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SAINT PETERSBURG STATE UNIVERSITY STUDIES IN MATHEMATICS

Renat V. Yuldashev

Nonlinear Analysis and Synthesis of Phase-Locked Loops

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The series *Saint Petersburg State University Studies in Mathematics* presents final results of research carried out in postgraduate mathematics programs at St. Petersburg State University. Most of this research is here presented after publication in leading scientific journals.

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ABSTRACT

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In the modern science and technology, devices that automatically adjust the frequency of quasiperiodic processes to achieve a certain phase relationships between them are very important. Examples of such devices are electric generators and motors, synchrophasotrons, and artificial cardiac pacemakers.

In electronic systems, similar problems of automatic frequency adjustment are encountered in radiolocation, telecommunication, computer architecture, navigation, and other like systems. One of the technical solutions used in the context of these problems is to utilise a phase-locked loop (PLL).

The widespread applications of PLL called for the development of rigorous mathematical models in the phase-frequency space for different waveforms of high-frequency signals. Towards this end, in the present study, several conditions that enable high-frequency signals have been formulated. Also, methods of asymptotic analysis for the signals with discontinuous waveforms have been developed. The main result of the study is the development of formulas for the characteristics of the phase detector (PD) for a PLL and for a PLL system with a squarer that allow one to derive differential equations for the circuits of that kind. Also, in the case of typical signal waveforms, a table of PD characteristics has been created. Moreover, the asymptotic equivalence between the phase-frequency model and the signal space model has been rigorously established. One of the major approaches to the study of a PLL in the modern engineering literature is numerical modeling. This approach is usually very time-consuming because of the high frequencies (up to 10Ghz) of the signals involved. Therefore, the discretization step has to be chosen sufficiently small for a clear observation of the dynamics of non-linear elements of a PLL. This makes the full modeling of a PLL in the signal space almost impossible for the high-frequency signals.

In this work, a new approach to overcoming this obstacle is proposed. The approach is based on the transition from PLL models on the level of electronic realization to asymptotically equivalent PLL models in the phase-frequency space. This makes it possible using a large discretization step, which reduces the time of numerical analysis by hundred-fold. The theoretical results developed in this study are in full agreement with the results of modeling PLL and PLL system with a squarer in the signal and the phase-frequency spaces.

The results of the study have been published in 22 papers (8 of which are indexed in Scopus)

Keywords: phase-locked loop, phase detector, characteristic, pll with squarer

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1 INTRODUCTION

1.1 Intellectual merit

A PLL is a classic control system widely used in telecommunication (Wendt and Fredentall, 1943; Richman, 1954; Gardner, 1966; Lindsey, 1972; Bullock, 2000; Manassewitsch, 2005) and computer architecture (Kung, 1988; Smith, 1999; Gardner et al., 1993; Simpson, 1944; Lapsley et al., 1997; Buchanan and Wilson, 2001; Xanthopoulos et al., 2001; Wainner and Richmond, 2003; Bindal et al., 2003; Young, 2004; Shu and Sanchez-Sinencio, 2005). It was invented by a French engineer Anri de Bellescize in the 1930s (Bellescize, 1932). After that, intensive studies of PLL were carried out (Lindsey, 1972; Fines and Aghvami, 1991; Djordjevic et al., 1998; Djordjevic and Stefanovic, 1999; Tomasi, 2001; Best, 2007; Couch, 2007). Following paper PIII, note that one of the first applications of PLL was in the wireless communication. In radioengineering, PLL-based circuits (e.g., a PLL system with a squarer and Costas loop) are used for demodulation, carrier recovery, and frequency synthesis (Costas, 1962; Miyazaki et al., 1991; Fines and Aghvami, 1991; Fiocchi et al., 1999; Bullock, 2000; Kaplan and Hegarty, 2006; Tomkins et al., 2009; Kim et al., 2010). Although a PLL is inherently a nonlinear circuit, in the modern literature, the main approaches to its analysis are based on the use of simplified linear models (Viterbi, 1966; Shakhgil'dyan and Lyakhovkin, 1972; Gardner, 1993; Egan, 2000; Best, 2003; Shu and Sanchez-Sinencio, 2005; Tretter, 2007)), the methods of linear analysis (Gardner, 1993; Egan, 2000; Best, 2003), empirical rules, and numerical simulation on the level of electronic realization (see a plenary lecture of D. Abramovitch at the 2002 American Control Conference (Abramovitch, 2002)). However it is known that the application of linearization methods and linear analysis for control systems can lead to untrue results and, therefore, requires special justifications (Kuznetsov and Leonov, 2001; Leonov et al., 2010d,b,a,b; Kuznetsov et al., 2010; Bragin et al., 2010; Leonov et al., 2011b; Leonov and Kuznetsov, 2011; Kuznetsov et al., 2011b,a,c; Leonov et al., 2011c; Bragin et al., 2011; Leonov et al., 2011a; Leonov and Kuznetsov, 2012; Leonov et al., 2012; Kiseleva et al., 2012; Andrievsky et al., 2012; Leonov G. A., 2013a;

Leonov and Kuznetsov, 2013; Kuznetsov et al., 2013; Leonov G. A., 2013b). A rigorous mathematical analysis of PLL models is often a very challenging task (Suarez and Quere, 2003; Margaris, 2004; Benarjee, 2006; Feely, 2007; Banerjee and Sarkar, 2008; Feely et al., 2012), so for the analysis of a nonlinear PLL a numerical simulation is often used (see, e.g., (Lai et al., 2005; Best, 2007)). However, in the context of high-frequency signals, complete numerical simulation of PLL-based circuit on the level of electronic realization (in the signal/time space), which is described by a nonlinear non-autonomous system of differential equations, is a very challenging task. Here, it is necessary to observe simultaneously "very fast time scale of the input signals" and "slow time scale of signal's phases" (Abramovitch, 2008a,b). Therefore, a relatively small discretization step in the simulation procedure does not allow one to consider phase locking processes for the high-frequency signals (up to 10Ghz) in a reasonable time period.

1.2 Goal of the work

The goals of this work include:

– derivation and justification of nonlinear models for the classic PLL and a PLL system with a squarer in the phase-frequency space;
– computing the phase detector characteristic for various types of signal waveforms;
– developing an efficient numerical simulation method for the classic PLL and a PLL system with a squarer.

1.3 Methods of investigation

In this work, an approach enabling the development of adequate mathematical models of PLLs is considered. The main idea can be traced back to the works (Viterbi, 1966; Gardner, 1966; Lindsey, 1972), carried out in the context of sinusoidal and square-wave signals only, and it consists in the construction of special nonlinear models of PLL circuits in the phase-frequency space. This approach requires a mathematical description of the circuit components and a proof of the reliability of the considered model (Leonov et al., 2006; Kudrewicz and Wasowicz, 2007b; Kuznetsov et al., 2008; Kuznetsov, 2008; Leonov et al., 2009; Kuznetsov et al., 2009b; Leonov et al., 2010c). This new approach, extended to a broader class of signals, requires a mathematical description of the circuit components and a proof of the reliability of the considered model.

In addition, the asymptotic analysis of high-frequency oscillations, the averaging method (Krylov and Bogolubov, 1947; Mitropolsky and Bogolubov, 1961), and numerical simulation methods are used to create PLL models for various non-sinusoidal signal waveforms.

1.4 The main results

– Nonlinear mathematical models of the classic PLL with piecewise-differentiable waveforms have been constructed (see papers PI,PII,PIV-PVIII);
– Nonlinear mathematical models of a PLL system with a squarer in the case of piecewise-differentiable waveforms have been derived (see paper PVII);
– An effective method of numerical simulation of a PLL has been developed (see paper PIII).

1.5 Adequacy of the results

The characteristics of a phase detector obtained in this work are in line with the well-known characteristics for the sinusoidal and square waveforms. All main results are rigorously proved (see proofs of the theorems in PI and PV). PLL models in the phase-frequency domain are justified using the averaging method by Krylov-Bogolyubov. Numerical simulation confirms the adequacy of the derived models.

1.6 Novelty

In this work, for the first time, two different approaches had been used jointly: nonlinear analysis of PLL models (Saint-Petersburg State University) and effective numerical analysis of dynamical systems (University of Jyväskylä). This made it possible to create an analytical method for calculating the characteristics of the phase detector (multiplier/mixer), to build a mathematical model of a PLL for various non-sinusoidal signal waveforms, and propose an effective method for numerical simulation of the classic PLL and a PLL system with a squarer.

1.7 Practicability

The obtained results allow one to calculate the phase detector characteristics and can be used to analyze the stability of PLL-based circuits. Obtained PLL models can significantly reduce time required for numerical simulation and allow one to determine such important features of the systems involved as pull-in range, pull-out range, and other properties, thereby, significantly reducing time needed for the development of PLL-based circuits and their analysis.

A practical application of the obtained results is presented in Patents AX–AXIV.

1.8 Appraisal of the work and publications

The results of this work were presented at a plenary lecture at an international conference Dynamical Systems and Applications (Kiev, Ukrane – 2012), IEEE international Congress on Ultra Modern Telecommunications and Control Systems and Workshops (St.Petersburg, Russia – 2012), at regular lectures at IEEE 4th International Conference on Nonlinear Science and Complexity (Budapest, Hungary – 2012), 9th International Conference on Informatics in Control, Automation and Robotics (Rome, Italy – 2012), IEEE 10-th International Symposium on Signals, Circuits and Systems (Iasi, Romania – 2011), 8th International Conference on Informatics in Control, Automation and Robotics (Noordwijkerhout, The Netherlands – 2011), 4th IFAC Workshop on Periodic Control System (Antalya, Turkey – 2010), International Workshop "Mathematical and Numerical Modeling in Science and Technology" (Jyväskylä, Finland – 2010); at the seminars of the Department of Applied Cybernetics (St. Petersburg State University, Faculty of Mathematics and Mechanics) and at the seminars of the Department of Mathematical Information Technology (University of Jyväskylä, Finland).

The results of this work appeared in 22 publications (8 in Scopus database, 4 patents) The main content of this work can be found in the included articles PI-PVIII. In articles PI-PV and PVII-PVIII co-authors formulated the problems, the author formulated and proved the theorems. In article PVI co-authors formulated the problems and proved the theorems for a Costas loop, the author formulated the problems and proved the theorems for a PLL.

Also, the material of the dissertation is presented in (Yuldashev, 2012, 2013a) and its extended version is in preparation in (Yuldashev, 2013b).

2 THE MAIN CONTENT

2.1 Asymptotic equivalence of PLL models

Consider the block diagram of the classic PLL on the level of electronic realization (signal space) shown in Fig. 1

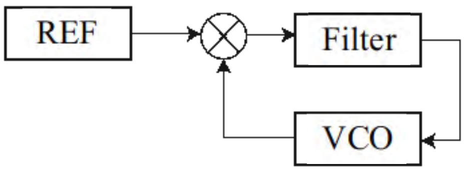


FIGURE 1 Block diagram of classical phase-locked loop on the level of electronic realization

The diagram comprise the following blocks (Boensel, 1967; Best, 2003; Kroupa, 2003): a multiplier used as the phase detector (PD), a low-pass linear filter, and a voltage-controlled oscillator (VCO). The phase detector compares the phase of an input signal against the phase of the VCO signal; the output of the PD is the measure of the phase error between its two inputs. The error voltage is then filtered by the loop filter, whose control output is applied to the VCO by changing its frequency in the direction that reduces the phase error between the input signal and the VCO.

To carry out a rigorous mathematical analysis of a PLL, it is necessary to consider PLL models in the signal and phase-frequency spaces (Leonov and Seledzhi, 2002; Abramovitch, 2002; Leonov, 2006; Kuznetsov et al., 2009a), and PVI. For constructing an adequate nonlinear mathematical model of a PLL in the phase-frequency space, it is necessary to find characteristic of the PD (Shakhgil'dyan and Lyakhovkin, 1972; Lindsey, 1972; Lindsey and Simon, 1973; Best, 2007).

Consider the block-diagram shown in Fig. 2.

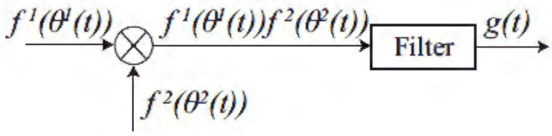


FIGURE 2 Multiplier and filter

Here VCO is a tunable voltage-control oscillator which generates high-frequency oscillations $f^2(\theta^2(t))$. The product $f^1(\theta^1(t))f^2(\theta^2(t))$ enters the linear low pass filter and $g(t)$ is the output of the filter.

Suppose that the waveforms $f^{1,2}(\theta)$ are 2π- periodic piecewise differentiable functions. Also, let us assume that the phases $\theta^{1,2}(t)$ are smooth, monotonically increasing functions with the derivatives (frequencies) satisfying the inequalities

$$\dot{\theta}^p(\tau) \geq \omega_{min} >> 1, \ \ p = 1, 2, \tag{1}$$

where ω_{min} is a fixed positive number. Note that in the modern devices, oscillators can reach frequencies up to tens of gigahertz.

Consider the block diagram shown in Fig. 3.

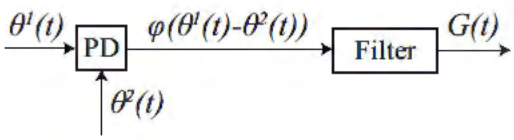


FIGURE 3 Phase detector and filter

Here, the PD is a nonlinear block with the characteristic $\varphi(\theta)$. The phases $\theta^{1,2}(t)$ are PD block inputs, and the output is the function $\varphi(\theta^1(t) - \theta^2(t))$. The PD characteristic $\varphi(\theta)$ depends on waveforms of the input signals. Impulse response functions and initial conditions of the filters in Fig. 2 and Fig. 3 coincide.

Definition. The block diagrams shown in Fig. 2 and Fig. 3 are called asymptotically equivalent, if for a large fixed time interval $[0, T]$ the following relation holds

$$G(t) - g(t) = O(\delta), \quad \delta = \delta(\omega_{min}), \quad \forall t \in [0, T], \tag{2}$$

where $\delta(\omega_{min}) \to 0$ as $\omega_{min} \to \infty$.

The notion of asymptotic equivalence allows one to pass from the analysis of the signal space PLL models to the analysis of the phase-frequency domain PLL models (Fig. 4)

The equivalence of the block-diagrams shown in Fig. 2 and Fig. 3 was demonstrated by Viterbi and Gardner (Viterbi, 1966; Gardner, 1966), yet only for the sinusoidal signals and without a formal justification. The first mathematically rigorous conditions of high-frequency signals and proof of the asymptotic equivalence of the block-diagrams in the case of the sinusoidal and square-wave signals

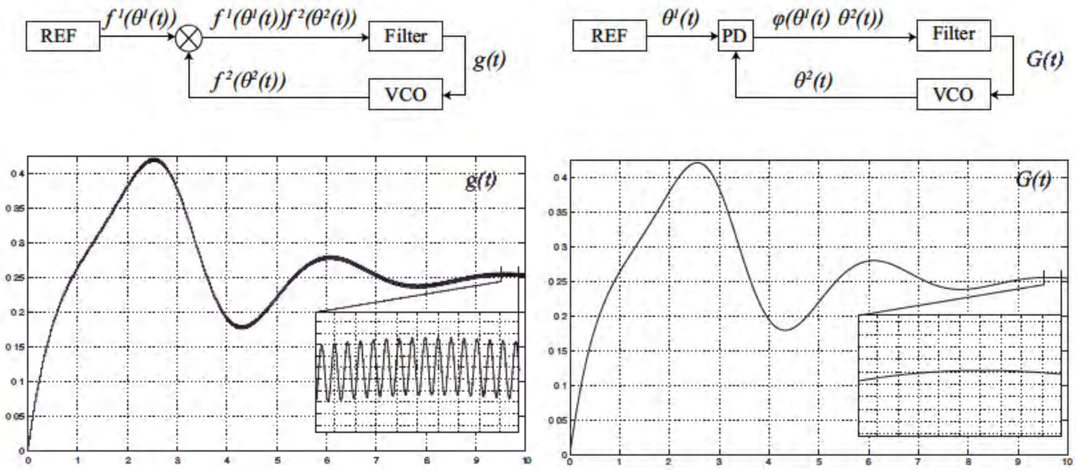


FIGURE 4 Asymptotic equivalence of PLL model on the level of electronic realization and phase-frequency domain PLL model

were presented in (Leonov and Seledzhi, 2005). It should be noted that for the sinusoidal signals the PD characteristic is also sinusoidal, while for the square waveform signal the PD characteristic is a continuous piecewise-linear function.

In this work, an effective analytical method for computing the characteristic of the multiplier-mixer of the phase detector is proposed. For various waveforms of high-frequency signals, including non-sinusoidal waveforms widely used in practice (Henning, 1981; Fiocchi et al., 1999), new classes of phase-detector characteristics are obtained, and a dynamical model of a PLL is constructed.

2.2 The basic assumptions

Following the works (Leonov et al., 1992; Leonov and Seledzhi, 2005; Leonov and Seledghi, 2005; Kuznetsov et al., 2008; Leonov, 2008; Leonov et al., 2009, 2010c), PI–PIV, AI, and AVI–AIX we assume that the following conditions hold true:

$$f^p(\theta) = \frac{a_0^p}{2} + \sum_{i=1}^{\infty} \left(a_i^p \cos(i\theta) + b_i^p \sin(i\theta) \right), \quad p = 1, 2,$$
$$a_0^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) dx, \tag{3}$$
$$a_i^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) \cos(ix) dx, \quad b_i^p = \frac{1}{\pi} \int_{-\pi}^{\pi} f^p(x) \sin(ix) dx, \quad i \in \mathbb{N}.$$

The relationship between the input $\xi(t)$ and the output $\sigma(t)$ of the linear filter is as follows:

$$\psi(t) = \alpha_0(t) + \int_0^t \gamma(t - \tau) \xi(\tau) d\tau, \tag{4}$$

where $\alpha_0(t)$ is an exponentially damped function, linearly dependent on the initial state of the filter at $t = 0$, $\gamma(t)$ is an impulse response function of the linear filter. Furthermore, we assume that $\gamma(t)$ is a differentiable function with a bounded derivative. From equation (4) one can obtain $g(t)$

$$g(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau) f^1\big(\theta^1(\tau)\big) f^2\big(\theta^2(\tau)\big) d\tau. \tag{5}$$

Let us assume that difference between frequencies is uniformly bounded

$$\left|\dot\theta^1(\tau) - \dot\theta^2(\tau)\right| \le \Delta\omega, \ \forall \tau \in [0, T], \tag{6}$$

where $\Delta\omega$ is a constant.

