The Geometric Series in Calculus

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1. INTRODUCTION. One of the fairly easily established facts from high school algebra is the Finite Geometric Series:

$$1 + r + r^2 + \cdots + r^n = \sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}. \tag{1.1}$$

This fact is made convincingly clear to all concerned by direct multiplication

$$\begin{array}{rrrrrr}
 & r^n & +r^{n-1} & + \cdots + & r^2 & +r+1 \\
 & \times & & & & -r+1 \\
\hline
 & r^n & +r^{n-1} & + \cdots + & r^2 & +r+1 \\
-r^{n+1} & -\, r^n & -r^{n-1} & - + \cdots + - & r^2 & -r \\
\hline
-\, r^{n+1} & & & & & +1
\end{array}$$

Unfortunately this elementary result is often skipped in algebra and is often first mentioned when infinite series arise in the second semester of calculus.

The object here is to show that the Geometric Series can play a very useful role in simplifying some important but complex topics in calculus. Most of the ideas in this note can be found in only slightly different guise sprinkled throughout Otto Toeplitz's charming book [**6**], which, unfortunately, is out of print.

2. THE DERIVATIVE OF x^n. Showing students that

$$\frac{d}{dx} x^n = nx^{n-1}$$

usually poses a dilemma. From the standard definition of a derivative, we see that

$$\frac{d}{dx} x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

How should we proceed?

One approach [**2**, p. 162] is to use the Binomial Theorem without saying much save for a few examples. Not exactly convincing!

Or should we prove the Binomial Theorem at this point? Probably not!

Perhaps we could prove a Weak Binomial Theorem:

$$(x+h)^n = x^n + nhx^{n-1} + h^2(\cdots).$$

Again, we have a distraction, at best.

Let us bring the Finite Geometric Series to the rescue. The standard definition of the derivative, viz.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{2.1}$$

is easily seen (both algebraically and geometrically) to be equivalent to

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}, \tag{2.2}$$

and if $x \neq 0$ and we look at the ratio of y to x, we find a third equivalent formulation ($y = qx$):

$$f'(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x}, \quad x \neq 0. \tag{2.3}$$

We may now use this third definition of $f'(x)$ to determine the derivative of x^n:

$$\begin{aligned}\lim_{q \to 1} \frac{(xq)^n - x^n}{qx - x} &= x^{n-1} \lim_{q \to 1} \frac{q^n - 1}{q - 1} \\ &= x^{n-1} \lim_{q \to 1} \left(1 + q + q^2 + \cdots + q^{n-1}\right) \quad (\text{by } (1.1)) \\ &= nx^{n-1}.\end{aligned} \tag{2.4}$$

While this is valid only if $x \neq 0$, the original definition (2.1) easily treats $x = 0$. The derivative of $x^{n/m}$ can be handled in the same manner by a simple change of the variable q.

3. INTEGRALS AND THE FUNDAMENTAL THEOREM OF CALCULUS. We often hope to say compelling things about Riemann sums when we define

$$\int_a^b f(x)\, dx = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}). \tag{3.1}$$

However, when we try to compute examples of simple integrals with a uniform partition P of $[a, b]$, we can wind up with expressions such as

$$\begin{aligned}\int_0^1 x^2\, dx &= \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \\ &= \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2.\end{aligned}$$

The problem is now analogous to our problem for taking the derivative of x^n. We must either pull

$$\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$$

out of a hat, or else spend a substantial amount of time motivating and proving it.

Again the Finite Geometric Series can come to our rescue. As an alternative to the Riemann sum, we can examine a geometric dissection of our interval (see Figure 1).

The area of the rectangles indicated is

$$\begin{aligned}A_q(X) &= f(X)(X - qX) + f(Xq)(Xq - Xq^2) \\ &\quad + f(Xq^2)(Xq^2 - Xq^3) + \cdots \\ &= \sum_{i=0}^{\infty} f(Xq^i)(Xq^i - Xq^{i+1}).\end{aligned} \tag{3.2}$$

As $q \to 1^-$, it is visually convincing that $A_q(X)$ converges to the area under the curve, and (probably in an appendix) an actual proof that this definition is equivalent to the standard Riemann sum definition is no more difficult than any other portion of the rigorous treatment of Riemann sums.

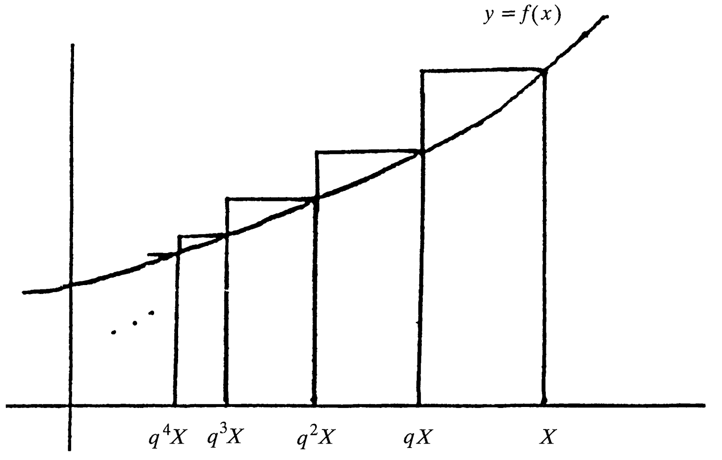


Figure 1. $A_q(X)$ = area of rectangles

In any event, now it is possible to integrate not just x^2, but, indeed, any positive integral power of x. First we note that the Finite Geometric Series directly leads to the Infinite Geometric Series. If $|r| < 1$, then

$$\sum_{i=0}^{\infty} r^i = \lim_{n\to\infty} \sum_{i=0}^{n} r^i = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}. \tag{3.3}$$

The subtleties of infinite series in general need not be introduced here because we have the explicit formula for the partial sums.

