem 5 (see, e.g. (Zygmund, 1968))**.** *Let $F(x)$ be a function with bounded variation over $0 \leq x \leq 2\pi$, and let C_n be the coefficients of F and $c_n = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{-int}dF(t)$ be the Fourier coefficients of dF (the integrals being Rieman-Stieltjes integral). If V is the total variation of F over $(0, 2\pi)$, then*

$$|C_n| \leq \frac{V}{\pi|n|}, |c_n| \leq \frac{V}{2\pi}. \tag{54}$$

Proof. The second inequatlity follows from the formula

$$2\pi|c_n| = \left|\int\limits_0^{2\pi} e^{-inx}dF(x)\right| \leq \int\limits_0^{2\pi} |dF(x)| = V. \tag{55}$$

Integrating by parts we see that

$$2\pi c_n = \int_0^{2\pi} e^{-inx} F(x)dx = \frac{F(2\pi) - F(0)}{-in} + \frac{2\pi c_n}{in} \tag{56}$$

for $n \neq 0$, and last sum is absolutely $\leq 2V/|n|$. Thus the coefficients of a function of bounded variation are $O(1/n)$.

APPENDIX 1.3 Uniform Convergence

We say that the series $S_n(x)$ converges uniformly to $f(x)$ on interval I, if

$$\lim_{n\to\infty} \left\{ \max_{x\in I} |f(x) - S_n(x)| \right\} = 0. \tag{57}$$

Theorem 6 ((Dirichlet-Jordan test, see e.g. (Zygmund, 1968)). *If f has period 2π and has bounded variation on $[0, 2\pi]$, then*

i. At every point x_0, Fourier series for f converges to the value $\frac{1}{2}[f(x_0+0) + f(x_0-0)]$; in particular, Fourier series for f converges to $f(x)$ in every point of continuity of f;
ii. If further f is contiunous at every point of a closed interval I, then Fourier series for f converges uniformly in I.

Proof.
Lemma
The integrals

$$\frac{2}{\pi}\int_0^{\xi} D_n(t)dt, \quad \frac{2}{\pi}\int_0^{\xi} D_n^*(t)dt, \quad \frac{2}{\pi}\int_0^{\xi} \frac{\sin(nt)}{t}dt \quad (0 \leq \xi \leq \pi) \tag{58}$$

are all uniformly bounded in n and ξ. The difference between any two of these integrals tends to 0 with $1/n$, uniformly in ξ.

Proof of Lemma.
Let us denote these integrals respectively by $\alpha_n(\xi)$, $\beta_n(\xi)$, $\gamma_n(\xi)$. Plainly, $\beta_n - \alpha_n$ is uniformly bounded and tends uniformly to 0 as $n \to \infty$. Furthermore,

$$\gamma_n - \beta_n = \frac{2}{\pi}\int_0^{n\xi} \frac{\sin u}{u} du = \frac{2}{\pi} G(n\xi), \tag{59}$$

where

$$G(v) = \int_0^{v} \frac{sinu}{u} du; \tag{60}$$

and this will follow if we show that $G(v)$ tends to a limit as $v \to +\infty$. Since the integrand tends to 0 it is enough to prove the existence of $\lim G(n\pi)$. But

$\alpha(\pi) = 1$ and $\alpha(\pi) - \gamma(\pi) \to 0$ together imply $G(n\pi) \to \frac{1}{2}\pi$, which proves lemma. □

Now return to Theorem. Since Fourier series for $\frac{1}{2}[f(x_0+t) + f(x_0-t)]$ is $\sum_{n=0}^{\infty} A_n(x_0)\cos(nt)$, and Fourier series for $\frac{1}{2}[f(x_0+t) - f(x_0-t)]$ is $-\sum_{n=0}^{\infty} B_n(x_0)\sin(nt)$, we conclude that Fourier series for f at $x = x_0$ is the same as the Fourier series at $t = 0$ of the even function $\frac{1}{2}[f(x_0+t) + f(x_0-t)]$, and conjugate Fourier series for f at $x = x_0$ is the series conjugate to the Fourier series at $t = 0$ of the odd function $\frac{1}{2}[f(x_0+t) - f(x_0-t)]$. Here

$$\begin{aligned} A_0(x) &= \frac{1}{2}a_0, \\ A_n(x) &= a_n\cos(nx) + b_n\sin(nx), \\ B_n(x) &= a_n\sin(nx) - b_n\cos(nx), \end{aligned} \tag{61}$$

Thus we may assume that $x_0 = 0$ and that $f(x)$ is even. We have to show that $S_n(0) \to f(+0)$.

Suppose first that $f(x)$ is non-negative and non-decreasing in $(0, 2\pi)$. Let C be a number greater than $|\beta_n(\xi)|$ fo all n and ξ. We write

$$S_n^*(0) - f(+0) = \frac{2}{\pi}\int_0^\pi [f(t) - f(+0)]D_n^*(t)dt = \frac{2}{\pi}\left(\int_0^\eta + \int_\eta^\pi\right) = A + B, \tag{62}$$

say, where η is so chosen that $|f(\eta) + f(+0)| < \epsilon/4C$. Since $f(t) - f(+0)$ is non-negative and non-decreasing the second mean value theorem gives

$$|A| = \left|\{f(\eta) - f(+0)\}\frac{2}{\pi}\int_{\eta'}^{\eta} D_n^*(t)dt\right| \leq \frac{\epsilon}{4C}2C = \frac{1}{2}\epsilon \quad (0 < \eta' < \eta). \tag{63}$$

For fixed η, B is a sine coefficient of the function $\omega_\eta(t)$ equal to 0 in $(0, \eta)$ and to $\{f(t) - f(+0)\}\frac{1}{2}\cot(\frac{1}{2}t)$ in (η, π). Thus, by theorem (4.12),

$$B \to 0, |A + B| < \epsilon \text{ for } n > n_0, \quad S_n(0) \to f(+0). \tag{64}$$

In general case f is in $(0, \pi)$, the difference $f_1 - f_2$ of two non-negative and non-decreasing functions (the positive and negative variations of f). If we define f_1 and f_2 in $(-\pi, 0)$ by condition evenness, the formula $f = f_1 - f_2$ becomes valid in $(-\pi, \pi)$ and the general result follow from the special case just proved.

Second assertion of the theorem follow from the argument just used if we note that continuity of f in I implies the continuity of the positive and negative variations of f in that interval, and that all the estimates obtained above hold uniformly for $x \in I$. ■