We split the interval $[0, T]$ into the intervals of length δ

$$\delta = \frac{1}{\sqrt{\omega_{min}}}. \tag{7}$$

Let us assume that

$$|\dot\theta^p(\tau) - \dot\theta^p(t)| \le \Delta\Omega, \ p = 1, 2, \quad |t - \tau| \le \delta, \quad \forall \tau, t \in [0, T], \tag{8}$$

where $\Delta\Omega$ does not depend on t and τ. Conditions (6)–(8) imply that the functions $\dot\theta^p(t)$ are almost constant and the functions $f^p(\theta^p(t))$ are rapidly oscillating within small intervals $[t, t+\delta]$. The boundedness of the derivative of $\gamma(t)$ implies that there exists a constant C, such that

$$|\gamma(\tau) - \gamma(t)| \le C\delta, \quad |t - \tau| \le \delta, \quad \forall \tau, t \in [0, T]. \tag{9}$$

2.3 Phase detector characteristics of PLL

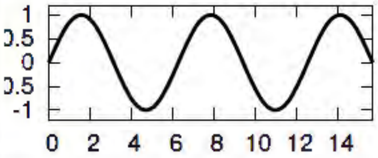
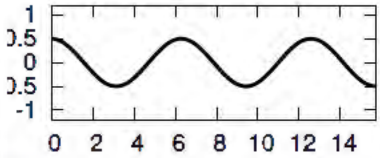
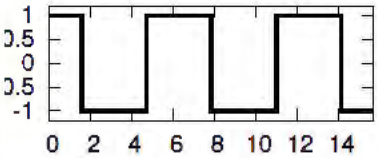
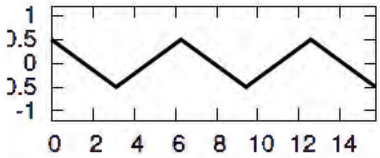
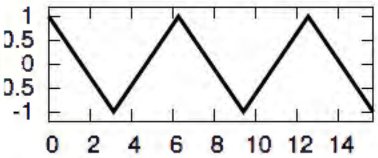
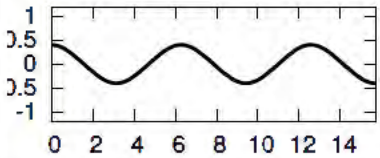
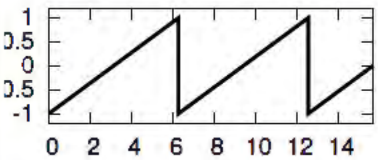
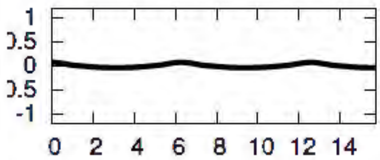
The following theorem was formulated for various types of signal waveforms in papers PI–PVII

Theorem 1. Let conditions (1), (3), (6) – (9) be satisfied. Then the systems shown in Fig. 2 and Fig. 3 are asymptotically equivalent, where

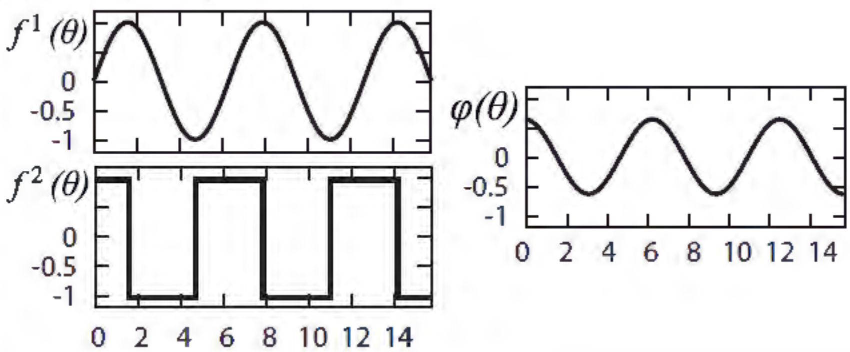
$$\varphi(\theta) = \frac{a_0^1 a_0^2}{4} + \frac{1}{2}\sum_{l=1}^{\infty}\Big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + (a_l^1 b_l^2 - b_l^1 a_l^2)\sin(l\theta) \Big). \tag{10}$$

See included papers PII, PIV for the full proof.

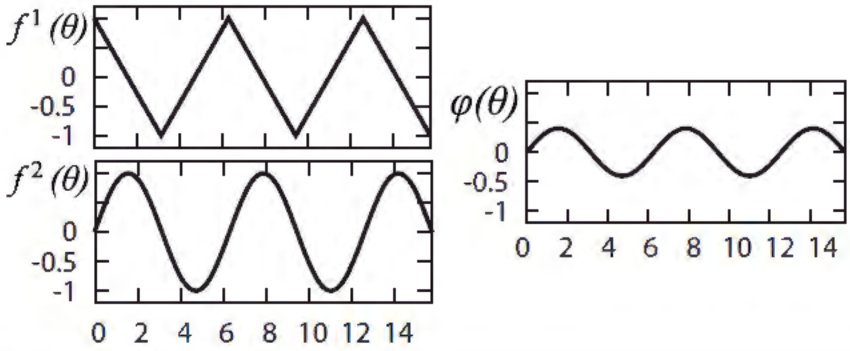
Theorem 1 allows one to compute the phase detector characteristic for the following typical signals shown in the table below. To the left, the waveforms $f^{1,2}(\theta)$ of the input signals are shown; depicted to the right are the corresponding PD characteristics $\varphi(\theta)$.

waveforms, $f^{1,2}(\theta)$	charactersitics, $\varphi(\theta)$
$f^{1,2}(\theta) = \sin(\theta)$ 	$\varphi(\theta) = \frac{1}{2}\cos(\theta)$ 
$f^{1,2}(\theta) = \mathrm{sign}(\sin(\theta))$ 	$\varphi(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right)$ 
$f^{1,2}(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right)$ 	$\varphi(\theta) = \frac{32}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\left((2n-1)\theta\right)$ 
$f^{1,2}(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\theta\right)$ 	$\varphi(\theta) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(n\theta\right)$ 

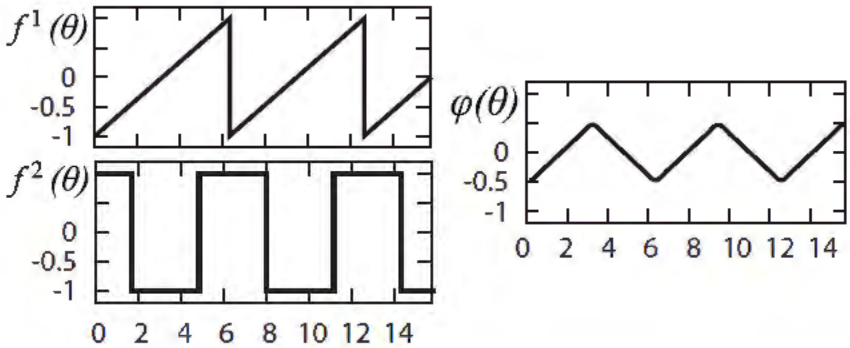
$f^1(\theta) = \sin(\theta)$,
$f^2(\theta) = \mathrm{sign}\,\sin(\theta)$,
$\varphi(\theta) = \frac{2}{\pi}\cos(\theta)$



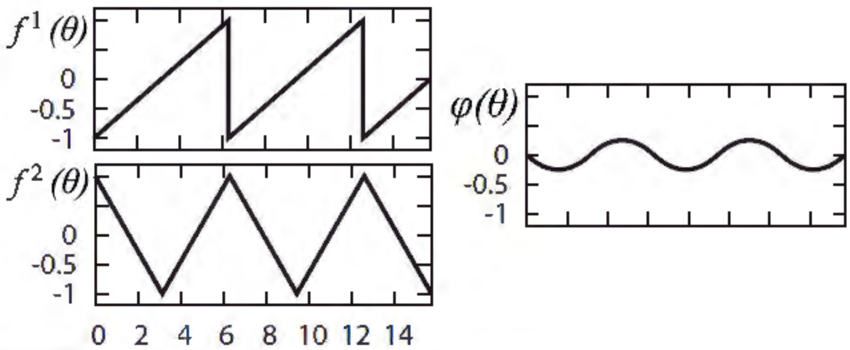
$$f^1(\theta) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left((2n-1)\theta\right),$$
$$f^2(\theta) = \sin(\theta),$$
$$\varphi(\theta) = \frac{4}{\pi^2} \sin(\theta)$$



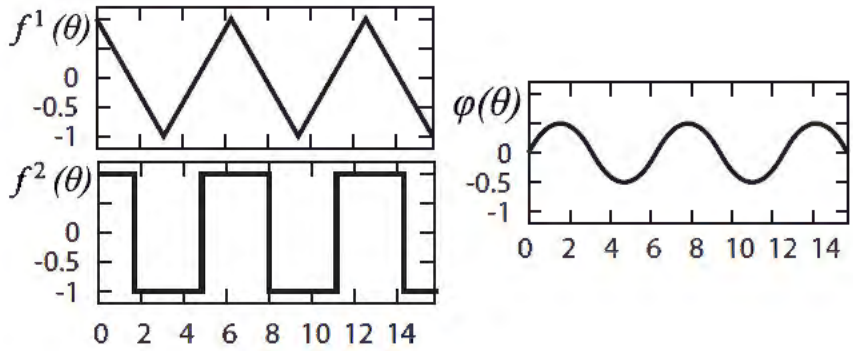
$$f^1(\theta) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\theta),$$
$$f^2(\theta) = \operatorname{sign} \sin(\theta),$$
$$\varphi(\theta) = -\frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\theta)}{(2n-1)^2}$$



$$f^1(\theta) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\theta)$$
$$f^2(\theta) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left((2n-1)\theta\right)$$
$$\varphi(\theta) = -\frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\theta)}{(2n-1)^3}$$



$$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right)$$
$$f^2(\theta) = \operatorname{sign}\sin(\theta),$$
$$\varphi(\theta) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\theta)}{(2n-1)^3}$$



2.4 Phase detector characteristics of PLL with squarer

Consider the block diagram of a PLL system with a squarer (see, e.g., (Hershey et al., 2002; Goradia et al., 1990), and PVIII) on the level of electronic realization shown in Fig. 5.

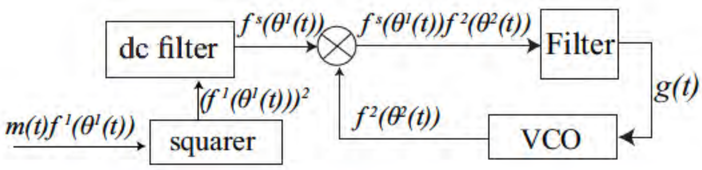


FIGURE 5 PLL with squarer on the level of electronic realization

This circuit is used for the carrier recovery during the data transfer (see, e.g., (Lindsey and Simon, 1973)). The circuit consists of the following blocks: a multiplier used as the phase detector (PD), a low-pass linear filter, a voltage-controlled oscillator (VCO), and a squarer. Here $m(t) = \pm 1$ is the transmitted data, and $f^1(t) = f^1(\theta^1(t))$ is the carrier. The VCO generates the signal $f^2(t) = f^2(\theta^2(t))$. The squarer multiplies the input signal by itself and the subsequent filter erases the DC offset (Best, 2007)

$$f^s(\theta^1(t)) = m(t)f^1(\theta^1(t))m(t)f^1(\theta^1(t)) = f^1(\theta^1(t))f^1(\theta^1(t)). \qquad (11)$$

Consequently, the data signal $m(t)$ does not affect the tunable oscillator frequency.

Let us consider the block-diagram in Fig. 6

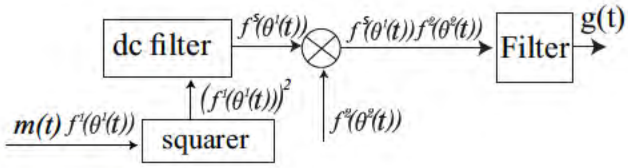


FIGURE 6 PLL with squarer

From equation (4) it follows that

$$g(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau) f^s(\theta^1(\tau)) f^2(\theta^2(\tau)) d\tau. \tag{12}$$

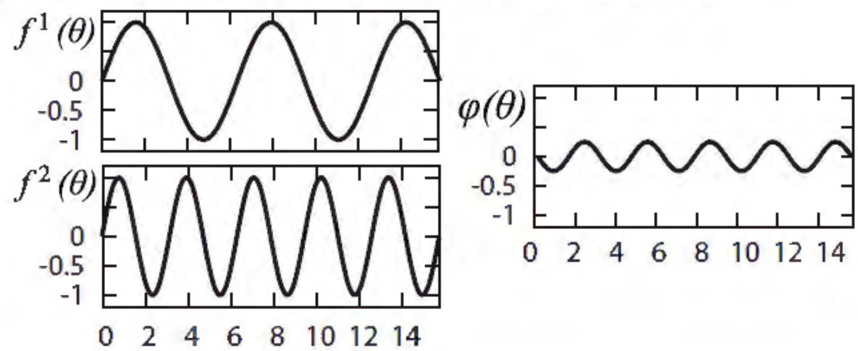
Theorem 2. Let conditions (1), (3), (6) – (9) be satisfied. Then the systems in Fig. 6 and Fig. 3 are asymptotically equivalent, where

$$\begin{aligned}
\varphi(\theta) &= \frac{1}{2}\sum_{l=1}^{\infty}\Big((A_l^1 a_l^2 + B_l^1 b_l^2)\cos(l\theta) + (A_l^1 b_l^2 - B_l^1 a_l^2)\sin(l\theta)\Big),\\
A_l^1 &= \frac{1}{2}\sum_{m=1}^{\infty}\big(a_m^1(a_{m+l}^1 + a_{m-l}^1) + b_m^1(b_{m+k} + b_{m-k}^1)\big),\\
B_l^1 &= \frac{1}{2}\sum_{m=1}^{\infty}\big(a_m^1(b_{m+l}^1 - b_{m-l}^1) - b_m^1(a_{m+k} + a_{m-k}^1)\big).
\end{aligned} \tag{13}$$

For the full proof, see paper PIX.

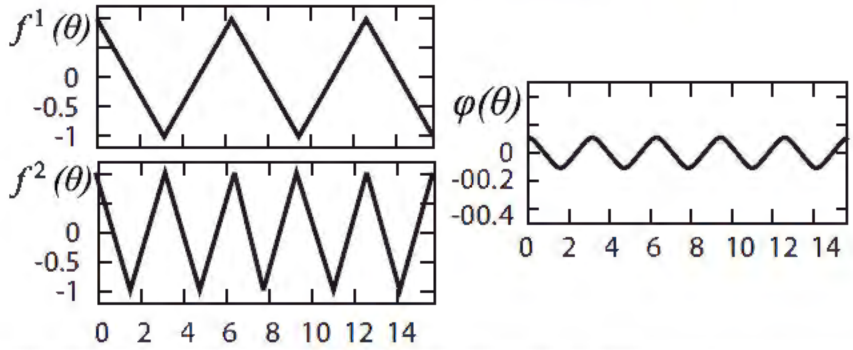
Phase detector characteristic of PLL with squarer for standard signal waveforms.

$f^1(\theta) = \sin(\theta)$, $f^2(\theta) = \sin(2\theta)$, $\varphi(\theta) = -\frac{1}{4}\sin(2\theta)$

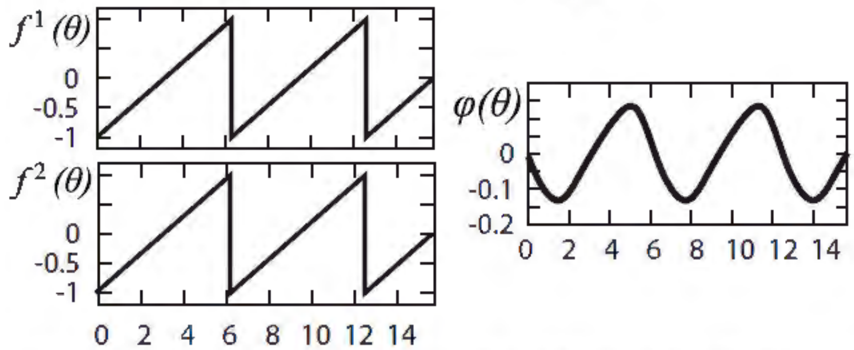


$$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big),$$

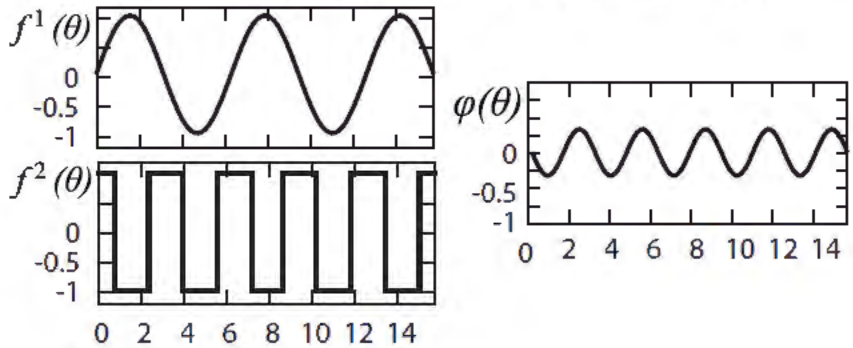
$$f^2(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((4n-2)\theta\big),\ \varphi(\theta) = \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{4}{(4n-2)^4}\cos\big((4n-2)\theta\big)$$



$$f^{1,2}(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\theta),\ \varphi(\theta) = -\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin(n\theta)$$

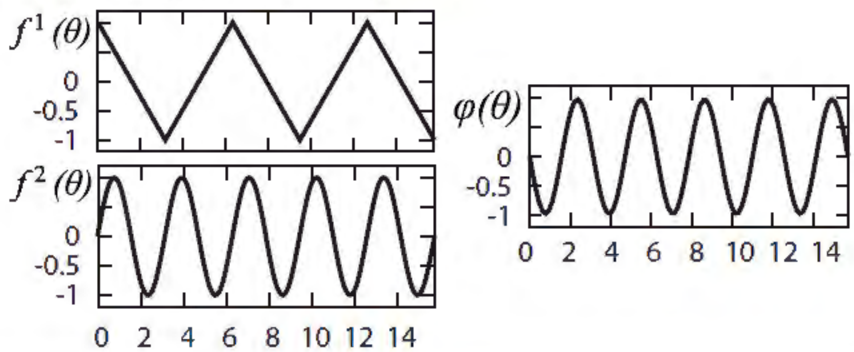


$$f^1(\theta) = \sin(\theta),\ f^2(\theta) = \operatorname{sign}\sin(2\theta),\ \varphi(\theta) = -\frac{1}{\pi}\sin(2\theta)$$



$$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big)$$

$$f^2(\theta) = \sin(2\theta),\ \varphi(\theta) = -\frac{8}{\pi^2}\sin(2\theta)$$



2.5 Differential equations of phase-lock loop

Let us consider a system of differential equations for a PLL on the level of electronic realization and in the phase-frequency domain (see papers PVI and PIII). From a mathematical point of view, a linear low-pass filter can be described by the system of linear differential equations

$$\frac{dx}{dt} = Ax + b\xi(t),\ \psi(t) = c^* x, \tag{14}$$

a solution of which takes the form

$$\gamma(t-\tau) = c^* e^{A(t-\tau)} b, \quad \alpha_0(t) = c^* e^{At} x_0. \tag{15}$$

The model of a tunable generator is usually assumed to be a linear one (Viterbi, 1966; Stensby, 1997; Kroupa, 2003; Best, 2007):

$$\dot{\theta}^2 = \omega^2_{free} + Lc^* x(t), \tag{16}$$

where ω^2_{free} is a free-running frequency of tunable generator and L is the oscillator's gain. Here, it is also possible to use nonlinear models of VCO; see, e.g., (Demir et al., 2000)

$$\begin{aligned} \dot{x} &= Ax + bf^1(\theta^1(t))f^2(\theta^2(t)), \\ \dot{\theta}^2 &= \omega^2_{free} + Lc^* x. \end{aligned} \tag{17}$$

Suppose that the frequency of the master generator is constant $\dot{\theta}^1(t) \equiv \omega^1$. Let us denote

$$\theta(t) = \theta^2(t) - \omega^1 t. \tag{18}$$

Then

$$\begin{aligned} \dot{x} &= Ax + bf^1(\omega^1 t)f^2(\theta + \omega^1 t), \\ \dot{\theta} &= \omega^2_{free} - \omega^1 + Lc^* x. \end{aligned} \tag{19}$$

For the classic PLL and PLL system with a squarer, system (19) is nonautonomous, thereby, being difficult for investigation (Margaris, 2004; Kudrewicz and Wasowicz, 2007a). Theorems 1 and 2, and the averaging method (Krylov and Bogolubov, 1947; Mitropolsky and Bogolubov, 1961) allow one to study the following, more simple, autonomous systems of differential equations

$$\begin{aligned} \dot{x} &= Ax + b\varphi(\theta), \\ \dot{\theta} &= \omega^2_{free} - \omega^1 + Lc^* x, \end{aligned} \tag{20}$$

where $\varphi(\theta)$ is the corresponding characteristic of the phase detector.

The system of differential equations for a PLL with a squarer is the same as the equations for the classic PLL.

2.6 Effective numerical simulation of PLL

According to (Abramovitch, 2008a,b), numerical simulation of a PLL in the signal space is a very slow process. However, the derived models of a PLL in the phase-frequency space enable for an effective numerical simulation of these circuits (see paper PV).

