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Dimension of spatially embedded networks

INTRODUCTION

We present supplementary material on our paper: "Dimension of Spatially Embedded Networks". The supplementary material is organized as follows : In Section I we present details on the Monte Carlo algorithms used to generate the spatially embedded networks presented in our paper. In Section II we discuss the stability of dimension under changes of the system size and of the degree distribution. In Section III, we present statistical details on the spatially embedded real network. In Section IV, we show the simulation method and finite size effect on the diffusion processes. Finally, in Section V we discuss percolation in spatially embedded networks.

I. GENERATION OF THE SPATIAL NETWORKS

To construct the networks we follow an algorithm similar to the one used in [1]. We initially arrange the nodes in a d dimensional regular lattice (where for example, $d = 2$ is the square lattice and $d = 3$ is the cubic lattice). To start, we choose a node i randomly and assume that it will try to connect to k_i other nodes, where k_i is drawn from a given degree distribution $P(k)$. Then, we select a distance r from node i with probability $\Phi(r) = r^{d-1}P(r) = cr^{d-1}r^{-\delta}$, where c is determined from the normalization condition $\int_1^L \Phi(r)dr = 1$, with $L = N^{1/d}$. Next, we connect randomly one of the N_r sites that are at distance r within the underlying lattice. We repeat this process for all nodes i in the underlying lattice and then remove multiple connections. Since, in d dimensions, the number of nodes at distance r from a node i scales as the surface of a d dimensional hypersphere i.e. with exponent $d-1$, the presence of the r^{d-1} is needed for proper normalization.

In this paper we have used two functional forms for $P(k)$ distribution, namely Poissonian distribution as in Erdös Rényi (ER) networks and a power law distribution in scale free (SF) networks. Fig. 1 presents four examples of network configurations embedded in 2 dimensional space for $\delta = 1$, $\delta = 3$ and $\delta = 4.5$ for ER networks and $\delta = 3$ for SF networks. By inspection one can see three qualitatively different structures. In the ER network for $\delta = 1$, there is a large number of long links which are responsible for the connectivity of the

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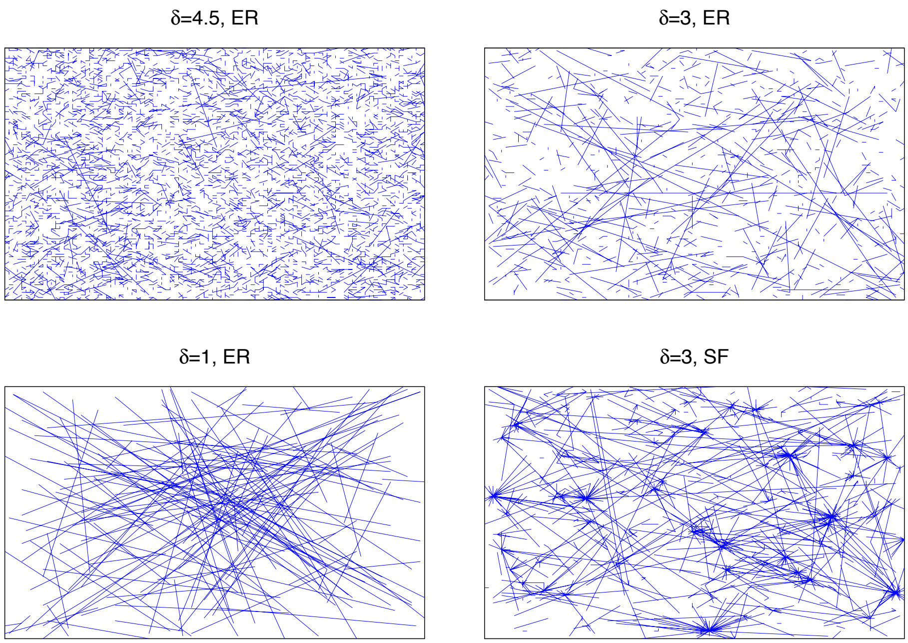