Hence

$$\begin{aligned} \int_0^X x^n\,dx &= \lim_{q\to 1^-} A_q(X) \\ &= \lim_{q\to 1^-} \sum_{i=0}^{\infty} (Xq^i)^n (Xq^i - Xq^{i+1}) \\ &= X^{n+1} \lim_{q\to 1^-} (1-q) \sum_{i=0}^{\infty} q^{i(n+1)} \\ &= X^{n+1} \lim_{q\to 1^-} \frac{(1-q)}{1-q^{n+1}} \qquad \text{(by (3.3))} \\ &= X^{n+1} \lim_{q\to 1^-} \frac{1}{1+q+q^2+\cdots+q^n} \qquad \text{(by (1.1))} \\ &= \frac{X^{n+1}}{n+1}. \end{aligned} \tag{3.4}$$

Again a simple change of the variable q allows the integration of $x^{n/m}$.

In addition to performing this integration of x^n, the shape of the Fundamental Theorem of Calculus is now much more transparent. From (3.2) (or Figure 1), we see that

$$A_q(X) = f(X)(X - qX) + A_q(Xq).$$

Hence

$$\frac{A_q(Xq) - A_q(X)}{qX - X} = f(X), \tag{3.5}$$

and by recalling (2.3) we see that (3.5) clearly and convincingly suggests the Fundamental Theorem of Calculus.

Although a fully rigorous proof of the Fundamental Theorem can be effected from (3.5), one probably does not really want to do so in a first calculus course.

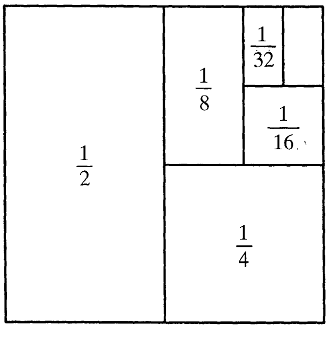
4. POWER SERIES. I won't dwell on the use of the Infinite Geometric Series in proving the Root Test and Ratio Test. This is well-known and practised in almost all calculus books.

I remark only that when one finally arrives at infinite series, the Infinite Geometric Series is an old and trusted friend rather than something that first arises as the case $p = -1$ of the binomial series [**4**, p. 605]:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3}x^3 + \cdots$$

5. EXERCISES ON THE GEOMETRIC SERIES. Given the great utility of the Geometric Series, any exercise that makes it more familiar will be useful. There are countless "plug and chug" type exercises. We close with three more "modern" exercises.

1. ([**1**, p. 4], Don Cohen): Observe the following dissection of a unit square



Show how this illustrates an instance of the Infinite Geometric Series.

2. The following is from W. Edwards Deming's *The New Economics* [**3**, p. 136].

The secret for reduction in time of development is to put more effort into the early stages, and to study the interaction between stages. Each stage should have the benefit of more effort than the next stage.

We content ourselves here to adopt a constant ratio of cost from one stage to the next. Specifically, let the cost of any stage be $1 - x$ times the cost of the preceding stage. Then if K be the cost of the opening stage (the 0-th stage, concepts and proposals), then the cost of the n-th stage would be

$$K_n = K(1-x)^n.$$

The total cost through the n-th stage would be

$$T_n = K\left[1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots + (1-x)^n\right].$$

We note that the series in the brackets is merely $1/x$ expanded in powers of $1 - x$. This is easily seen by writing $x = 1 - (1 - x)$. This series will converge if $0 < x \leq 1$, which satisfies our requirements. Further,

$$T_n = K\left\{\left[1 + (1-x) + (1-x)^2 + (1-x)^3 + \cdots \text{to infinity}\right] - \frac{(1-x)^{n+1}}{x}\right\}$$
$$= \frac{K}{x}\left[1 - (1-x)^{n+1}\right].$$

Assignment. Rewrite Deming's argument so that the role of the Finite Geometric Series is **clear**. Why did Deming use the Infinite Geometric Series?

3. A problem attributed by R. Raimi to a Professor Sleator at the University of Michigan in 1941: Two trees are one mile apart. A drib (it is not necessary that you know what a drib is) flies from one tree to the other and back, making the first trip at 10 miles per hour, the return at 20 miles per hour, the next at 40 and so on, each trip at twice the speed of the preceding. When will the drib be in both trees at the same time? Do not spend time wondering or arguing about the drib, but solve the problem.

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