Two types of signals will be considered here: the "sawtooth" waveform and the "triangle" waveform signals. The Fourier series expansion of the "sawtooth" waveform is as follows

$$f^1(\theta) = \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin(i\theta), \quad a_i^1 = 0, \quad b_i^1 = \frac{2}{i\pi}. \tag{21}$$

Likewise, the Fourier series expansion of the "triangle" waveform is as follows

$$f^2(\theta) = \frac{8}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^2}\cos((2i-1)\theta), \quad a_{2i-1}^2 = \frac{8}{(2i-1)^2\pi^2}, \tag{22}$$
$$a_{2i}^2 = 0, \quad b_i^2 = 0, \quad i \in \mathbb{N}.$$

Numerical simulation of the classic PLL in the signal domain (on the level of electronic realization) and in the phase-frequency domain have been considered.

All models were implemented in Matlab. The phase-frequency domain model of a PLL made it possible to significantly speed up the simulation of the circuit (see Fig. 8 and Fig. 7).

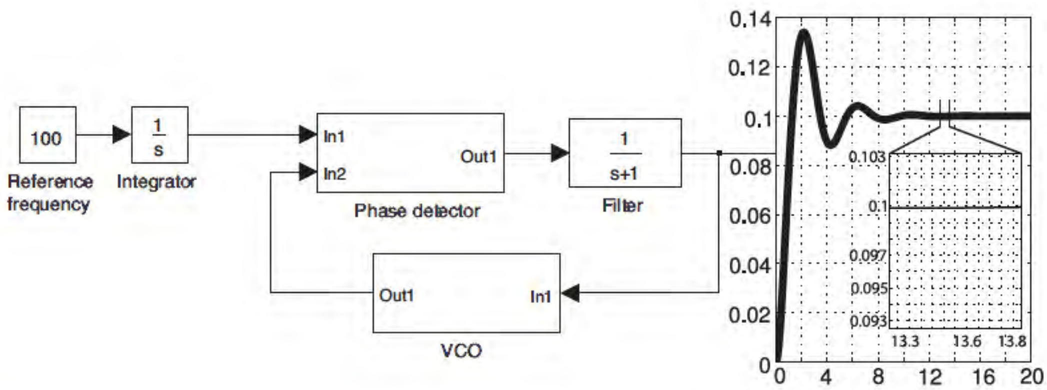


FIGURE 7 Phase-frequency domain model of PLL. Simulation time — 0.3 seconds

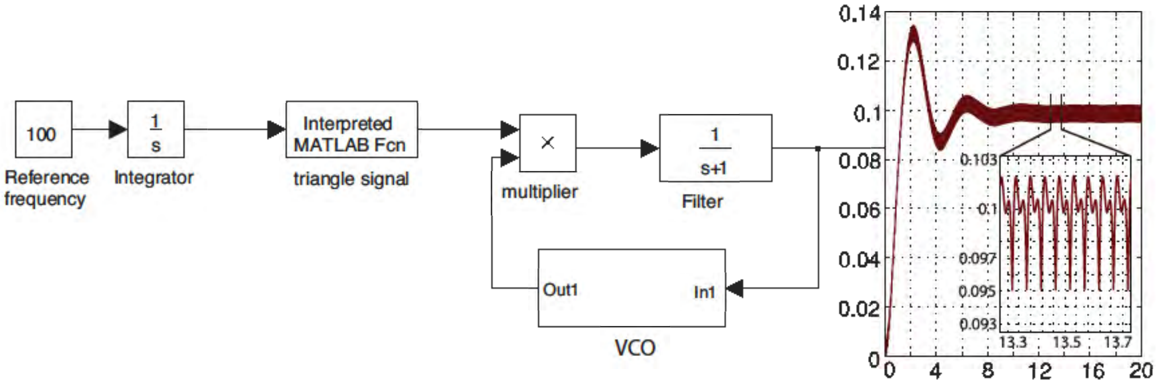


FIGURE 8 Signal domain model of PLL. Simulation time — 30 seconds

In Fig. 7 and Fig. 8 VCO frequency is 99Hz; reference frequency – 100Hz; filter transfer function — $\frac{1}{1+s}$; VCO input gain — 10; time interval — 20 seconds; $M = 10$.

Nowadays, oscillators can reach frequencies up to 10Ghz. Therefore, the discretization step has to be chosen sufficiently small for a clear observation of the dynamics of the phase detector. Depending on application, the time period required for the synchronization of oscillators can vary from several seconds to several minutes. This makes the full numerical simulation of a PLL in the signal space almost impossible for the high-frequency signals. According to the engineers of the ST-Ericsson firm, the full simulation of a PLL can take up two weeks. On the other hand, numerical simulation in the phase-frequency space allows one to consider only low-frequency signals, something that allows one to increase a discretization step and decrease time needed for numeriacal simulation.

The numeriacal simulation method is described in included article PV and patents AVIII, AVII.

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ORIGINAL PAPERS

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NONLINEAR ANALYSIS OF PHASE-LOCKED LOOP BASED CIRCUITS

by

R.E. Best, N.V. Kuznetsov, G.A. Leonov, M.V. Yuldashev, R.V. Yuldashev 2013

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Nonlinear Analysis of Phase-locked Loop Based Circuits

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Phase-locked loop (PLL) is a classical circuit widely used in telecommunication and computer architectures. PLL was invented in the 1930s-1940s (Bellescize, 1932) and then intensive studies of the theory and practice of PLL were carried out (Viterbi, 1966; Gardner, 1966; Lindsey and Simon, 1973). One of the first applications of phase-locked loop (PLL) is related to the problems of wireless data transfer. In radio engineering, PLL-based circuits (e.g. Costas Loop, PLL with squarer) are used for carrier recovery, demodulation, and frequency synthesis (see, e.g., (Stiffler, 1964; Kroupa, 2003; Best, 2007)).

Although PLL is essentially a nonlinear control system, in modern literature, devoted to the analysis of PLL-based circuits, the main direction is the use of simplified linear models, the methods of linear analysis, empirical rules, and numerical simulation (see plenary lecture of D. Abramovich at American Control Conference 2002 (Abramovitch, 2002)). Rigorous nonlinear analysis of PLL-based circuit models is often a very difficult task (Feely, 2007; Banerjee and Sarkar, 2008; Suarez et al, 2012; Feely et al, 2012), so for analysis of nonlinear PLL models it is widely used, in practice, numerical simulation (see, e.g., (Best, 2007)). However for high-frequency signals, complete numerical simulation of *physical model of PLL-based circuit in signals/time space*, which is described by a nonlinear non-autonomous system of differential equations, is very challenging task (Abramovitch, 2008a,b) since it is necessary to observe simultaneously "*very fast time scale of the input signals*" and "*slow time scale of signal's phases*". Here relatively small discretization step in numerical procedure does not allow one to consider phase locking processes for high-frequency signals in reasonable time.

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Here two approachers, which allow one to overcome these difficulties, are considered. The first idea is traced back to the works of Viterbi (1966) and consists in construction of mathematical models of PLL-based circuits in phase-frequency space. This approach requires to determine mathematical characteristics of the circuit components and to prove reliability of considered mathematical model. The second idea is traced back to the works of Emura (1982) and consists in design of circuit components in such a way that there is no oscillation with double frequency in the loop.

1 Phase-frequency model of classical PLL

To overcome simulation difficulties for PLL-based circuits it is possible to construct *mathematical model in phase-frequency/time space* (Leonov et al, 2012a), which can be described by nonlinear dynamical system of differential equations, so here it is investigated only slow time scale of signal's phases and frequencies. That, in turn, requires (Abramovitch, 2002) the computation of phase detector characteristic (nonlinear element used to math reference and controllable signals), which depends on waveforms of considered signals. Using results of analysis of this mathematical model for conclusions on behavior of the physical model requires rigorous justification.

Consider a classical PLL on the level of electronic realization (Fig. 1)

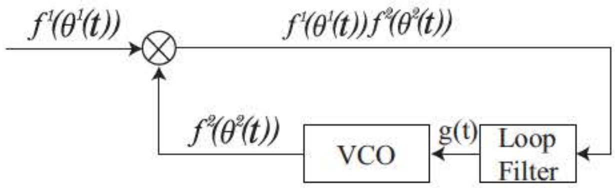


Fig. 1: Block diagram of PLL on the level of electronic realization.

Here signals $f^p(t) = f^p(\theta^p(t)),\ p = 1,2$ with $\theta^p(t)$ as a phases are oscillations generated by reference oscillator and tunable voltage-control oscillator (VCO), respectively.

The block $\otimes$ is a multiplier (used as a phase detector) of oscillations $f^1(t)$ and $f^2(t)$, and the signal $f^1(\theta^1(t))f^2(\theta(t))$ is its output. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter has the form:

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,\mathrm{d}\tau, \tag{1}$$

where $\gamma(t)$ is an impulse response function of filter and $\alpha_0(t)$ is an exponentially damped function depending on the initial data of filter at moment $t = 0$. By assumption, $\gamma(t)$ is a differentiable function with bounded derivative (this is true for the most considered filters (Thede, 2005)).

1.1 High-frequency property of signals

Suppose that the waveforms $f^{1,2}(\theta)$ are bounded 2π-periodic piecewise differentiable functions[1]. Consider Fourier series representation of such functions

$$f^p(\theta) = \sum_{i=1}^{\infty} \left(a_i^p \sin(i\theta) + b_i^p \cos(i\theta)\right),\ p = 1,2,$$

$$a_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\sin(i\theta)d\theta, b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\cos(i\theta)d\theta.$$

A high-frequency property of signals can be reformulated in the following way. By assumption, the phases $\theta^p(t)$ are smooth functions (this means that frequencies are changing continuously, what corresponds to classical PLL analysis (Kroupa, 2003; Best, 2007)). Suppose also that there exists a sufficiently large number ω_{min} such that the following conditions are satisfied on a fixed time interval $[0,T]$:

$$\dot\theta^p(\tau) \geq \omega_{min} > 0,\ \ p = 1,2, \tag{2}$$

where T is independent of ω_{min} and $\dot\theta^p(\tau) = \frac{d\theta^p(\tau)}{d\tau}$ denotes frequencies of signals. The frequencies difference is assumed to be uniformly bounded

$$\left|\dot\theta^1(\tau) - \dot\theta^2(\tau)\right| \leq \Delta\omega,\ \forall\tau \in [0,T]. \tag{3}$$

Requirements (2) and (3) are obviously satisfied for the tuning of two high-frequency oscillators with close frequencies. Denote $\delta = \omega_{min}^{-\frac{1}{2}}$. Consider the following relations

$$\begin{aligned} &|\dot\theta^p(\tau) - \dot\theta^p(t)| \leq \Delta\Omega,\ p = 1,2,\\ &|t-\tau| \leq \delta, \quad \forall\tau,t \in [0,T], \end{aligned} \tag{4}$$

where $\Delta\Omega$ is independent of δ. Conditions (2)–(4) mean that the functions $\dot\theta^p(\tau)$ are almost constant and the functions $f^p(\theta^p(\tau))$ are rapidly oscillating on small intervals $[t,t+\delta]$.

The boundedness of derivative of $\gamma(t)$ implies

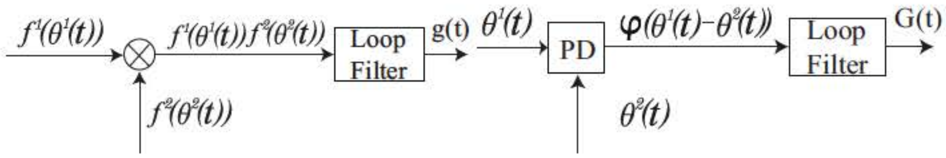
$$|\gamma(\tau) - \gamma(t)| = O(\delta), \quad |t-\tau| \leq \delta, \quad \forall\tau,t \in [0,T]. \tag{5}$$

---

[1] the functions with a finite number of jump discontinuity points differentiable on their continuity intervals

1.2 Phase detector characteristic computation for classical PLL

Consider two block diagrams shown in Fig. 2a and Fig.2b. Here, PD is a nonlinear



(a) Multiplier and filter (b) Phase detector and filter

Fig. 2: Phase detector models

block with characteristic $\varphi(\theta)$. The phases $\theta^p(t)$ are PD block inputs and the output is a function $\varphi(\theta^1(t)-\theta^2(t))$. The PD characteristic $\varphi(\theta)$ depends on waveforms of input signals.

The signal $f^1(\theta^1(t))f^2(\theta^2(t))$ and the function $\varphi(\theta^1(t)-\theta^2(t))$ are the inputs of the same filters with the same impulse response function $\gamma(t)$ and with the same initial state. The outputs of filters are the functions $g(t)$ and $G(t)$, respectively. By (1) one can obtain $g(t)$ and $G(t)$:

$$\begin{aligned} g(t) &= \alpha_0(t) + \int_0^t \gamma(t-\tau) f^1\big(\theta^1(\tau)\big) f^2\big(\theta^2(\tau)\big)\, d\tau, \\ G(t) &= \alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\big(\theta^1(\tau)-\theta^2(\tau)\big)\, d\tau. \end{aligned} \tag{6}$$

Using the approaches outlined in (Leonov, 2008; Kuznetsov et al, 2010, 2011b, 2012b), the following result can be proved.

Theorem 1. [Leonov et al (2012a, 2011); Kuznetsov et al (2012a)] *Let conditions* (2)–(5) *be satisfied and*

$$\varphi(\theta)=\frac{1}{2}\sum_{l=1}^{\infty}\Big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + (a_l^1 b_l^2 - b_l^1 a_l^2)\sin(l\theta)\Big). \tag{7}$$

Then the following relation

$$|G(t)-g(t)| = O(\delta), \quad \forall t \in [0,T]$$

is valid.

See Appendix for a proof of this theorem.

Broadly speaking, this theorem separates low-frequency error-correcting signal from parasite high-frequency oscillations. This theorem allows one to compute a phase detector characteristic for various typical waveforms of signals.

1.3 Description of classical Costas Loop

Nowadays BPSK and QPSK modulation techniques are used in telecommunication. For these techniques it is used different modifications of PLL with squarer and Costas Loop (Gardner, 1966; Lindsey and Simon, 1973; Best, 2007). However, the realization of some parts of PLL with squarer, used in analog circuits, can be quite difficult (Best, 2007). In the digital circuits, maximum data rate is limited by the speed of analog-to-digital converter (ADC) (Gardner, 1993; Gardner et al, 1993). Here, it will be considered analog Costas Loop, which are easy for implementation and effective for demodulation.

Various methods for analysis of Costas loop are well developed by engineers and considered in many publications (see, e.g., (Kroupa, 2003; Lindsey, 1972; Gardner, 1966)). However, the problems of construction of adequate nonlinear models and nonlinear analysis of such models are still far from being resolved. Further it will be considered only classical BPSK Costas loop, but similar analysis could be done for QPSK Costas Loop.

Consider physical model of classical Costas Loop (Fig. 3)

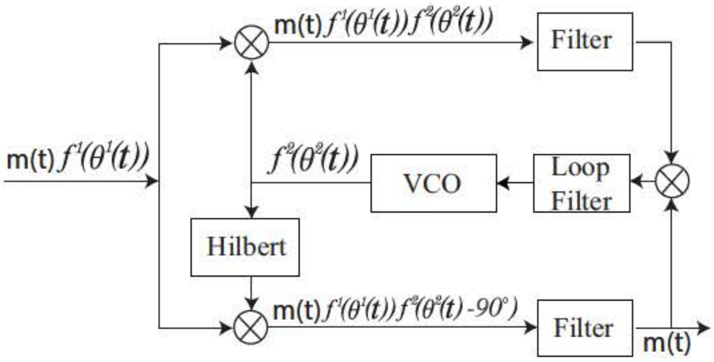


Fig. 3: Block diagram of Costas Loop at the level of electronic realization.

Here $f^1(t)$ is a carrier and $m(t) = \pm 1$ is data signal. Hilbert transform block shifts phase of input signal by $-\frac{\pi}{2}$.

In the simplest case when

$$\begin{aligned}
&f^1\left(\theta^1(t)\right) = \cos\left(\omega^1 t\right), f^2\left(\theta^2(t)\right) = \sin\left(\omega^2 t\right)\\
&m(t)f^1(\theta^1(t))f^2(\theta^2(t)) = \tfrac{m(t)}{2}\left(\sin(\omega^2 t - \omega^1 t) - \sin(\sin(\omega^2 t + \omega^1 t))\right) \qquad (8)\\
&m(t)f^1(\theta^1(t))f^2(\theta^2(t) - \tfrac{\pi}{2}) = \tfrac{m(t)}{2}\left(\cos(\omega^2 t - \omega^1 t) + \cos(\sin(\omega^2 t + \omega^1 t))\right)
\end{aligned}$$

standard engineering assumption is that a low pass filter removes the upper sideband with frequency from the input but leaves the lower sideband without change. Thus, after synchronization one gets demodulated data $m(t)\cos((\omega^1 - \omega^2)t) = m(t)\cos(0) = m(t)$ on the output of the lower filter (see Fig. 3).

Further, to avoid these assumption, a rigorous mathematical approach for the analysis of Costas loop will be demonstrated.

1.4 Computation of phase detector characteristic for Costas loop

From a theoretical point of view, since two arm filters in Fig. 3 are used for demodulation, for analysis of synchronization processes one can study Costas loop with only central Loop filter. Also since $m(t)^2 = 1$, the transmitted data $m(t)$ do not affect the operation of VCO. Thus one can consider the following equivalent block diagrams of Costas loop in signals/time and phase-frequency spaces (Fig. 4).

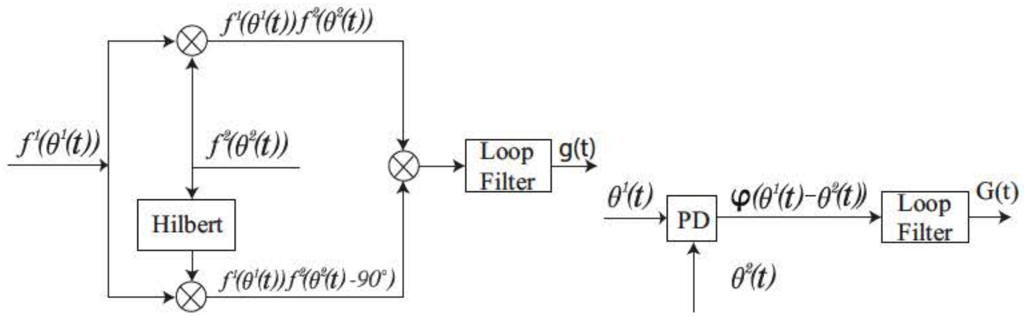


Fig. 4: Equivalent block diagrams of Costas loop in signals/time and phase-frequency spaces.

In both diagrams the filters are the same and have the same impulse transient function $\gamma(t)$ and the same initial data. The filters outputs are the functions $g(t)$ and $G(t)$, respectively.

Consider a case of non-sinusoidal piecewise-differentiable carrier oscillation $f^1(\theta^1(t))$ and tunable harmonic oscillation

$$
\begin{aligned}
f^1(\theta) &= \sum_{i=1}^{\infty} \left(a_i^1 \cos(i\theta) + b_i^1 \sin(i\theta)\right), \\
f^2(\theta) &= b_1^2 \sin(\theta).
\end{aligned}
\tag{9}
$$

The following assertion is valid.

Theorem 2. [Kuznetsov et al (2012b); Leonov et al (2012b)] *If conditions* (2)–(5) *are satisfied and*

$$\varphi(\theta) = \frac{(b_1^2)^2}{8}\Bigg[(a_1^1)^2 \sin(2\theta) + 2\sum_{q=1}^{\infty} a_q^1 a_{q+2}^1 \sin(2\theta) - \\ - 2a_1^1 b_1^1 \cos(2\theta) + 2\sum_{q=1}^{\infty} a_{q+2}^1 b_q^1 \cos(2\theta) - 2\sum_{q=1}^{\infty} a_q^1 b_{q+2}^1 \cos(2\theta) - \\ - (b_1^1)^2 \sin(2\theta) + 2\sum_{q=1}^{\infty} b_q^1 b_{q+2}^1 \sin(2\theta)\Big)\Bigg]. \qquad (10)$$

then the following relation

$$G(t) - g(t) = O(\delta), \quad \forall t \in [0,T] \qquad (11)$$

is valid.