FIG. 1: Demonstration of four different configurations of the embedded networks in 2 dimensional space with Poissonian degree distribution and $\delta = 4.5, 3, 1$ (top left,top right,bottom left respectively) and with scale free(SF) degree distribution and $\delta = 3, \alpha = 1.8$ (bottom right). As δ decreases, the network has more long range connections.

whole lattice. For $\delta = 3$, the ER network is made up from several localized subgraphs which are connected to each other by a few long links, while in the embedded SF network these subgraphs are mainly connected by some hubs having many short links and few long links. Finally, for $\delta = 4.5$ the network is generated from local connections mainly where long links are rare and the network behaves similarly to the well known regular lattices.

II. THE NETWORK DIMENSION

To obtain the dimension d of our spatially constrained networks we use the scaling relation $M \sim r^d$ (Eq. 1) in the paper. We have determined $M(r)$ for various system sizes L, various average degrees $\bar{k}$ in ER networks, and various degree exponents α in scale free networks. Representative results for networks embedded in two dimensions are given in Fig. 2, where we show that for a Poissonian network with $\bar{k} = 3$ and $\delta = 3, 3.5$, the dimension d does not depend on the system size (Fig. 2a,b). Figure 2c shows that for a ER network of size

$L = 1000$ with $\delta = 3$, the dimension does not depend on $\bar{k}$. Figure 2d finally yields, for a scale free network with $\alpha = 3$, the dimension of the network for $\delta = 2.5, 3.5$. The result shows that the dimensions of these networks are the same as for the ER networks, suggesting that the network dimension d depends mainly on the exponent δ.