In general, the proof of this result repeats the proof of Theorem 1. The details of the proof can be found in (Kuznetsov et al, 2012b; Leonov et al, 2012b). Note that this result could be easily extended to the case of two non-sinusoidal signals.

2 Engineering solutions for elimination of high-frequency oscillations

Consider engineering solution for elimination of high-frequency oscillations on the output of PD for harmonic signals. Further it is considered special analog PLL and analog Costas loop implementations, which allow one to effectively solve this problem.

2.1 Two-phase PLL

Consider a special modification of phase-locked loop (two phase PLL) suggested in (Emura, 1982).

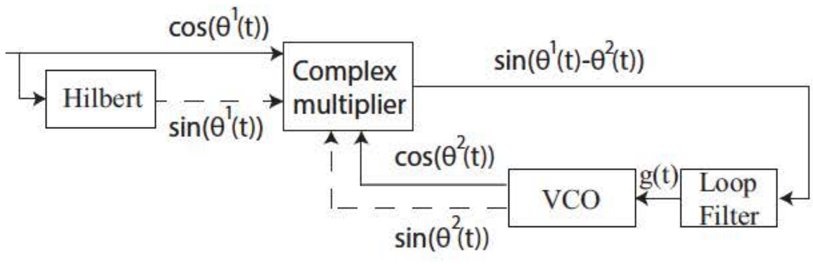


Fig. 5: Two-phase PLL

Here, a carrier is $\sin(\theta^1(t))$ with $\theta^1(t)$ as a phase and the output of Hilbert block is $\cos(\theta^1(t))$. VCO generates oscillations $\sin(\theta^2(t))$ and $\cos(\theta^2(t))$ with $\theta^2(t)$ as a phase. Figure 6 shows the structure of phase detector – complex multiplier.

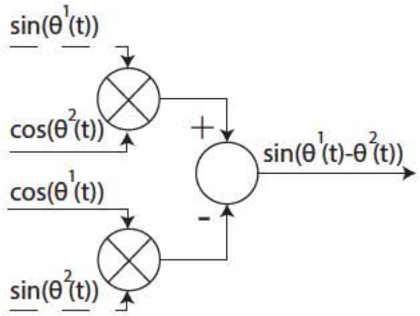


Fig. 6: Phase detector in two-phase PLL

The phase detector consists of two analog multipliers and analog subtractor. Here

$$\sin(\theta^1(t))\cos(\theta^2(t)) - \cos(\theta^1(t))\sin(\theta^2(t)) = \sin(\theta^1(t) - \theta^2(t))$$

In this case there is no high-frequency component at the output of phase detector. Thus block-scheme in Fig. 6 is equivalent to block-scheme in Fig. 2b, where phase detector characteristic is $\varphi(\theta) = \sin(\theta)$.

2.2 Two-phase Costas loop

Consider now engineering solution (Tretter, 2007) for the problem of elimination of high-frequency oscillations in Costas Loop (Fig. 7).

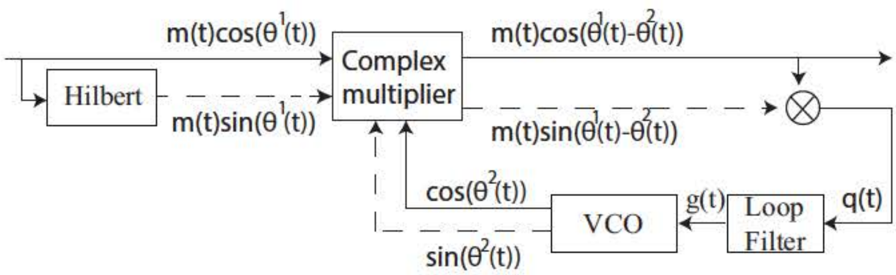


Fig. 7: Two-phase Costas loop

Here a carrier is $\cos(\theta^1(t))$ with $\theta^1(t)$ as a phase. VCO generates the oscillations $\cos(\theta^2(t))$ and $\sin(\theta^2(t))$ with $\theta^2(t)$ as a phase, and $m(t) = \pm 1$ is a relatively slow data signal (carrier period is several orders of magnitude smaller than time between data transitions). In Fig. 8 is shown a structure of phase detector.

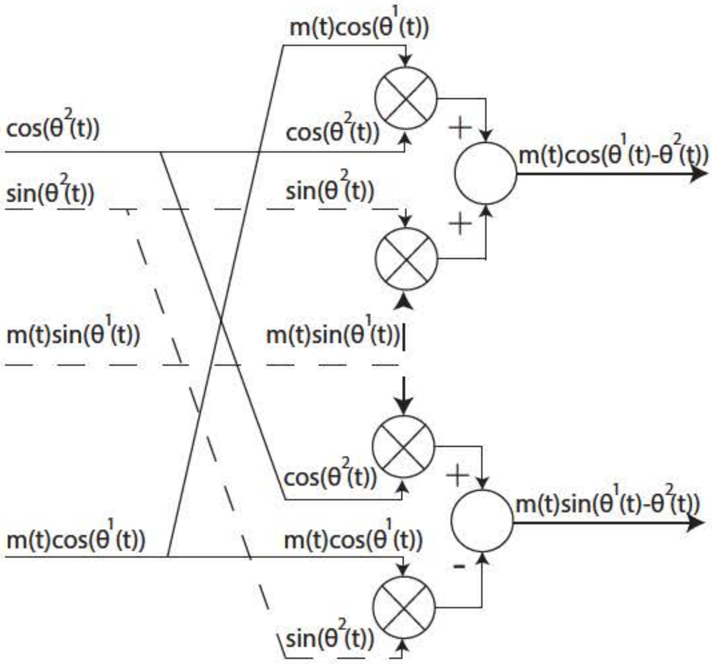


Fig. 8: Phase detector in two-phase Costas loop

Here the outputs of phase detector are the following

$$
\begin{aligned}
m(t)\left(\cos(\theta^1(t))\cos(\theta^2(t))+\sin(\theta^1(t))\sin(\theta^2(t))\right) &= m(t)\cos(\theta^1(t)-\theta^2(t))\\
m(t)\left(\sin(\theta^1(t))\cos(\theta^2(t))-\cos(\theta^1(t))\sin(\theta^2(t))\right) &= m(t)\sin(\theta^1(t)-\theta^2(t))
\end{aligned} \tag{12}
$$

If oscillators are synchronized (i.e. $\theta^1(t)=\theta^2(t)$), one of the outputs of phase detector contains only data signal $0.5m(t)$. Therefore, tacking into account $m(t)=\pm 1$, the input of the Loop filter takes the form

$$m(t)\cos(\theta^1(t)-\theta^2(t))m(t)\sin(\theta^1(t)-\theta^2(t))=\frac{1}{2}\sin(2(\theta^1(t)-\theta^2(t)))$$

and it depends only on phase difference of VCO and carrier. Thus block-scheme in Fig. 8 is equivalent to block-scheme in Fig. 2b, where phase detector characteristic is $\varphi(\theta)=\frac{1}{2}\sin(2(\theta))$.

3 Differential equation for PLL and Costas loop

Here differential equations for considered PLL-based circuits are derived.

From a mathematical point of view, a linear low-pass filter can be described by a system of linear differential equations

$$\dot{x} = Ax + p\xi(t),\ \sigma = c^* x, \tag{13}$$

a solution of which takes the form (1). Here, A is a constant matrix, $x(t)$ is a state vector of filter, b and c are constant vectors.

The model of tunable generator is usually assumed to be linear (Kroupa, 2003; Best, 2007):

$$\dot{\theta}^2(t) = \omega^2_{free} + LG(t),\ t \in [0, T]. \tag{14}$$

where ω^2_{free} is a free-running frequency of tunable generator and L is an oscillator gain. Here it is also possible to use nonlinear models of VCO; see e.g. (Suarez and Quere, 2003; Demir et al, 2000).

Suppose that the frequency of master generator is constant $\dot{\theta}^1(t) \equiv \omega^1$. Equation of tunable generator (14) and equation of filter (13) yield

$$\dot{x} = Ax + p\xi(t),\ \dot{\theta}^2 = \omega^2_{free} + Lc^* x. \tag{15}$$

For classical PLL circuit

$$\xi(t) = f^1(\theta^1(t)) f^2(\theta^2(t)), \tag{16}$$

for classical Costas loop

$$\xi(t) = f^1(\theta^1(t)) f^2(\theta^2(t) - \frac{\pi}{2}) f^1(\theta^1(t)) f^2(\theta^2(t)), \tag{17}$$

for two-phase PLL

$$\xi(t) = \sin(\theta^1(t) - \theta^2(t)), \tag{18}$$

and for two-phase Costas loop

$$\xi(t) = \frac{1}{2}\sin\left(2(\theta^1(t) - \theta^2(t))\right), \tag{19}$$

While for two phase PLL and Costas loop, system (15) is autonomous, for classical PLL and Costas loop, system (15) is nonautonomous and rather difficult for investigation (Kudrewicz and Wasowicz, 2007; Margaris, 2004). Here, Theorems 1 and 2 allow one to study more simple autonomous system of differential equations

$$\begin{aligned} &\dot{x} = Ax + p\varphi(\Delta\theta),\ \Delta\dot{\theta} = \omega^2_{free} - \omega^1 + Lc^* x, \\ &\Delta\theta = \theta^2 - \theta^1, \end{aligned} \tag{20}$$

where $\varphi(\theta)$ is the corresponding characteristic of phase detector. By well-known averaging method (Krylov and Bogolyubov, 1947) one to show that solutions of (15) and (20) are close under some assumptions. Thus, by Theorems 2 and 1, the block-schemes of PLL and Costas Loop in signals/time space (Fig. 1,3) can be asymptotically changed [for high-frequency generators, see conditions (2)–(4)] for the block-scheme in phase-frequency space (Fig. 9).

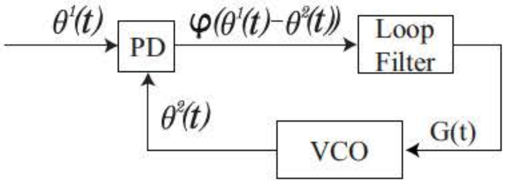


Fig. 9: Block scheme of phase-locked loop in phase-frequency space

Methods of nonlinear analysis for system (20) are well developed (see, e.g., (Leonov, 2006; Leonov et al, 2006; Kuznetsov et al, 2009a,b, 2008; Leonov et al, 2009)). The simulation approach for PLL analysis and design, based on the obtained analytical results, is discussed in (Kuznetsov et al, 2011a).

It should be noted that instead of conditions (3) and (5) for simulations of real system, it is necessary to consider the following conditions

$$|\Delta\omega| << \omega_{min}, \ |\lambda_A| << \omega_{min},$$

where λ_A is the largest (in modulus) eigenvalue of matrix A. Also, for correctness of transition from equation (21) to (25) it is necessary to consider $T << \omega_{min}$. It is easy to see that for sinusoidal waveforms operations of classical PLL and two-phase PLL are very similar because the phase detector characteristic and corresponding phase-frequency models are the same. Theoretical results are justified by simulation of classical PLL and two-phase PLL (Fig. 10).

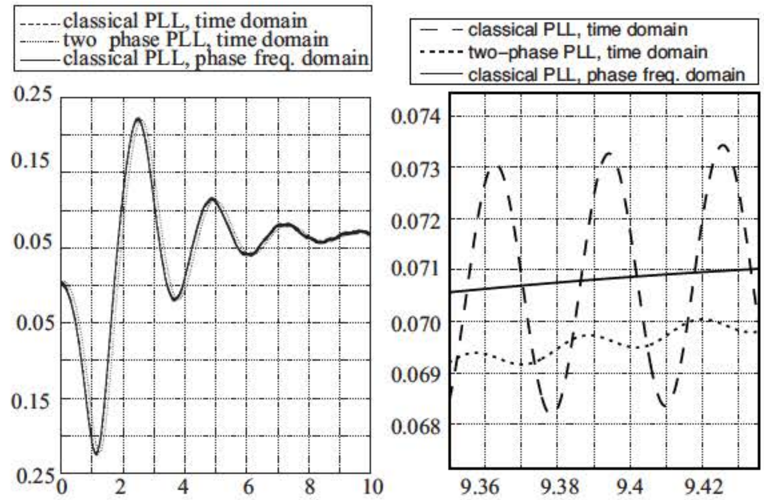


Fig. 10: Classical PLL in signals/time space; two-phase and classical PLLs in phase-frequency space, $\omega^2_{free} = 99$ Hz, $\omega^1 = 100$ Hz, $L = 15$, filter transfer functions $\frac{1}{s+1}$

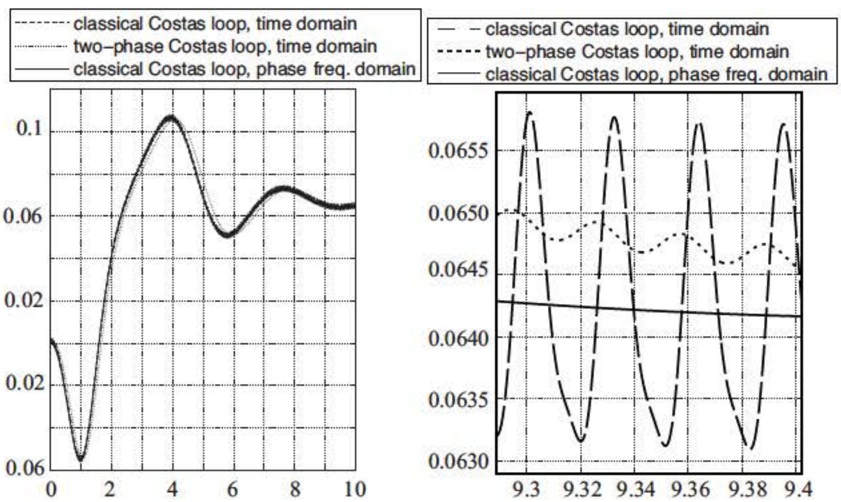


Fig. 11: Classical Costas Loop in signals/time space, two-phase and classical Costas Loops in phase-frequency space, $\omega^2_{free} = 99$ Hz, $\omega^1 = 100$ Hz, $L = 15$, filter transfer functions $\frac{1}{s+1}$

Unlike the filter output for the phase-frequency model of classical and two-phase PLLs, for signals/time space model of classical PLL the outputs of filter and phase detector contains additional high-frequency oscillations. These high-frequency oscillations interfere with qualitative analysis and efficient simulation of PLL. Filter output of two-phase PLL is delayed compared to the classical one because of non-ideality of Hilbert transform. Similar results can be obtained for Costas loop (see Fig.11).

Appendix

Proof. Suppose that $t \in [0,T]$. Consider a difference

$$g(t) - G(t) = \int_0^t \gamma(t-s)\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big) - \varphi\big(\theta^1(s)-\theta^2(s)\big)\Big]ds. \tag{21}$$

Suppose that there exists $m \in \mathbb{N}\cup\{0\}$ such that $t \in [m\delta,(m+1)\delta]$. By definition of δ, one has $m < \frac{T}{\delta}+1$. The continuity condition implies that $\gamma(t)$ is bounded on $[0,T]$ and $f^1(\theta), f^2(\theta)$ are bounded on $\mathbb{R}$. Since $f^{1,2}(\theta)$ are piecewise differentiable, one can obtain

$$a_i^p = O(\frac{1}{i}),\ b_i^p = O(\frac{1}{i}). \tag{22}$$

Hence $\varphi(\theta)$ converges uniformly and $\varphi(\theta)$ is continuous, piecewise differentiable, and bounded. Then the following estimates

$$\int_t^{(m+1)\delta} \gamma(t-s)f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big)ds = O(\delta),$$
$$\int_t^{(m+1)\delta} \gamma(t-s)\varphi\big(\theta^1(s)-\theta^2(s)\big)ds = O(\delta)$$

are satisfied. It follows that (21) can be represented as

$$g(t)-G(t) = \sum_{k=0}^{m}\int_{[k\delta,(k+1)\delta]}\gamma(t-s)$$
$$\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big) - \varphi\big(\theta^1(s)-\theta^2(s)\big)\Big]ds + O(\delta). \tag{23}$$

Prove now that on each interval $[k\delta,(k+1)\delta]$ the corresponding integrals are equal to $O(\delta^2)$.

Condition (5) implies that on each intervals $[k\delta,(k+1)\delta]$ the following relation

$$\gamma(t-s) = \gamma(t-k\delta)+O(\delta),\ t>s,\ s,t \in [k\delta,(k+1)\delta] \tag{24}$$

is valid. Here $O(\delta)$ is independent of k and the relation is satisfied uniformly with respect to t. By (23), (24), and the boundedness of $f^1(\theta), f^2(\theta)$, and $\varphi(\theta)$,

$$g(t)-G(t) = \sum_{k=0}^{m}\gamma(t-k\delta)\int\limits_{[k\delta,(k+1)\delta]}$$
$$\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big) - \varphi\big(\theta^1(s)-\theta^2(s)\big)\Big]ds + O(\delta). \tag{25}$$

Denote

$$\theta_k^p(s) = \theta^p(k\delta)+\dot{\theta}^p(k\delta)(s-k\delta),\ p=1,2.$$

Then for $s \in [k\delta, (k+1)\delta]$, condition (4) yields

$$\theta^p(s) = \theta^p_k(s) + O(\delta).$$

From (3) and the boundedness of derivative $\varphi(\theta)$ on $\mathbb{R}$ it follows that

$$\int\limits_{[k\delta,(k+1)\delta]} \big|\varphi\big(\theta^1(s) - \theta^2(s)\big) - \varphi\big(\theta^1_k(s) - \theta^2_k(s)\big)\big| ds = O(\delta^2). \tag{26}$$

If $f^1(\theta)$ and $f^2(\theta)$ are continuous on $\mathbb{R}$, then for $f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big)$ the following relation holds

$$\begin{aligned}&\int_{[k\delta,(k+1)\delta]} f^1(\theta^1(s))f^2(\theta^2(s))ds = \\ &= \int_{[k\delta,(k+1)\delta]} f^1(\theta^1_k(s))f^2(\theta^2_k(s))ds + O(\delta^2).\end{aligned} \tag{27}$$

Consider the validity of this estimate for the considered class of piecewise-differentiable waveforms. Since the conditions (2) and (4) are satisfied and the functions $\theta^{1,2}(s)$ are differentiable and satisfy (3), for all $k = 0, ..., m$ there exist sets E_k [the union of sufficiently small neighborhoods of discontinuity points of $f^{1,2}(t)$] such that the following relation $\int\limits_{E_k} ds = O(\delta^2)$ is valid, in which case the relation is satisfied uniformly with respect to k. Then from the piecewise differentiability and the boundedness of $f^{1,2}(\theta)$ it is possible to obtain (27) (see Corollary 1.