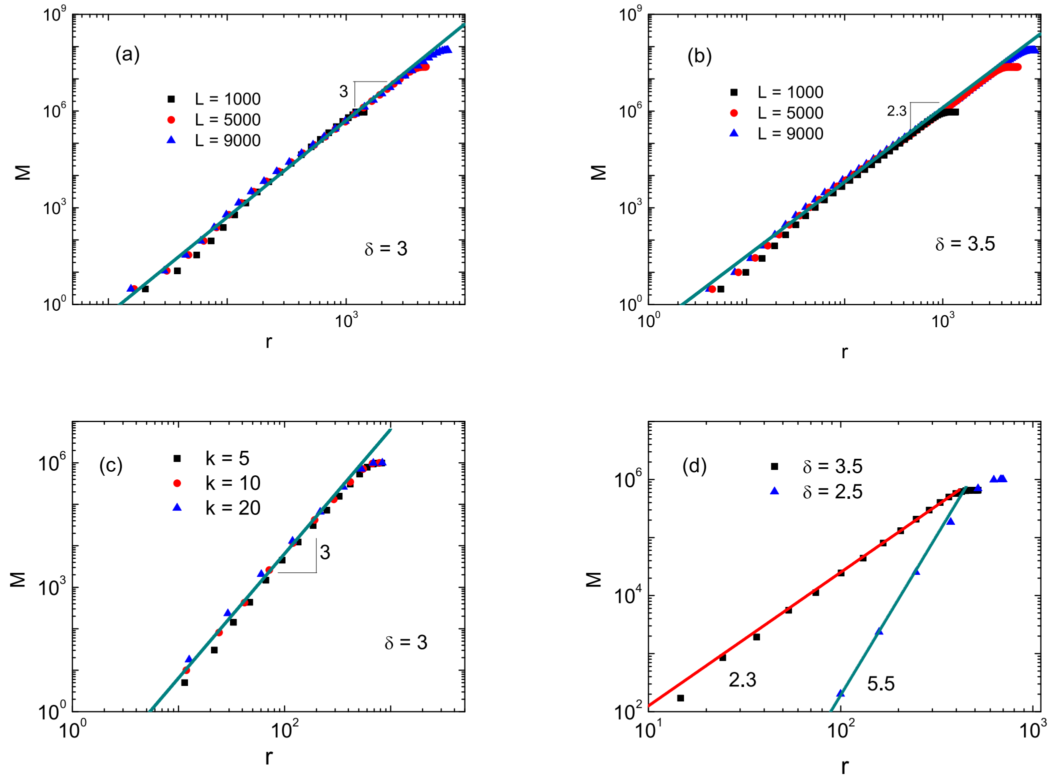

FIG. 2: The scaling relations between the mass M and the metric distance r for (a,b) different network sizes of embedded ER networks with $\delta = 3$ (a) and $\delta = 3.5$ (b), (c) different average degrees of embedded ER networks with $\delta = 3$, and (d) embedded SF networks with $\alpha = 3$ for $\delta = 2.5$ and 3.5. The dimension for the corresponding ER networks have been obtained in (b) and in Fig. 3a of the article. For $\delta = 3$ we have shown in Fig. 3b of the article that the dimension is independent of α, for α between 1.8 and 3.5.

III. REAL NETWORKS

The degree distribution and link length distribution of the airline network have been studied and reported in [2, 3]. Here we analyze two other real networks, the Internet and the European power grid. Figure 3 shows the degree distribution $P(k)$ and the link length

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distribution $P(r)$ for the Internet [4] (Figs. 3a,c) and the European power grid [5] (Figs. 3b,d). The figure reveals that $P(k) \sim k^{-\alpha}$ and $P(r) \sim r^{-\delta}$, with $\alpha = 2.1$ and $\delta = 2.6$ for the Internet. Figs. 3b,d show P(k) and P(r) for the European power grid. The result suggests that both distributions decay exponentially. Accordingly, we expect that the dimension of the European power grid is equal to the embedding dimension 2. Figure 4 shows that this is the case.

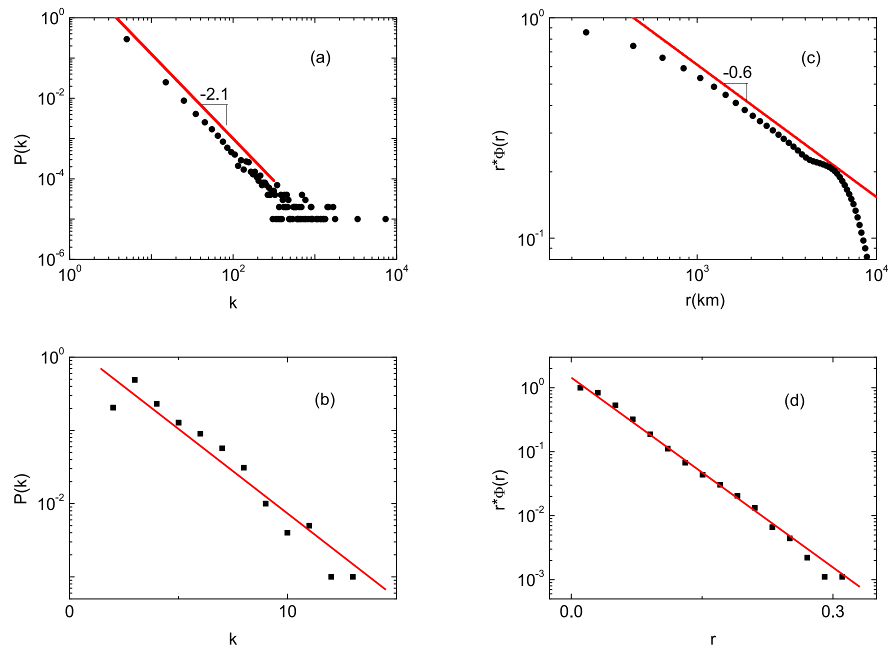

FIG. 3: The statistical properties of the Internet and the power grid network. (a, c) show the degree and link length distribution of the Internet suggesting that $P(k) \sim k^{-\alpha}$ with $\alpha \simeq 2.1$ and $P(r) \sim r^{-\delta}$ with $\delta \simeq 2.6$; (b) shows the degree distribution of the power grid network in a semi log plot suggesting that $P(k)$ decays exponentially with k. (d) shows the link length distribution of the power grid network in a semi log plot suggesting an exponential decay of $P(r)$.