By (27) and (26), relation (25) can be rewritten as

$$\begin{aligned} g(t) - G(t) = &\sum_{k=0}^{m} \gamma(t - k\delta) \int\limits_{[k\delta,(k+1)\delta]} \\ &[f^1\big(\theta^1_k(s)\big)f^2\big(\theta^2_k(s)\big) - \varphi\big(\theta^1_k(s) - \theta^2_k(s)\big)]ds + O(\delta) \\ &= \sum_{k=0}^{m} \gamma(t - k\delta) \int\limits_{[k\delta,(k+1)\delta]} \\ &\Bigg[\Bigg(\sum_{i=1}^{\infty} a^1_i \cos\big(i\theta^1_k(s)\big) + b^1_i \sin\big(i\theta^1_k(s)\big)\Bigg) \\ &\times \Bigg(\sum_{j=1}^{\infty} a^2_j \cos\big(j\theta^2_k(s)\big) + b^2_j \sin\big(j\theta^2_k(s)\big)\Bigg) - \\ &- \varphi\big(\theta^1_k(s) - \theta^2_k(s)\big)\Bigg]ds + O(\delta)\ . \end{aligned} \tag{28}$$

Since conditions (2)–(4) are satisfied, it is possible to choose $O(\frac{1}{\delta})$ of sufficiently small time intervals of length $O(\delta^3)$, outside of which the functions $f^p(\theta^p(t))$ and $f^p(\theta^p_k(t))$ are continuous. It is known that on each interval, which has no discontinuity points, Fourier series of the functions $f^1(\theta)$ and $f^2(\theta)$ converge uniformly. Then there exists a number $M = M(\delta) > 0$ such that outside sufficiently small neighborhoods of discontinuity points of $f^p(\theta^p(t))$ and $f^p(\theta_k^p(t))$, the sum of the first M series terms approximates the original function with accuracy to $O(\delta)$. In this case by relation (28) and the boundedness of $f^1(\theta)$ and $f^2(\theta)$ on $\mathbb{R}$, it can be obtained

$$\begin{gathered} g(t) - G(t) = \sum_{k=0}^{m} \gamma(t - k\delta) \int\limits_{[k\delta,(k+1)\delta]} \sum_{i=1}^{M} \sum_{j=1}^{M} \\ \Big[\mu_{i,j}(s) - \varphi\big(\theta_k^1(s) - \theta_k^2(s)\big)\Big] ds + O(\delta), \end{gathered} \tag{29}$$

where

$$\begin{aligned} \mu_{i,j}(s) = \frac{1}{2}\Big(& (a_i^1 a_j^2 + b_i^1 b_j^2)\cos(i\theta^1 - j\theta^2) + \\ & + (-a_i^1 b_j^2 + b_i^1 a_j^2)\sin(i\theta^1 - j\theta^2) + \\ & + (-b_i^1 b_j^2 + a_i^1 a_j^2)\cos(i\theta^1 + j\theta^2) + \\ & + (a_i^1 b_j^2 + b_i^1 a_j^2)\sin(i\theta^1 + j\theta^2)\Big). \end{aligned}$$

From definition of δ and (22) it follows that $\forall i \in \mathbb{N}, j \in \mathbb{N}$ the relation

$$\int\limits_{[k\delta,(k+1)\delta]} \frac{1}{i} \cos\big(j(\omega_{min} s + \theta_0)\big) ds = \frac{O(\delta^2)}{ij} \tag{30}$$

is valid. Taking into account (30) and (2), one obtains the estimate

$$\int\limits_{[k\delta,(k+1)\delta]} b_j^p \cos\big(j\theta_k^p(s)\big) ds = \frac{O(\delta^2)}{j^2}.$$

Similar estimate is also valid for the addends with *sin*.

Consider the addend involving $\cos(i\theta_k^1(s) + j\theta_k^2(s))$ in $\mu_{i,j}(s)$. By (2) it can be obtained $i\dot\theta^1(k\delta) + j\dot\theta^2(k\delta) \geq (i+j)\omega_{min}$. Then (30) yields the following relation

$$
\begin{aligned}
&\int\limits_{[k\delta,(k+1)\delta]} \cos\Big(i\big(\theta^1(k\delta)+\dot\theta^1(k\delta)(s-k\delta)\big)+ \\
&+j\big(\theta^2(k\delta)+\dot\theta^2(k\delta)(s-k\delta)\big)\Big)ds= \\
&=\int\limits_{[k\delta,(k+1)\delta]} \cos\Big(\big(i\dot\theta^1(k\delta)+j\dot\theta^2(k\delta)\big)s \\
&-\big(i\theta^1(k\delta)+j\theta^2(k\delta)\big)k\delta \\
&+\big(i\dot\theta^1(k\delta)+j\dot\theta^2(k\delta)\big)k\delta\Big)ds=O\left(\frac{\delta^2}{i+j}\right).
\end{aligned} \tag{31}
$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{M}\sum_{j=1}^{M}\int\limits_{[k\delta,(k+1)\delta]}\frac{-b_i^1b_j^2+a_i^1a_j^2}{2}\cos\Big(i\big(\theta_k^1(s)\big)+j\big(\theta_k^2(s)\big)\Big)ds= \\
&=\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{O(\delta^2)}{ij(i+j)}\ .
\end{aligned}
$$

The convergence of series $\sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{\infty}\frac{1}{ij(i+j)}$ implies that the above expression is $O(\delta^2)$. Obviously, a similar relation occurs for the addend $\sin(i\theta_k^1(s)+j\theta_k^2(s))$.

Thus, by (29)

$$
\begin{aligned}
&g(t)-G(t)=\sum_{k=0}^{m}\gamma(t-k\delta)\int\limits_{[k\delta,(k+1)\delta]}\Bigg[\sum_{i=1}^{M}\sum_{j=1}^{M} \\
&\Bigg\{\frac{a_i^1a_j^2+b_i^1b_j^2}{2}\cos\Big(i\theta_k^1(s)-j\theta_k^2(s)\Big)+ \\
&+\frac{a_i^1b_j^2-b_i^1a_j^2}{2}\sin\Big(i\theta_k^1(s)-j\theta_k^2(s)\Big)\Bigg\}- \\
&-\varphi\big(\theta_k^1(s)-\theta_k^2(s)\big)\Bigg]ds+O(\delta)\ .
\end{aligned}
$$

Note that, here, the addends with indices $i=j$ give, in sum, $\varphi\big(\theta_k^1(s)-\theta_k^2(s)\big)$ with accuracy to $O(\delta)$. Consider the addends with indices $i<j$, involving cos (for the addends with indices $i>j$, involving sin, similar relations are satisfied). By (3), similar to (31), the following relation

$$
\begin{aligned}
&\sum_{i=2}^{M}\sum_{j=1}^{i-1}\frac{a_i^1a_j^2+b_i^1b_j^2}{2}\int\limits_{[k\delta,(k+1)\delta]}\cos\Big(i\big(\theta_k^1(s)\big)-j\big(\theta_k^2(s)\big)\Big)ds= \\
&=\sum_{i=2}^{M}\sum_{j=1}^{i-1}O(\delta^2)O\left(\frac{1}{ij|i-j|}\right)=O(\delta^2)
\end{aligned}
$$

is valid (see Lemma 2). ■

Let us now proceed to the proof of the used auxiliary lemmas.

One can show that it is possible to choose sufficiently small neighborhoods of points of discontinuity of $f(\theta(t))$, in which there are also points of discontinuity of $f(\theta_k(s))$ (see. Fig. 12, indicated neighborhoods are hatched). Then $f(\theta(s))$ can be approximated with the help of $f(\theta_k(s))$ (see. Corollary 1).

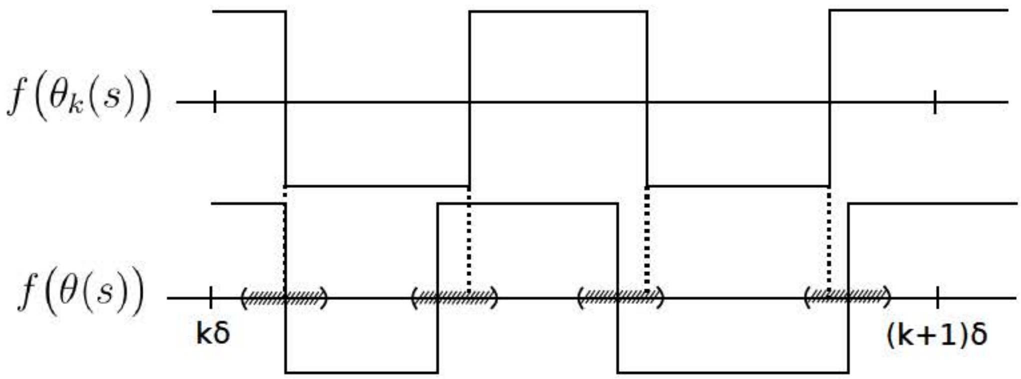


Fig. 12: Approximation of function $f(\theta(s))$ with the help of $f(\theta_k(s))$

Lemma 1. *Suppose, $f(\theta)$ is piecewise-differentiable 2π-periodic bounded function. $\theta(t)$ is a smooth function such that the conditions of high-frequency property (2)–(4) are satisfied. Then there exist sets $E_{\varepsilon,k}$ such that any ε-neighborhood of point of discontinuity of $f(\theta)$ acted by $\theta^{-1}(s)$ and $\theta_k^{-1}(s)$, attains the same interval, completely contained in $E_{\varepsilon,k}$, where*

$$\theta_k(s) = \theta(k\delta) + \dot{\theta}(k\delta)(s - k\delta), \tag{32}$$

in which case these sets are small:

$$\int\limits_{E_{\varepsilon,k}} ds = O(\delta^2). \tag{33}$$

Proof. By the data, $f(\theta)$ is bounded on $\mathbb{R}$. If $f(\theta)$ is continuous on $\mathbb{R}$, then the assertion of Lemma is obvious. Consider the case when $f(\theta)$ has at least 1 point of discontinuity.

Taking into account (4) and a smoothness of θ, it is possible to introduce the following notion

$$\omega_{min} \leq m_k = \min_{[k\delta,(k+1)\delta]} \dot{\theta}(s),$$
$$\omega_{min} \leq M_k = \max_{[k\delta,(k+1)\delta]} \dot{\theta}(s).$$

Then for $s \in [k\delta, (k+1)\delta]$ one obtains

$$\theta(k\delta) + m_k(s - k\delta) \le \theta(s) \le \theta(k\delta) + M_k(s - k\delta). \tag{34}$$

Then (32) implies

$$\begin{aligned} &\theta(s) \in [\theta(k\delta), \theta(k\delta) + M_k\delta],\\ &\theta_k(s) \in [\theta(k\delta), \theta(k\delta) + M_k\delta]. \end{aligned} \tag{35}$$

Suppose, $a_1, a_2, ...a_N$ are discontinuity points of $f(\theta)$ such that

$$a_j \in [\theta(k\delta), \theta(k\delta) + M_k\delta]. \tag{36}$$

Here there are altogether $O(\frac{1}{\delta})$ intervals of length δ on $[0,T]$, on each of which the increase of $\dot{\theta}(s)$ is less than $\Delta\Omega$. Thus, $\dot{\theta}(s) \le \omega_{min} + \Delta\Omega O(\frac{1}{\delta})$, i.e. $M_k = O(\frac{1}{\delta^2})$. If on interval $[0,2\pi]$ the function $f(\theta)$ has $N_{[0,2\pi]}$ discontinuities, then on interval of the length $M_k\delta$ there are $N = \frac{1}{2\pi}M_k\delta N_{[0,2\pi]}$ discontinuities. However $M_k\delta = O(\frac{1}{\delta^2})\delta = O(\frac{1}{\delta})$. Thus, $N = O(\frac{1}{\delta})$.

Consider ε-neighborhoods

$$V^j_{\varepsilon,k} = (a_j - \varepsilon, a_j + \varepsilon),\ 0 < \varepsilon < \delta.$$

The choice of such neighborhoods becomes clear in proving Lemma from the latter relations of (42).

Introduce the following notion

$$\left(\frac{a_j - \varepsilon - \theta(k\delta) + M_k k\delta}{M_k}, \frac{a_j + \varepsilon - \theta(k\delta) + m_k k\delta}{m_k}\right) = \widetilde{E}^j_{\varepsilon,k}\,, \tag{37}$$

$$E^j_{\varepsilon,k} = \widetilde{E}^j_{\varepsilon,k} \cap [k\delta, (k+1)\delta]. \tag{38}$$

In this case if $s \in [k\delta, (k+1)\delta] \setminus E^j_{\varepsilon,k}$, then $\theta(s), \theta_k(s)$ do not attain ε - neighborhoods of a_j, denoted by $V^j_{\varepsilon,k}$. Denote

$$E_{\varepsilon,k} = \bigcup_{j=1}^{N} E^j_{\varepsilon,k}. \tag{39}$$

This implies that the condition (48) is satisfied.

Further, for $E_{\varepsilon,k}$, it will be proved that property (33) is satisfied. The following estimation

$$\int\limits_{E^j_{\varepsilon,k}} ds \le \left(\frac{a_j + \varepsilon - \theta(k\delta) + m_k k\delta}{m_k} - \frac{a_j - \varepsilon - \theta(k\delta) + M_k k\delta}{M_k}\right) \tag{40}$$

is valid. Using (34), one obtains $|M_k - m_k| \le C$. Then

$$
\begin{aligned}
&\frac{a_j+\varepsilon-\theta(k\delta)+m_k k\delta}{m_k}-\frac{a_j-\varepsilon-\theta(k\delta)+M_k k\delta}{M_k}\leq\\
&\leq\frac{\big(a_j-\theta(k\delta)\big)\big(M_k-m_k\big)}{M_k m_k}+\varepsilon\left(\frac{1}{m_k}+\frac{1}{M_k}\right)+\left(\frac{m_k k\delta}{m_k}-\frac{M_k k\delta}{M_k}\right)\leq\\
&\leq\big(a_j-\theta(k\delta)\big)\frac{M_k-m_k}{M_k m_k}+\frac{2\varepsilon}{\omega_{min}}\leq\\
&\leq\big(a_j-\theta(k\delta)\big)\frac{C}{M_k m}+\frac{2\varepsilon}{\omega_{min}}\ .
\end{aligned}
\tag{41}
$$

By (36)

$$
\begin{aligned}
&\big(a_j-\theta(k\delta)\big)\frac{C}{M_k m_k}+\frac{2\varepsilon}{\omega_{min}}\leq\\
&\leq M_k\delta\frac{C}{M_k m_k}+\frac{2\varepsilon}{\omega_{min}}\leq\\
&\leq\frac{C\delta}{m_k}+\frac{2\varepsilon}{\omega_{min}}\leq\\
&\leq\frac{C\delta}{\omega_{min}}+\frac{2\varepsilon}{\omega_{min}}=\\
&=O(\delta^3)+\varepsilon O(\delta^2).
\end{aligned}
\tag{42}
$$

The relations (40), (41), and (42) imply

$$
\int\limits_{E^j_{\varepsilon,k}} ds = O(\delta^3). \tag{43}
$$

Taking into account that the number of points of discontinuity are equal to $N = O\left(\frac{1}{\delta}\right)$, one proves the assertion of Lemma 1:

$$
\int\limits_{E_{\varepsilon,k}} ds = \sum_{j=1}^{N}\int\limits_{E^j_{\varepsilon,k}} ds = O(\delta^2)\ . \tag{44}
$$

■

Corollary 1. *Suppose, $f(\theta)$ is a piecewise-differentiable 2π-periodic bounded function. $\theta(t)$ is a smooth function and the conditions of high-frequency property* (2)–(4) *are satisfied.*

Then

$$
\int\limits_{k\delta}^{(k+1)\delta} f(\theta(s))ds = \int\limits_{k\delta}^{(k+1)\delta} f(\theta_k(s)) + O(\delta^2), \tag{45}
$$

where

$$
\theta_k(s) = \theta(k\delta) + \dot\theta(k\delta)(s-k\delta). \tag{46}
$$

Proof. By the data, $f(\theta)$ is bounded on $\mathbb{R}$. If $f(\theta)$ is continuous on $\mathbb{R}$, then the assertion of Lemma is obvious. Consider the case when $f(\theta)$ has at least 1 point of discontinuity. Since the conditions of Lemma 1 are satisfied, there exist sets $E_{\varepsilon,k}$. Then the use of (33) and the boundedness of $f(\theta)$ gives

$$\begin{aligned}\int\limits_{[k\delta,(k+1)\delta]} f(\theta(s))ds &= \int\limits_{[k\delta,(k+1)\delta]\setminus E_{\varepsilon,k}} f(\theta(s))ds + O(\delta^2),\\ \int\limits_{[k\delta,(k+1)\delta]} f(\theta_k(s))ds &= \int\limits_{[k\delta,(k+1)\delta]\setminus E_{\varepsilon,k}} f(\theta_k(s))ds + O(\delta^2).\end{aligned} \tag{47}$$

In addition, according to assertion of Lemma 1, the functions $f(\theta)$ are differentiable with respect to θ and their derivatives are bounded for

$$\theta \in \{\theta(s) | s \in [k\delta,(k+1)\delta]\setminus E_{\varepsilon,k}\} \cup \{\theta_k(s) | s \in [k\delta,(k+1)\delta]\setminus E_{\varepsilon,k}\}, \tag{48}$$

i.e. on this set, $f(\theta)$ is Lipschitzian.

By (46) and (4)

$$\theta(s) = \theta(k\delta) + \int\limits_{k\delta}^{(k+1)\delta} \dot{\theta}(v)dv = \theta_k(s) + O(\delta). \tag{49}$$

Then (48) and (49) yeld

$$\begin{aligned}\int\limits_{[k\delta,(k+1)\delta]\setminus E_{\varepsilon,k}} \big|f\big(\theta(s)\big) - f\big(\theta_k(s)\big)\big|ds &= O(\delta^2),\\ \int\limits_{[k\delta,(k+1)\delta]\setminus E_{\varepsilon,k}} f\big(\theta(s)\big)ds &= \int\limits_{[k\delta,(k+1)\delta]\setminus E_{\varepsilon,k}} f\big(\theta_k(s)\big)ds + O(\delta^2),\end{aligned} \tag{50}$$

Then (47) implies the assertion of Lemma. ■

Lemma 2. *The following series* $\sum\limits_{i=1}^{\infty}\sum\limits_{j=1,j\neq i}^{\infty}\frac{1}{ij|i-j|}$ *converges.*

Proof. For Lemma to be proved, it is sufficient to prove convergence of the following series

$$\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{1}{ij(i-j)} \tag{51}$$

Consider $i = 9$

$$\frac{1}{9}\Big(\frac{1}{1(9-1)}+\frac{1}{2(9-2)}+\frac{1}{3(9-3)}+\frac{1}{4(9-4)}+\frac{1}{5(9-5)}+\frac{1}{6(9-6)}+\frac{1}{7(9-7)}+\frac{1}{8(9-8)}\Big)=$$
$$=\frac{1}{9}\Big(\frac{1}{1(9-1)}+\frac{1}{2(9-2)}+\frac{1}{3(9-3)}+\frac{1}{4(9-4)}+\frac{1}{(9-4)4}+\frac{1}{(9-3)3}+\frac{1}{(9-2)2}+\frac{1}{(9-1)1}\Big) \tag{52}$$

This implies that it is sufficient to prove convergence of the following series

$$\sum_{i=2}^{\infty}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor}\frac{1}{ij(i-j)}\,, \tag{53}$$

where $\lfloor x\rfloor = \max\limits_{d\in\mathbb{Z},\, d\le x} d$. Since

$$\frac{1}{iy(i-y)} \tag{54}$$

decreases for $y\in(0,\frac{i}{2}], i\geq 2$, one obtains that

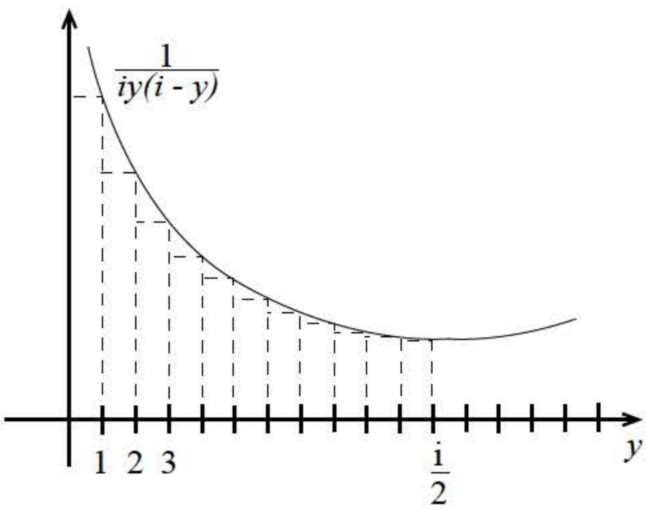


Fig. 13

$$\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor}\frac{1}{ij(i-j)}\leq\frac{1}{i(i-1)}+\int_{1}^{\frac{i}{2}}\frac{1}{iy(i-y)}dy,\ i\geq 2, \tag{55}$$

However

$$\int_1^{\frac{i}{2}} \frac{1}{iy(i-y)}dy =$$
$$= \frac{1}{i^2}\Big(\log(y) - \log(i-y)\Big)\Big|_1^{\frac{i}{2}} =$$
$$= \frac{1}{i^2}\Big(\log(\frac{i}{2}) - \log(i-\frac{i}{2}) - \log(1) + \log(i-1)\Big) =$$
$$= \frac{\log(i-1)}{i^2}\ . \tag{56}$$

It follows that a series

$$\sum_{i=2}^{\infty} \frac{\log(i-1)}{i^2} \tag{57}$$

converges. ■

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PII

NONLINEAR ANALYSIS OF PHASE-LOCKED LOOP WITH SQUARER

by

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Nonlinear analysis of phase-locked loop with squarer

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Abstract: The phase-locked loop with squarer is a classical phase-locked loop (PLL) based carrier recovery circuit. Simulation of the loop is nontrivial task, because of high-frequency properties of considered signals. Simulation in space of signals' phases allows one to overcome these difficulties, but it is required to compute phase detector characteristic. In this paper for various waveforms of high-frequency signals new classes of phase detector characteristics are computed for the first time. The problems of rigorous mathematical analysis of the control signal of voltage-controlled oscillator are discussed. Nonlinear model of PLL with squarer is constructed.

Keywords: Squaring circuits, Squarer, Nonlinear circuits, Circuit models, Nonlinear analysis, Phase-locked loop, PLL, Non-sinusoidal signal, Non-harmonic signal

1. INTRODUCTION

A PLL with squarer is a classical phase-locked loop (PLL) based circuit for carrier recovery [Bullock (2000); Best (2007); Hershey et al. (2002)]. Nowadays PLL with squarer is used in Global Positioning Systems (GPS) [Goradia et al. (1990); Kaplan and Hegarty (2006)], power systems control [Chang and Chen (2008); Sarkar and Sengupta (2010)] etc.

Although PLL is essentially a nonlinear control system, in modern engineering literature, devoted to the analysis of PLL-based circuits, the main direction is the use of simplified linear models, the methods of linear analysis, empirical rules, and numerical simulation (see a plenary lecture of D. Abramovich at American Control Conference 2002 [Abramovitch (2002)]). While linearization and analysis of linearized models of control systems may lead to incorrect conclusions [1] attempts to prove the reliability of conclusions, based on the application such simplified approaches, are quite rare (see, e.g., [Suarez and Quere (2003); Margaris (2004); Feely (2007); Banerjee and Sarkar (2008); Feely et al. (2012); Suarez et al. (2012)]). Rigorous nonlinear analysis of PLL-based circuit models is often very difficult task, so for analysis of nonlinear PLL models numerical simulation is widely used. However for high-frequency signals, complete numerical simulation of *physical model of PLL-based circuit in space of signals*, described by nonlinear non-autonomous system of differential equations, is highly complicated [Abramovitch (2008a,b)] since it is necessary to simultaneously observe "*very fast time scale of the input signals*" and "*slow time scale of signal's phases*". Here a relatively small discretization step in numerical procedure does not allow one to consider transition processes for high-frequency signals in a reasonable time.

To overcome these difficulties, it is possible to construct *mathematical model of PLL-based circuit in space of signals' phases* described by nonlinear dynamical system of differential equations. In this case it is investigated only slow time scale of signal's phases and frequencies. That, in turn, requires [Leonov et al. (2012b)] the computation of phase detector characteristic (nonlinear element which is used to match reference and controllable signals), which depends on waveforms of considered signals [Kuznetsov et al. (2011c,b, 2010)]. However, the use of results of analysis of this mathematical model for the conclusions, concerning the behavior of the considered physical model, requires rigorous justification [Leonov et al. (2011b, 2012b)].

All in all it should be remarked here that the requirement of convergence in regarding to nonlinear non-autonomous control systems with input [van den Berg et al. (2006)] for certainty of limit solution computation, and also the discovery of hidden oscillations (which cannot be found by standard simulation) [Leonov et al. (2011a, 2012a)] in nonlinear dynamical models of PLL [Leonov and Kuznetsov (2013)] make unreliable the use of simple modeling and show the importance of development and application of analytical methods for analysis of nonlinear models of PLL-based circuit.

In this paper effective approaches to rigorous nonlinear analysis of classical analog PLL with squarer are discussed. An effective analytical method for computation of

[1] see, e.g., counterexamples to filter hypothesis, to Aizerman's and Kalman's conjectures on absolute stability, [Leonov and Kuznetsov (2011a,b); Kuznetsov et al. (2011a); Bragin et al. (2011); Leonov and Kuznetsov (2013)] and Perron effects of Lyapunov exponents sign inversions for time-varying linearizations [Kuznetsov and Leonov (2005); Leonov and Kuznetsov (2007)]),

multiplier/mixer phase-detector characteristics is demonstrated. For various non-sinusoidal waveforms of high-frequency signals, new classes of phase-detector characteristics are obtained, and dynamical model of PLL with squarer is constructed.

2. DESCRIPTION OF PLL WITH SQUARER IN SPACE OF SIGNALS

Consider classical PLL with squarer at the level of electronic realization (Fig. 1)

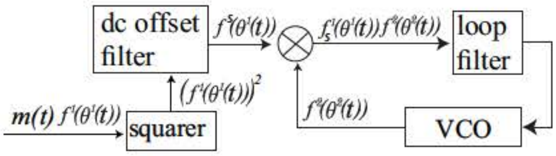


Fig. 1. Block diagram of PLL with squarer at the level of electronic realization.

Here $m(t)f^1(\theta^1(t))$ is an input signal, where $m(t) = \pm 1$ is transmitted data and $f^1(t) = f^1(\theta^1(t))$ is carrier oscillation. VCO is a voltage-controlled oscillator, which generates oscillations $f^2(t) = f^2(\theta^2(t))$. Here $\theta^{1,2}(t)$ are phases. Squarer block multiplies input signal by itself and the following filter removes dc offset [Best (2007)].

The block $\otimes$ is a multiplier (used as phase-detector (PD)) of oscillations $f^s(t)$ and $f^2(t)$, the signal $f^s(\theta^1(t))f^2(\theta(t))$ is its output. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter is as follows:

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,\mathrm{d}\tau, \tag{1}$$

where $\gamma(t)$ is an impulse response function of filter and $\alpha_0(t)$ is an exponentially damped function depending on the initial data of filter at moment $t = 0$. By assumption, $\gamma(t)$ is a differentiable function with bounded derivative (this is true for the most considered filters Best (2007)). The operation of circuit in Fig. 1 without data $m(t)$ corresponds to the work of classical PLL.

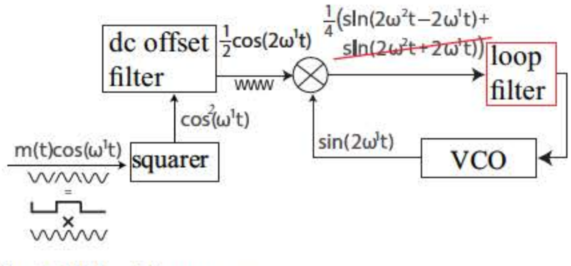


Fig. 2. PLL with squarer

In the simplest case of input signal with harmonic carrier (see Fig. 2)

$$m(t)f^1(\theta^1(t)) = m(t)\cos(\omega^1 t),\ m(t) = \pm 1.$$

The output of the squarer block after filtration dc offset is equal to

$$f^s(\theta^1(t)) = \frac{m^2(t)}{2}\cos(2\omega^1 t) = \frac{1}{2}\cos(2\omega^1 t).$$

To obtain synchronization for harmonic signals, the output of VCO should also have doubled frequency [Best (2007)] (this is true for all signal waveforms for which the squaring results in a doubling of frequency, e.g., sinusoidal and triangular waveforms; for the squared sawtooth signal its frequency does not change, see Fig. 3 and Fig. 4):

$$f^2(\theta^2(t)) = \sin(2\omega^2 t).$$

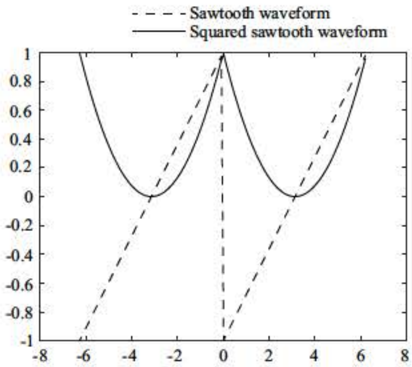


Fig. 3. Sawtooth waveform, frequency of the signal is not doubled after squaring

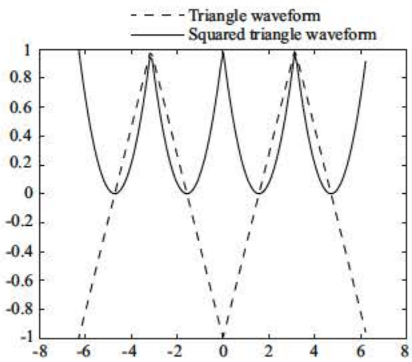


Fig. 4. Triangle waveform, frequency of the signal is doubled after squaring

Thus the output of multiplier becomes

$$f_s^1(\theta^1(t))f^2(\theta^2(t)) = \frac{1}{2}\cos(2\omega^1 t)\sin(2\omega^2 t) = \frac{1}{4}\big(\sin(2(\omega^2-\omega^1)t) + \sin(2(\omega^2+\omega^1)t)\big).$$

Standard engineering assumption is that the filter removes the upper sideband with frequency from the input but leaves the lower sideband without change. Thus it is assumed that VCO input is $\frac{1}{4}\sin(2(\omega^2-\omega^1)t)$.

This result was known to engineers (see, e.g., [Gardner (1966); Best (2007)]) but had no rigorous mathematical justification. Here this result is generalized and rigorously proved for the general case of signal waveforms. Nowadays PLL with squarer with non-sinusoidal waveform is used, for example, in power systems (see, e.g., [Sarkar and Sengupta (2010); Chang and Chen (2008)]).

3. COMPUTATION OF PHASE DETECTOR CHARACTERISTIC

Suppose that the waveforms $f^{1,2}(\theta)$ are the bounded 2π-periodic piecewise differentiable functions(what is true for the most considered waveforms). Consider Fourier series representation of such functions

$$f^p(\theta) = \sum_{i=1}^{\infty} \left(a_i^p \sin(i\theta) + b_i^p \cos(i\theta)\right),\ p = 1, 2,$$
$$a_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\sin(i\theta)d\theta, b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\cos(i\theta)d\theta.$$

A high-frequency property of signals can be reformulated in the following way. By assumption, the phases $\theta^p(t)$ are smooth functions (this means that frequencies are changed continuously, what corresponds to classical PLL analysis [Best (2007); Kroupa (2003)]). Suppose also that there exists a sufficiently large number ω_{min} such that on fixed time interval $[0, T]$ the following conditions

$$\dot\theta^p(\tau) \geq \omega_{min} > 0,\ \ p = 1, 2 \tag{2}$$

are satisfied. Here T is independent of ω_{min} and $\dot\theta^p(t)$ denotes frequencies of signals. The frequencies difference is assumed to be uniformly bounded

$$\left|\dot\theta^1(\tau) - \dot\theta^2(\tau)\right| \leq \Delta\omega,\ \forall\tau \in [0, T]. \tag{3}$$

Requirements (2) and (3) are obviously satisfied for the tuning of two high-frequency oscillators with close frequencies [Best (2007); Kroupa (2003)]. Let us introduce $\delta = \omega_{min}^{-\frac{1}{2}}$ Consider relations

$$\begin{aligned} &|\dot\theta^p(\tau) - \dot\theta^p(t)| \leq \Delta\Omega,\ p = 1, 2, \\ &|t - \tau| \leq \delta, \quad \forall\tau, t \in [0, T], \end{aligned} \tag{4}$$

where $\Delta\Omega$ is independent of δ and t. Conditions (2)–(4) mean that the functions $\dot\theta^p(\tau)$ are almost constant and the functions $f^p(\theta^p(\tau))$ are rapidly oscillating on small intervals $[t, t+\delta]$.

The boundedness of derivative of $\gamma(t)$ implies

$$|\gamma(\tau) - \gamma(t)| = O(\delta), \quad |t - \tau| \leq \delta, \quad \forall\tau, t \in [0, T]. \tag{5}$$

Consider two block diagrams shown in Fig. 5 and Fig. 6.

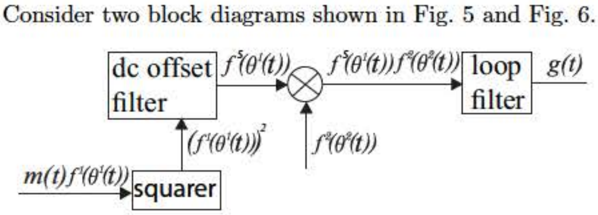


Fig. 5. Physical model of PLL with squarer in space of signals.

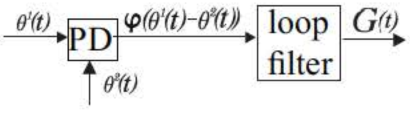


Fig. 6. Model in space of signals' phases.

In Fig. 6 $\theta^{1,2}(t)$ are shown the phases of oscillations $f^{1,2}(\theta^{1,2}(t))$. Here phase detector is nonlinear block with characteristic $\varphi(\theta)$. The phases $\theta^{1,2}(t)$ are the inputs of PD block and the function $\varphi(\theta^1(t) - \theta^2(t))$ is the output. The waveform of phase detector characteristic depends on the waveforms of input signals.

In both diagrams the loop filters are the same. They have the same impulse transient function $\gamma(t)$ and the same initial conditions. The slave oscillator inputs are the functions $g(t)$ and $G(t)$, respectively.

The signal $f^s(\theta^1(t))f^2(\theta^2(t))$ and the function $\varphi(\theta^1(t) - \theta^2(t))$ are the inputs of the same loop filters with the same impulse response function $\gamma(t)$ and with the same initial state. The outputs of loop filters are the functions $g(t)$ and $G(t)$, respectively. By (1), one can obtain $g(t)$ and $G(t)$:

$$\begin{aligned} g(t) &= \alpha_0(t) + \int_0^t \gamma(t-\tau) f^s\big(\theta^1(\tau)\big) f^2\big(\theta^2(\tau)\big) d\tau, \\ G(t) &= \alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi(\theta^1(\tau) - \theta^2(\tau)) d\tau. \end{aligned} \tag{6}$$

Then, by the approaches, outlined in [Leonov (2008)] and [Kuznetsov et al. (2011c,b, 2010)], the following result can be rigorously mathematically proved.

Based on the (2), for convenience, we introduce the following notation

$$a_{-h}^1 = a_h^1, \quad b_{-h}^1 = -b_h^1,\ h < 0,\ h \in \mathbb{N}. \tag{7}$$

Theorem. *Let conditions* (2)–(5) *be satisfied and*

$$\begin{aligned} &\varphi(\theta) = \frac{1}{2}\sum_{l=1}^{\infty}\Big((A_l^1 a_l^2 + B_l^1 b_l^2)\cos(l\theta) + \\ &(A_l^1 b_l^2 - B_l^1 a_l^2)\sin(l\theta)\Big), \\ &A_l^1 = \frac{1}{2}\sum_{m=1}^{\infty}\left(a_m^1(a_{m+l}^1 + a_{m-l}^1) + b_m^1(b_{m+k} + b_{m-k}^1)\right) \\ &B_l^1 = \frac{1}{2}\sum_{m=1}^{\infty}\left(a_m^1(b_{m+l}^1 - b_{m-l}^1) - b_m^1(a_{m+k} + a_{m-k}^1)\right) \end{aligned} \tag{8}$$

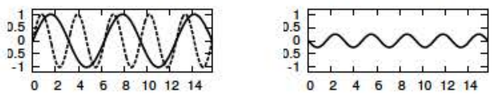
Then the following relation

$$|G(t) - g(t)| = O(\delta), \quad \forall t \in [0, T] \tag{9}$$

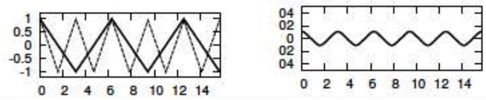
is valid.

3.1 Phase detector characteristics examples

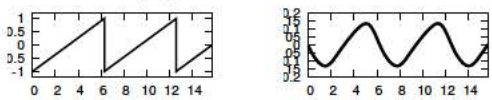
$f^1(\theta) = \sin(\theta)$, $f^2(\theta) = \sin(2\theta)$, $\varphi(\theta) = -\frac{1}{4}\sin(2\theta)$



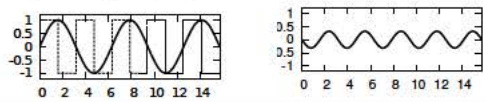
$$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right),$$
$$f^2(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((4n-2)\theta\right),$$
$$\varphi(\theta) = \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{4}{(4n-2)^4}\cos\left((4n-2)\theta\right)$$



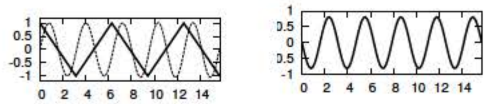
$$f^{1,2}(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(n\theta\right),$$
$$\varphi(\theta) = -\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\left(n\theta\right)$$



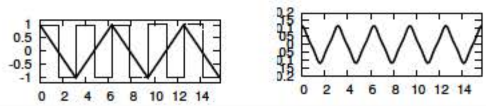
$f^1(\theta) = \sin(\theta)$, $f^2(\theta) = \operatorname{sgn}\sin(2\theta)$, $\varphi(\theta) = -\frac{1}{\pi}\sin(2\theta)$



$$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right),$$
$f^2(\theta) = \sin(2\theta)$, $\varphi(\theta) = -\frac{8}{\pi^2}\sin(2\theta)$



$$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left((2n-1)\theta\right),$$
$f^2(\theta) = \operatorname{sgn}\sin(2\theta)$, $\varphi(\theta) = \frac{32}{\pi^5}\sum_{n=1}^{\infty}\frac{\cos(2(2n-1)\theta)}{(2n-1)^3}$



4. EQUATIONS OF PLL WITH SQUARER IN SPACE OF SIGNALS' PHASES

From the obtained results it follows that block-scheme (Fig. 1) of PLL with squarer in space of signals (for high-frequency generators) can be asymptotically changed to block-scheme in space of signals' phases (Fig. 7).

Here PD is a phase detector with the corresponding characteristics $\varphi(\theta)$. One should make a remark concerning the derivation of differential equations of PLL with squarer.

Note that, by assumption, the control law of tunable oscillators is linear:

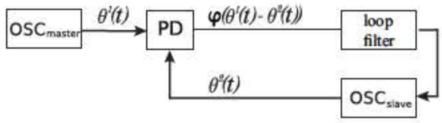


Fig. 7. Phase-locked loop with phase detector

$$\dot{\theta}^2(t) = \omega^2 + LG(t). \qquad (10)$$

Here ω^2 is a free-running frequency of tunable oscillator, L is a certain number, and $G(t)$ is a control signal, which is a loop filter output (Fig. 6). Thus, the equation of PLL with squarer is as follows

$$\dot{\theta}^2(t) = \omega^2 + \\ +L\Big(\alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\big(\theta^1(\tau) - \theta^2(\tau)\big)\mathrm{d}\tau\Big). \qquad (11)$$

Assuming that input signal carrier is such that $\dot{\theta}^1(t) \equiv \omega^1$, one can obtain the following equation for PLL with squarer

$$\big(\theta^1(t)-\theta^2(t)\big)' + \\ +L\Big(\alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\big(\theta^1(\tau)-\theta^2(\tau)\big)\mathrm{d}\tau\Big) = \\ = \omega^1 - \omega^2, \qquad (12)$$

which is an equation of classical PLL.

The analysis of PLL with squarer models is based on the theory of phase synchronization. Modification of direct Lyapunov method with the construction of periodic Lyapunov-like functions, the method of positively invariant cone grids, and the method of nonlocal reduction turned out to be most effective [Leonov et al. (1996); Kudrewicz and Wasowicz (2007); Leonov et al. (2009)].

CONCLUSION

In this paper an effective approach, based on construction of mathematical model in space of signals' phases, to investigation of phase-locked loop with squarer is discussed. Mathematical model of PLL with squarer for non-sinusoidal signals is constructed and new classes of phase detector characteristics are computed for the first time. Methods for rigorous mathematical analysis are discussed.