IV. SIMULATION AND FINITE SIZE EFFECTS IN THE DIFFUSION PROCESS

To simulate diffusion on the embedded networks we perform random walks on the graphs using the following algorithm:

At time $t = 0$ a random walker is placed on a randomly selected node i of the spatial network. At each time step the walker will move to one of the its neighbors randomly. After

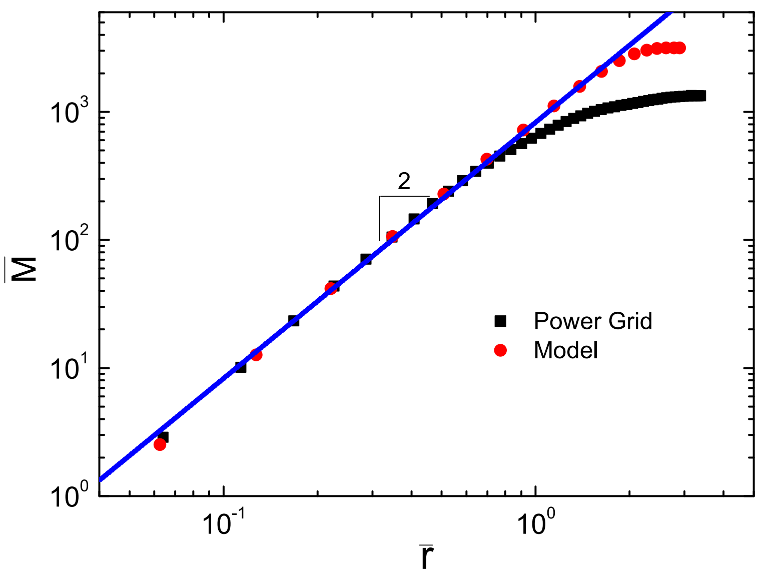

FIG. 4: The number of nodes M as a function of the metric distance r in the power grid network (squares) and a model network (circles) with the same distance distributions as the power grid network. The slope of the straight line is 2 which is equal to the embedding dimension of the lattice. The model network has 3200 nodes, power law distance distribution with $\delta = 4.5$ and Poissonian degree distribution with $\bar{k} = 5$.

each step the time t is increased by one unit. We repeat the process for different network realizations and monitor, for each time step, the fraction of walkers being at their origin position P_0 and the average metric distance r of the walkers from their origin. Figure 5 shows, for a ER network with $\bar{k} = 3$ and $\delta = 3.5$, the probability P_0 of a walker of being at the origin as a function of the mean distance r traveled by the random walker, for three system sizes $L = 30, 50,$ and 100. The result shows that the expected power law behavior only holds below a crossover value r_c which increases with increasing system size L. The strong decay of P_0 at large r values is thus a finite size effect, which vanishes in the limit of $L \to \infty$. The final decay of P_0 in the global airline network and in the Internet that is seen in Fig. 4 of the manuscript, can be therefore identified as a finite size effect.

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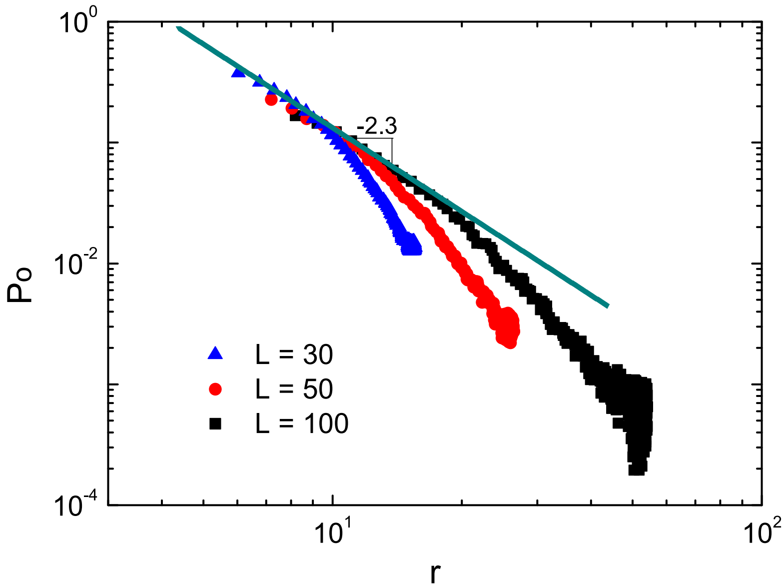