ACKNOWLEDGEMENTS

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PIII

SIMULATION OF PHASE-LOCKED LOOPS IN PHASE-FREQUENCY DOMAIN

by

N.V. Kuznetsov, G.A. Leonov, P. Neittaanmaki, S.M. Seledzhi, M.V. Yuldashev, R.V. Yuldashev 2012 [Scopus]

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Simulation of Phase-Locked Loops in Phase-Frequency Domain

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***Abstract*— This article is devoted to simulation of classical phase-locked loop (PLL). Based on new analytical method for computation of phase detector characteristics (PD), an realization in Simulink for simulation of classical PLL in phase space for general types of signal waveforms is done. This enables to avoid a number of numerical problems in the simulation of PLL in signal.**

I. INTRODUCTION

Various methods for theoretical and numerical analysis of phase-locked loops are well developed by engineers and considered in many publications (see, e.g., [1]–[4]), but the problems of construction of adequate nonlinear models and nonlinear analysis of such models are still far from being resolved.

As noted by D. Abramovitch in his keynote talk at American Control Conference [5], the main tendency in a modern literature on analysis of stability and design of PLL [6]–[9] is the use of simplified linearized models, the application of the methods of linear analysis, a rule of thumb, and simulation. However it is known that the application of linearization methods and linear analysis for control systems can lead to untrue results and, therefore, requires special justifications. In 50-60s of last century the investigations of widely known Markus-Yamabe's, Aizerman's conjecture (Aizerman problem) and Kalman's conjecture (Kalman problem) on absolute stability have led to the finding of hidden oscillations (which can not be found by the above methods) in automatic control systems with scalar piecewise-linear nonlinearity, which belongs to the sector of linear stability (see, e.g., [10]–[13] and others).

Standard numerical analysis also often can not reveal nontrivial regimes. In 1961, Gubar' [14] showed analytically the possibility of existence of hidden oscillation in two-dimensional system of phase locked-loop with piecewise-constant impulse nonlinearity. In the system considered from computational point of view all trajectories tend to equilibria, but, in fact, there is a bounded domain of attraction only. Recently the chaotic hidden oscillations (hidden attractors — a basin of attraction of which does not contain neighborhoods of equilibria) were discovered [15]–[19], in electronic Chua's circuit. Hidden attractors can not be found by standard computational procedure (in which after transient process a trajectory, started from a point in a neighborhood of equilibrium, reaches an oscillation and identifies it). Also investigation of bifurcations in phase locked-loop requires development and application of special numerical procedures [20]–[22].

Simulation of PLL can be performed in signal/time domain or in phase-frequency domain. Full simulation of PLL in signal/time domain is rare because of nonlinearities of phase detectors and high frequencies of considered signals. According to D. Abramovitch [23], [24]:"*Generally, a simulation step size, which is small enough to clearly observe the dynamics of the phase detector, makes it difficult to observe the dynamics of the entire loop*."

There is another approach, which allows one to simulate PLL systems in phase-frequency domain, consider only slow time scale and significantly reduces simulation time. But such consideration requires justification. Here for constructing an adequate nonlinear mathematical model of PLL in phase-frequency domain it is necessary to find a characteristic of phase detector. This characteristic depends on the realization of phase detector and the types of signals at the input. The characteristics of phase detector for harmonic and square wave signals are well known to engineers [1], [4], [5]. However, in a number of applications a non-sinusoidal signal must be considered [25]–[28]. Below, following [29]–[32], an effective approach for simulation of classical PLL in phase-frequency domain for general class of signals is considered.

II. MATHEMATICAL EXPLANATION OF SIMULATION IN PHASE-FREQUENCY DOMAIN

As mentioned above, PLL simulation in phase-frequency domain is highly efficient but strongly depends on signal waveforms and requires some mathematical justification.

In this section the required clarification, providing phase-detector characteristics for wide class of signal waveforms, is made.

Consider a passage of product of high-frequency oscillations through linear filter (Fig. 1).

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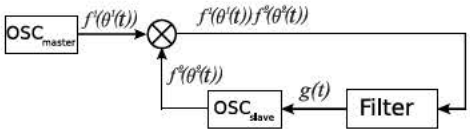

Fig. 1. PLL in signal space

The signals $f^1(\theta^1(t))$ and $f^2(\theta^2(t))$ are high-frequency oscillations [30], [33], [34] (the signals of master and slave generators, respectively); block $\otimes$ denotes an analog multiplier. The output of multiplier is connected with the Low Pass Filter (LPF, Loop Filter). The output of filter $g(t)$ is used as a control signal for slave oscillator (VCO) frequency.

Consider a Fourier series segment of signal waveforms $f^{1,2}(\theta)$:

$$f^p(\theta) = \sum_{i=1}^{M} \left(a_i^p \cos(i\theta) + b_i^p \sin(i\theta)\right), \quad p = 1, 2. \tag{1}$$

Assume that for phases of signals $\theta^p(t)$ the following condition

$$\dot\theta^p(t) \geq \omega_{min}, \quad p = \{1, 2\} \tag{2}$$

is valid, where ω_{min} is sufficiently large. Here ω_{min} is the low boundary of frequency range of signals. Following [29], [35], the number M should be chosen such that

$$\sum_{i=M}^{\infty} \left(a_i^p \cos(i\theta) + b_i^p \sin(i\theta)\right) \leq \frac{1}{\sqrt{\omega_{min}}}. \tag{3}$$

For smooth functions the estimation $M \geq \sqrt{\omega_{min}}$ guaranties this condition.

The frequency difference is assumed to be uniformly bounded

$$\left|\dot\theta^1(\tau) - \dot\theta^2(\tau)\right| \leq \Delta\omega, \ \forall\tau \in [0, T]. \tag{4}$$

Here T is independent of ω_{min}, $\dot\theta^p(t)$ are signal frequencies.

Introduce the following notation

$$\delta = \frac{1}{\sqrt{\omega_{min}}}. \tag{5}$$

Consider the relations

$$\begin{aligned} &|\dot\theta^p(\tau) - \dot\theta^p(t)| \leq \Delta\Omega, \ p = 1, 2, \\ &|t - \tau| \leq \delta, \quad \forall\tau, t \in [0, T], \end{aligned} \tag{6}$$

where $\Delta\Omega$ is independent of δ and t.

Consider now a block-scheme in Fig. 2. Here PD is a

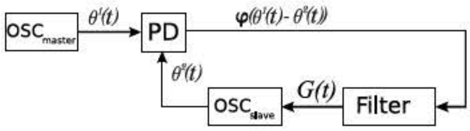

Fig. 2. PLL in phase-frequency domain

nonlinear block with the output $\varphi(\theta^1(t) - \theta^2(t))$, where

$$\varphi(\theta) = \frac{1}{2}\sum_{l=1}^{\infty}\Big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + \\ + (b_l^1 a_l^2 - a_l^1 b_l^2)\sin(l\theta)\Big).$$

The characteristics and initial states of filters, shown in Fig. 1 and Fig. 2, coincide. Then one can mathematically prove [29]–[31] that under conditions (1)–(6)

$$G(t) - g(t) \approx 0. \tag{7}$$

For general class of signals, relation (7) permits one to proceed simulation of PLL in phase-frequency domain.

III. Simulation Model

Let us demonstrate how the formulated above result allows one to improve simulation of PLL. For this purpose we consider several models of PLL: in phase-frequency domain, in signal/time domain, and linear model. Here it is assumed that the output signals of VCO and master oscillators have sawtooth and triangular waveforms (other waveforms can be analyzed in a similar manner).

Consider Matlab Simulink model of PLL in the signal/time domain shown in Fig. 3. Here we have standard Simulink

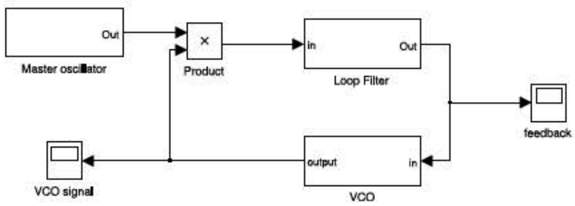

Fig. 3. Matlab Simulink model of PLL in signal/time domain

blocks: Product as a phase detector, feedback and VCO signal scopes for collecting results of simulation, master oscillator (Fig. 4), and VCO and Loop Filter subsystems.

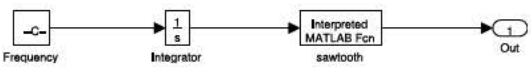

Fig. 4. Master oscillator design

Loop filter design is shown in Fig. 5. It consists of a block with $\frac{1}{s+1}$ transfer function. VCO block has straight forward

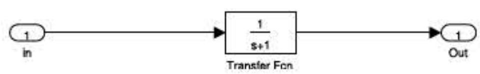

Fig. 5. Loop Filter block

structure (Fig. 6). Interpreted MATLAB Fcn block represents an output signal waveform. In the considered case it is equal to $-\text{sawtooth}(u, 0.5)$ ($-\text{sawtooth}(u)$ for master oscillator) function.

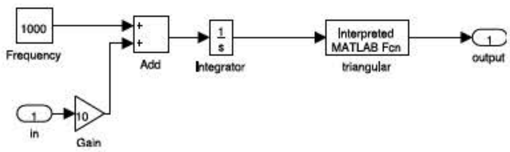

Fig. 6. VCO

In phase-frequency domain we have the same filter and scopes, the oscillator outputs are equal to phases of signals (Fig. 7).

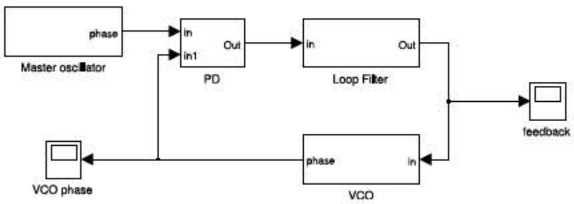

Fig. 7. Phase-frequency domain model

For sawtooth signal the Fourier coefficients take the form

$$f^1(\theta) = \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin(i\theta), \qquad a_i^1 = 0, \quad b_i^1 = \frac{2}{i\pi}. \tag{8}$$

Triangular signal has the following coefficients

$$f^2(\theta) = \frac{8}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{(2i-1)^2}\cos((2i-1)\theta), \qquad a_{2i-1}^2 = \frac{8}{(2i-1)^2\pi^2}, \qquad a_{2i}^2 = 0, b_i^2 = 0, \quad i \in \mathbb{N}. \tag{9}$$

PD block realizes the phase detector characteristic function $\varphi(\theta^1 - \theta^2)$ (Fig. 8). Corresponding MATLAB Function code

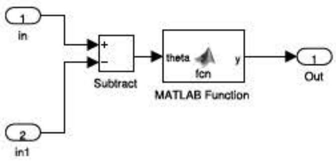

Fig. 8. PD Simulink block

is shown in Fig. 9.

```
function y = fcn(u)
%#codegen

y = 0;
for i=1:10
    y = y + sin((2*i-1)*u)/(2*i-1)/(2*i-1)/(2*i-1);
end
y = 8*y/pi/pi/pi;
```

Fig. 9. MATLAB code

Consider now linear model of PLL. It differs from signal/time domain model in PD design, which is shown in Fig. 10. By (7), phase detector characteristic is equal to

$$\varphi(\theta^1 - \theta^2) = \frac{8}{\pi^3}\sum_{l=1}^{M}\frac{1}{(2l-1)^3}\sin\left((2l-1)(\theta^1-\theta^2)\right). \tag{10}$$

Then the linearized PD characteristics are equal to

$$\varphi(\theta^1 - \theta^2) \approx (\theta^1 - \theta^2)\frac{8}{\pi^3}\sum_{l=1}^{\infty}\frac{1}{(2l-1)^2} = \frac{1}{\pi}(\theta^1 - \theta^2). \tag{11}$$

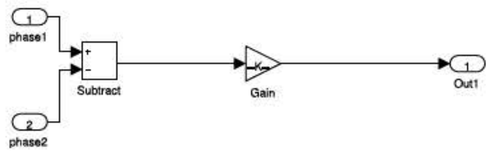

Fig. 10. Linear model of Phase Detector

IV. Simulation Results

Simulation parameters are shown in Table I. Since it is impossible to compute infinite sum of series in (7), only the first M terms of the sum are taken into the account. In Fig. 11 is shown the outputs of filters in both domains (signal/time and phase-frequency) simultaneously. Oscillating curve in the background corresponds to feedback in signal/time domain and thick line in the foreground corresponds to feedback in phase-frequency domain. As we can see, even for M small enough these curves are almost identical. But simulation in signal/time domain takes significant time (about 10 minutes) even for relatively low frequencies while phase-frequency domain simulation is almost instantaneous.

TABLE I
Simulation parameters

VCO frequency	99Hz
OSC_{master} frequency	100Hz
Filter transfer function	$\frac{1}{1+s}$
VCO input gain	10
Simulated time	20 seconds
M	10

For the same parameters, linear model behaviour is quite different. In Fig. 12 are shown simulation results of both signal/time domain and linear models. These results highly differ from each other, comparing to the case of phase-frequency domain.

As mentioned above, simulation in signal/time domain with high frequency signals takes significant time and computational power. So, in the following example the simulation in phase-frequency domain is fulfilled only. Table II shows simulation parameters.

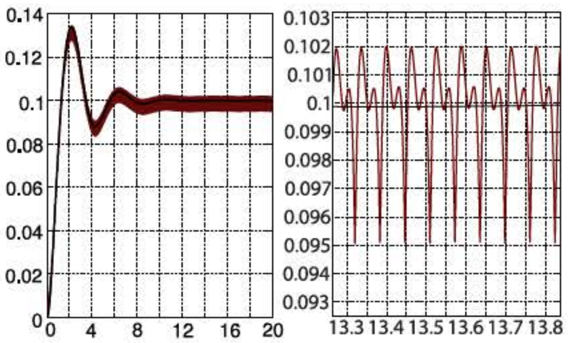

Fig. 11. Signal/time and phase-frequency domains models simulation. Filter output

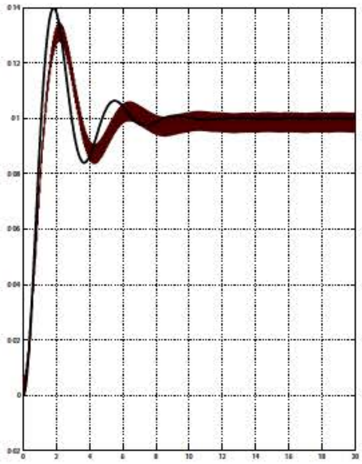

Fig. 12. Signal/time domain and linear models simulation. Filter output

TABLE II

SIMULATION PARAMETERS

VCO frequency	$10^9 + 1$Hz
OSC_{master} frequency	10^9Hz (1 Ghz)
Filter transfer function	$\frac{1}{1+s}$
VCO input gain	10
Simulated time	20 seconds
M	10

The results are shown in Fig. 13. Simulation takes less than a second.

CONCLUSION

Based on new analytical method for computation of phase detector characteristics, an realization in Simulink for simulation of classical PLL in phase space for general types of signal waveforms is considered. This enables to avoid a number of numerical problems in the simulation of PLL in signal. Corresponding numerical examples are presented.

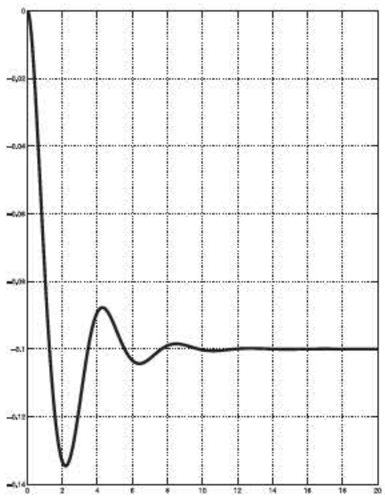

Fig. 13. Phase-frequency domain simulation for 1 Ghz signals. Filter output

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PIV

ANALYTICAL METHOD FOR COMPUTATION OF PHASE-DETECTOR CHARACTERISTIC

by

G.A. Leonov, N.V. Kuznetsov, M.V. Yuldashev, R.V. Yuldashev 2012 [Scopus]

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Analytical Method for Computation of Phase-Detector Characteristic

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***Abstract*—Discovery of undesirable hidden oscillations, which cannot be found by simulation, in models of phase-locked loop (PLL) showed the importance of development and application of analytical methods for the analysis of such models. Approaches to a rigorous nonlinear analysis of analog PLL with multiplier phase detector (classical PLL) and linear filter are discussed. An effective analytical method for computation of multiplier/mixer phase-detector characteristics is proposed. For various waveforms of high-frequency signals, new classes of phase-detector characteristics are obtained, and dynamical model of PLL is constructed.**

***Index Terms*—Analog integrated circuits, nonlinear analysis, phase-detector characteristic, phase-locked loop (PLL).**

I. Introduction

DISCOVERY OF undesirable hidden oscillations [1], which cannot be found by simulation, in phase-locked loop (PLL) models [2] showed the importance of development and application of analytical methods for the analysis of such models. To carry out the nonlinear analysis of PLL, it is necessary to consider PLL models in signal and phase–frequency spaces [3]–[6]. For constructing an adequate nonlinear mathematical model of PLL in phase–frequency space, it is necessary to find the characteristic of phase detector (PD) (PD is a nonlinear element used in PLL to match tunable signals). The PD inputs are high-frequency signals of reference and tunable oscillators, and the output contains a low-frequency error correction signal, corresponding to a phase difference of input signals. For the suppression of high-frequency component at PD output (if such component exists), a low-pass filter is applied. The characteristic of PD is a function defining a dependence of signal at the output of PD (in the phase–frequency space) on the phase difference of signals at the input of PD. PD characteristic depends on the realization of PD and waveforms of input signals.

The characteristics of classical PD—multiplier for typical sinusoidal signal waveforms are well known to engineers [3], [7]–[10].

Furthermore, following [11], on the examples of PD in the form of multiplier, the general principles of computing the PD characteristics for various types of signals, based on a rigorous mathematical analysis of high-frequency oscillations [12], [13], will be considered.

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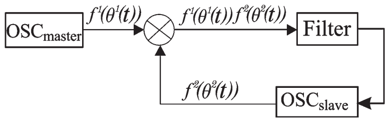


Fig. 1. Block diagram of PLL on the level of electronic realization.

II. Description of Classical PLL in Signal Space

Consider classical PLL on the level of electronic realization (Fig. 1).

Here, $\text{OSC}_{\text{master}}$ is a master oscillator, and $\text{OSC}_{\text{slave}}$ is a slave oscillator [tunable voltage-control oscillator (VCO)], which generate oscillations $f^p(t) = f^p(\theta^p(t))$, $p = 1, 2$ with $\theta^p(t)$ as phases, correspondingly.

The block $\otimes$ is a multiplier (used as PD) of oscillations $f^1(t)$ and $f^2(t)$, and the signal $f^1(t)f^2(t)$ is its output. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter is as follows:

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)d\tau \qquad (1)$$

where $\gamma(t)$ is an impulse response function of filter and $\alpha_0(t)$ is an exponentially damped function depending on the initial data of filter at moment $t = 0$. By assumption, $\gamma(t)$ is a differentiable function with bounded derivative (this is true for the most considered filters [9]).

A. High-Frequency Property of Signals

Suppose that the waveforms $f^{1,2}(\theta)$ are bounded 2π-periodic piecewise differentiable functions[1] (this is true for the most considered waveforms). Consider Fourier series representation of such functions

$$f^p(\theta) = \sum_{i=1}^{\infty} \left(a_i^p \sin(i\theta) + b_i^p \cos(i\theta)\right), \quad p = 1, 2$$

$$a_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\sin(i\theta)d\theta \quad b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\cos(i\theta)d\theta.$$

A high-frequency property of signals can be reformulated in the following way. By assumption, the phases $\theta^p(t)$ are smooth functions (this means that frequencies are changing continuously, which is corresponding to classical PLL analysis

[1]The functions with a finite number of jump discontinuity points differentiable on their continuity intervals.

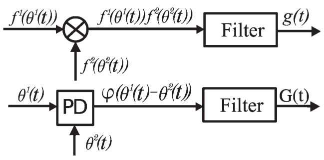


Fig. 2. Multiplier and filter; PD and filter.