FIG. 5: The probability P_0 that a diffusing particle is at its starting site, after traveling an average distance r for networks with $\delta = 3.5$ and three different sizes $L = 30, 50, 100$ (triangles, circles, squares). The finite size effects on the diffusion are clearly seen and are similar to those obtained in real networks (Fig. 4 in the manuscript).

V. THE CLUSTER SIZE DISTRIBUTION FOR PERCOLATION ON SPATIALLY EMBEDDED NETWORKS

To study the percolation process on spatially embedded networks we randomly remove a fraction q of the nodes of the network. We initially estimate the percolation transition q_c as the value of q where the second largest cluster on the network reaches a maximum. A more accurate estimation of the percolation threshold is then obtained by varying q around the initially estimated q_c and checking how the size of the largest cluster M_L scales as a power law with the system size N. At criticality, this scaling assumes the form of a power law and, thus, we get a more accurate estimation of q_c by considering it equal to the value of q for which a double logarithmic plot of M_L versus N becomes a straight line.

At the critical percolation threshold q_c, it is expected that the cluster size distribution $n(s)$ scale as $n(s) \sim s^{-\tau}$ [6]. In Fig. 6 we plot the cluster size distribution at q_c for four different networks with 10^6 nodes, Poissonian degree distribution with $\bar{k} = 4$ and $\delta = 4.5, 3.5, 3, 2.5$.

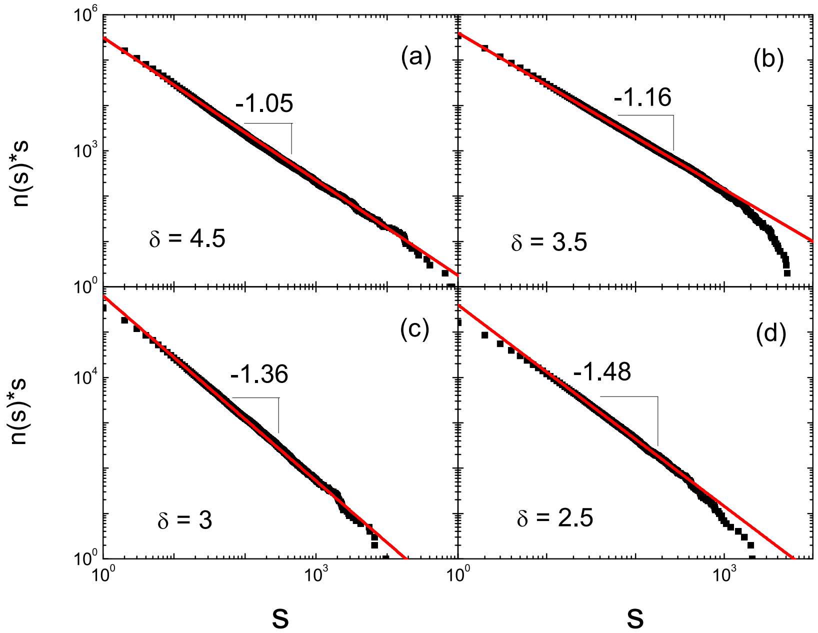

FIG. 6: The cluster size distribution of percolation at criticality is plotted for different values of δ for networks with 10^6 nodes and Poissonian degree distribution with $\bar{k} = 4$. The slope of this power law scaling is related with the dimension of the network via Eq. (3).

We observe in Fig. 6a that for $\delta > 4$, $n(s)$ is similar to the known value for percolation in 2d lattice. For $2 < \delta < 4$, the values of τ change with δ, where a new regime emerges from the competition between spatial constraints and the long range interactions (Fig. 6b,c). When δ approaches 2, τ tends to the mean field result (2.5) as shown in Fig. 6d.

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