[9], [10]). Suppose also that there exists a sufficiently large number $\omega_{\min}$ such that the following conditions are satisfied on the fixed time interval [0, T]:

$$\dot\theta^p(\tau) \geq \omega_{\min} > 0, \quad p = 1, 2 \tag{2}$$

where T is independent of $\omega_{\min}$ and $\dot\theta^p(t)$ denotes frequencies of signals. The frequencies difference is assumed to be uniformly bounded

$$\left|\dot\theta^1(\tau) - \dot\theta^2(\tau)\right| \leq \Delta\omega, \quad \forall\tau \in [0, T]. \tag{3}$$

Requirements (2) and (3) are obviously satisfied for the tuning of two high-frequency oscillators with close frequencies [9], [10]. Let us introduce $\delta = \omega_{\min}^{-(1/2)}$. Consider the relations

$$\begin{aligned} \left|\dot\theta^p(\tau) - \dot\theta^p(t)\right| &\leq \Delta\Omega, \quad p = 1, 2 \\ |t - \tau| &\leq \delta, \quad \forall\tau, t \in [0, T] \end{aligned} \tag{4}$$

where $\Delta\Omega$ is independent of δ and t. Conditions (2)–(4) mean that the functions $\dot\theta^p(\tau)$ are almost constant and the functions $f^p(\theta^p(\tau))$ are rapidly oscillating on small intervals $[t, t + \delta]$.

The boundedness of derivative of $\gamma(t)$ implies

$$|\gamma(\tau) - \gamma(t)| = O(\delta) \quad |t - \tau| \leq \delta, \quad \forall\tau, t \in [0, T]. \tag{5}$$

III. Phase-Detector Characteristic Computation

Consider the two block diagrams shown in Fig. 2. Here, PD is a nonlinear block with characteristic $\varphi(\theta)$. The phases $\theta^p(t)$ are PD block inputs, and the output is a function $\varphi(\theta^1(t) - \theta^2(t))$. The PD characteristic $\varphi(\theta)$ depends on the waveforms of input signals $f^p(\theta)$.

The signal $f^1(\theta^1(t))f^2(\theta^2(t))$ and the function $\varphi(\theta^1(t) - \theta^2(t))$ are the inputs of the same filters with the same impulse response function $\gamma(t)$ and with the same initial state. The outputs of filters are functions $g(t)$ and $G(t)$, respectively. By (1), one can obtain $g(t)$ and $G(t)$

$$\begin{aligned} g(t) &= \alpha_0(t) + \int_0^t \gamma(t-\tau) f^1\left(\theta^1(\tau)\right) f^2\left(\theta^2(\tau)\right) d\tau \\ G(t) &= \alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\left(\theta^1(\tau) - \theta^2(\tau)\right) d\tau. \end{aligned} \tag{6}$$

Then, using the approaches outlined in [11] and [14]–[16], the following result can be proved.

Theorem 1: Let conditions (2)–(5) be satisfied and

$$\varphi(\theta) = \frac{1}{2}\sum_{l=1}^{\infty}\left(\left(a_l^1 a_l^2 + b_l^1 b_l^2\right)\cos(l\theta) + \left(a_l^1 b_l^2 - b_l^1 a_l^2\right)\sin(l\theta)\right). \tag{7}$$

Then, the following relation:

$$|g(t) - G(t)| = O(\delta), \quad \forall t \in [0, T]$$

is valid.

Proof: Suppose that $t \in [0, T]$. Consider the difference

$$g(t) - G(t) = \int_0^t \gamma(t-s)\left[f^1\left(\theta^1(s)\right) f^2\left(\theta^2(s)\right) - \varphi\left(\theta^1(s) - \theta^2(s)\right)\right] ds. \tag{8}$$

Suppose that there exists $m \in \mathbb{N} \cup \{0\}$ such that $t \in [m\delta, (m+1)\delta]$. By definition of δ, we have $m < (T/\delta) + 1$. The continuity condition implies that $\gamma(t)$ is bounded on [0, T] and $f^1(\theta)$ and $f^2(\theta)$ are bounded on $\mathbb{R}$. Since $f^{1,2}(\theta)$ are piecewise-differentiable, one can obtain

$$a_i^{1,2} = O\left(\frac{1}{i}\right) \quad b_i^{1,2} = O\left(\frac{1}{i}\right). \tag{9}$$

Hence, $\varphi(\theta)$ converges uniformly, and $\varphi(\theta)$ is continuous, piecewise differentiable, and bounded. Then, the following estimates:

$$\int_t^{(m+1)\delta} \gamma(t-s) f^1\left(\theta^1(s)\right) f^2\left(\theta^2(s)\right) ds = O(\delta)$$

$$\int_t^{(m+1)\delta} \gamma(t-s)\varphi\left(\theta^1(s) - \theta^2(s)\right) ds = O(\delta).$$

are satisfied. It follows that (8) can be represented as

$$g(t) - G(t) = \sum_{k=0}^{m} \int_{[k\delta,(k+1)\delta]} \gamma(t-s) \times\left[f^1\left(\theta^1(s)\right) f^2\left(\theta^2(s)\right) - \varphi\left(\theta^1(s) - \theta^2(s)\right)\right] ds + O(\delta). \tag{10}$$

Prove now that, on each interval $[k\delta, (k+1)\delta]$, the corresponding integrals are equal to $O(\delta^2)$.

Condition (5) implies that, on each interval $[k\delta, (k+1)\delta]$, the following relation:

$$\gamma(t-s) = \gamma(t-k\delta) + O(\delta), \quad t > s; \quad s, t \in [k\delta, (k+1)\delta]. \tag{11}$$

is valid. Here, $O(\delta)$ is independent of k, and the relation is satisfied uniformly with respect to t. By (10), (11), and the boundedness of $f^1(\theta)$, $f^2(\theta)$, and $\varphi(\theta)$, it can be obtained that

$$g(t) - G(t) = \sum_{k=0}^{m} \gamma(t-k\delta) \int_{[k\delta,(k+1)\delta]} \times\left[f^1\left(\theta^1(s)\right) f^2\left(\theta^2(s)\right) - \varphi\left(\theta^1(s) - \theta^2(s)\right)\right] ds + O(\delta). \tag{12}$$

Denote

$$\theta_k^p(s) = \theta^p(k\delta) + \dot\theta^p(k\delta)(s - k\delta), \quad p = 1, 2.$$

Then, for $s \in [k\delta, (k+1)\delta]$, condition (4) yields

$$\theta^p(s) = \theta_k^p(s) + O(\delta).$$

From (3) and the boundedness of derivative $\varphi(\theta)$ on $\mathbb{R}$, it follows that

$$\int\limits_{[k\delta,(k+1)\delta]} \left|\varphi\left(\theta^1(s)-\theta^2(s)\right)-\varphi\left(\theta_k^1(s)-\theta_k^2(s)\right)\right| ds = O(\delta^2). \tag{13}$$

If $f^1(\theta)$ and $f^2(\theta)$ are continuous on $\mathbb{R}$, then, for $f^1(\theta^1(s))f^2(\theta^2(s))$, one obtains the following relation:

$$\int\limits_{[k\delta,(k+1)\delta]} f^1\left(\theta^1(s)\right) f^2\left(\theta^2(s)\right) ds = \int\limits_{[k\delta,(k+1)\delta]} f^1\left(\theta_k^1(s)\right) f^2\left(\theta_k^2(s)\right) ds + O\left(\frac{1}{\delta^2}\right). \tag{14}$$

Let us consider why this estimate is valid for the considered class of piecewise differentiable waveforms. Since the conditions (2) and (4) are satisfied and the functions $\theta^{1,2}(s)$ are differentiable and satisfy (3), for all $k = 0,\ldots,m$, there exist sets E_k [the union of sufficiently small neighborhoods of discontinuity points of $f^{1,2}(t)$] such that the following relation takes place: $\int_{E_k} ds = O(\delta^2)$, in which case the relation is satisfied uniformly with respect to k. Then, from the piecewise differentiability and the boundedness of $f^{1,2}(\theta)$, it can be proved (14).

By (14) and (13), relation (12) can be rewritten as

$$\begin{aligned} g(t)-G(t) =& \sum_{k=0}^{m}\gamma(t-k\delta) \int\limits_{[k\delta,(k+1)\delta]} \left[f^1\left(\theta_k^1(s)\right)f^2\left(\theta_k^2(s)\right)-\varphi\left(\theta_k^1(s)-\theta_k^2(s)\right)\right]ds+O(\delta)\\ =& \sum_{k=0}^{m}\gamma(t-k\delta) \int\limits_{[k\delta,(k+1)\delta]} \left[\left(\sum_{i=1}^{\infty} a_i^1\cos\left(i\theta_k^1(s)\right)+b_i^1\sin\left(i\theta_k^1(s)\right)\right)\right.\\ &\times\left(\sum_{j=1}^{\infty} a_j^2\cos\left(j\theta_k^2(s)\right)+b_j^2\sin\left(j\theta_k^2(s)\right)\right)\\ &\left.-\varphi\left(\theta_k^1(s)-\theta_k^2(s)\right)\right]ds+O(\delta). \end{aligned} \tag{15}$$

Since the conditions (2)–(4) are satisfied, it is possible to choose $O(1/\delta)$ of sufficiently small time intervals of length $O(\delta^3)$, outside of which the functions $f^p(\theta^p(t))$ and $f^p(\theta_k^p(t))$ are continuous. It is known that, on each interval, which has no discontinuity points, Fourier series of the functions $f^1(\theta)$ and $f^2(\theta)$ converge uniformly. Then, there exists a number $M = M(\delta) > 0$ such that, outside sufficiently small neighborhoods of discontinuity points of $f^p(\theta^p(t))$ and $f^p(\theta_k^p(t))$, the sum of the first M series terms approximates the original function with accuracy to $O(\delta)$. In this case, by relation (15) and the boundedness of $f^1(\theta)$ and $f^2(\theta)$ on $\mathbb{R}$, it can be obtained

$$\begin{aligned} g(t)-G(t) = \sum_{k=0}^{m}\gamma(t-k\delta) \int\limits_{[k\delta,(k+1)\delta]} \sum_{i=1}^{M}\sum_{j=1}^{M} \\ \times\left[\mu_{i,j}(s)-\varphi\left(\theta_k^1(s)-\theta_k^2(s)\right)\right]ds+O(\delta) \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mu_{i,j}(s) = \frac{1}{2}\Big(&\left(a_i^1a_j^2+b_i^1b_j^2\right)\cos(i\theta^1-j\theta^2)\\ &+\left(-a_i^1b_j^2+b_i^1a_j^2\right)\sin(i\theta^1-j\theta^2)\\ &+\left(-b_i^1b_j^2+a_i^1a_j^2\right)\cos(i\theta^1+j\theta^2)\\ &+\left(a_i^1b_j^2+b_i^1a_j^2\right)\sin(i\theta^1+j\theta^2)\Big). \end{aligned}$$

From definition of δ and (9), it follows that, $\forall i\in\mathbb{N}$, $j\in\mathbb{N}$

$$\int\limits_{[k\delta,(k+1)\delta]} \frac{1}{i}\cos\left(j(\omega_{\min}s+c_0)\right)ds = \frac{O(\delta^2)}{ij} \tag{17}$$

is valid. Taking into account (17) and (2), one obtains the estimate

$$\int\limits_{[k\delta,(k+1)\delta]} b_j^p\cos\left(j\theta_k^p(s)\right)ds = \frac{O(\delta^2)}{j^2}.$$

Similar estimate is also valid for the addends with sin.

Consider the addend involving $\cos(i\theta_k^1(s)+j\theta_k^2(s))$ in $\mu_{i,j}(s)$. By (2), it can be obtained $i\dot\theta^1(k\delta)+j\dot\theta^2(k\delta)\geq(i+j)\omega_{\min}$. Then, (17) yields the following relation:

$$\begin{aligned} &\int\limits_{[k\delta,(k+1)\delta]} \cos\Big(i\left(\theta^1(k\delta)+\dot\theta^1(k\delta)(s-k\delta)\right)\\ &\qquad+j\left(\theta^2(k\delta)+\dot\theta^2(k\delta)(s-k\delta)\right)\Big)ds\\ &=\int\limits_{[k\delta,(k+1)\delta]} \cos\Big(\left(i\dot\theta^1(k\delta)+j\dot\theta^2(k\delta)\right)s+i\theta^1(k\delta)+j\theta^2(k\delta)\\ &\qquad-i\left(i\dot\theta^1(k\delta)+j\dot\theta^2(k\delta)\right)k\delta\Big)ds\\ &=O\left(\frac{\delta^2}{i+j}\right). \end{aligned} \tag{18}$$

Then

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\int\limits_{[k\delta,(k+1)\delta]} \frac{-b_i^1b_j^2+a_i^1a_j^2}{2}\cos\left(i\left(\theta_k^1(s)\right)+j\left(\theta_k^2(s)\right)\right)ds = \sum_{i=1}^{M}\sum_{j=1}^{M}\frac{O(\delta^2)}{ij(I+j)}.$$

The convergence of series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(1/ij(i+j))$ implies that the above expression is $O(\delta^2)$. Obviously, a similar relation occurs for the addend $\sin(i\theta_k^1(s)+j\theta_k^2(s))$.

Thus, by (16)

$$\begin{aligned} &g(t)-G(t)\\ &=\sum_{k=0}^{m}\gamma(t-k\delta)\int\limits_{[k\delta,(k+1)\delta]} \left[\sum_{i=1}^{M}\sum_{j=1}^{M}\left(\frac{a_i^1a_j^2+b_i^1b_j^2}{2}\cos\left(i\theta_k^1(s)-j\theta_k^2(s)\right)\right.\right.\\ &\qquad\left.+\frac{a_i^1b_j^2-b_i^1a_j^2}{2}\sin\left(i\theta_k^1(s)-j\theta_k^2(s)\right)\right)\\ &\qquad\left.-\varphi\left(\theta_k^1(s)-\theta_k^2(s)\right)\right]ds+O(\delta). \end{aligned}$$

Note that, here, the addends with indices $i = j$ give, in sum, $\varphi(\theta_k^1(s) - \theta_k^2(s))$ with accuracy to $O(\delta)$. Consider the addends with indices $i < j$, involving $\cos$ (for the addends with indices $i > j$, involving $\sin$, similar relations are satisfied). By (3), similar to (18), the following relation:

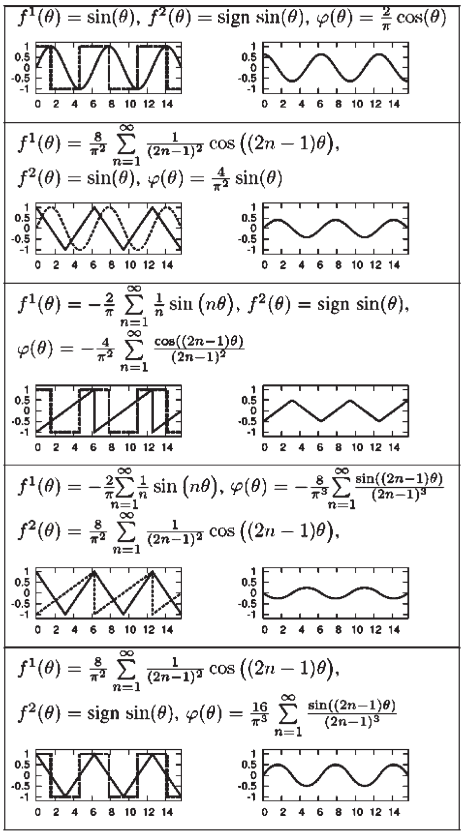
$$\sum_{i=2}^{M}\sum_{j=1}^{i-1}\frac{a_i^1 a_j^2 + b_i^1 b_j^2}{2}\int\limits_{[k\delta,(k+1)\delta]}\cos\left(i\left(\theta_k^1(s)\right) - j\left(\theta_k^2(s)\right)\right)ds$$
$$= \sum_{i=2}^{M}\sum_{j=1}^{i-1} O(\delta^2) O\left(\frac{1}{ij|i-j|}\right)$$
$$= O(\delta^2).$$

is valid. The proof of theorem is completed. ■

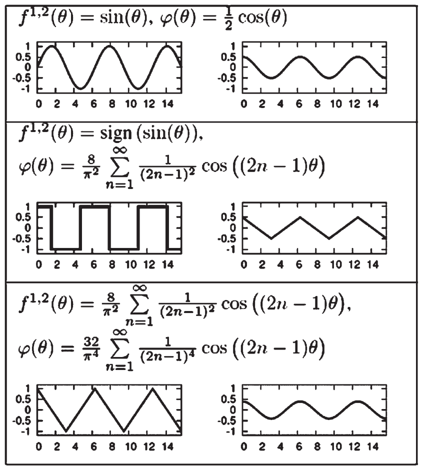
Roughly speaking, this theorem separates low-frequency error-correcting signal from parasite high-frequency oscillations. This result was known to engineers [9] for sinusoidal signals only.

This theorem, for example, allows one to compute a PD characteristic for the following typical signals [9] shown in the table hereinafter. The waveforms $f^{1,2}(\theta)$ of input signals are shown in the left diagram, and the corresponding PD characteristic $\varphi(\theta)$ is plotted in the right one.

A. Phase detector characteristics for equal signal waveforms



B. Phase detector characteristics for mixed signal waveforms



IV. Description of Classical PLL in Phase–Frequency Space

From the mathematical point of view, a linear filter can be described [9] by a system of linear differential equations

$$\dot{x} = Ax + b\xi(t) \quad \sigma = c^* x \tag{19}$$

a solution of which takes the form (1). Here, A is a constant matrix, $x(t)$ is a state vector of filter, and b and c are constant vectors.

The model of tunable generator is usually assumed to be linear [9], [10]

$$\dot{\theta}^2(t) = \omega_{\text{free}}^2 + LG(t), \quad t \in [0, T] \tag{20}$$

where ω_{free}^2 is a free-running frequency of tunable generator and L is an oscillator gain. Here, it is also possible to use nonlinear models of VCO; see, e.g., [17] and [18].

Suppose that the frequency of master generator is constant $\dot{\theta}^1(t) \equiv \omega^1$. Equation of tunable generator (20) and equation of filter (19), yield

$$\dot{x} = Ax + bf^1\left(\theta^1(t)\right)f^2\left(\theta^2(t)\right) \quad \dot{\theta}^2 = \omega_{free}^2 + Lc^* x. \tag{21}$$

The system (21) is nonautonomous and rather difficult for investigation [4]. Here, Theorem 1 allows one to study more simple autonomous system of differential equations [in place of nonautonomous (21)]

$$\dot{x} = Ax + b\varphi(\Delta\theta) \quad \Delta\dot{\theta} = \omega_{\text{free}}^2 - \omega^1 + Lc^* x$$
$$\Delta\theta = \theta^2 - \theta^1. \tag{22}$$

Well-known averaging method [19]–[21] allows one to show that solutions of (21) and (22) are close under some assumptions. Thus, by Theorem 1, the block scheme of PLL in signal space (Fig. 1) can be asymptotically changed [for high-frequency generators, see conditions (2)–(4)] to the block scheme on the level of phase–frequency relations (Fig. 3).

In Fig. 3, PD has the corresponding characteristics. Thus, using asymptotic analysis of high-frequency oscillations, the characteristics of PD can be computed. Methods of nonlinear analysis for this model are well developed [4].

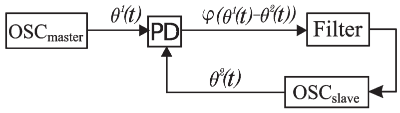


Fig. 3. Block scheme of PLL in phase–frequency space.

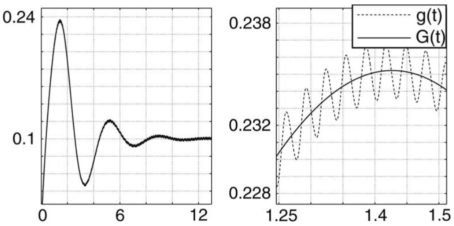


Fig. 4. $\omega^2_{\text{free}} = 99$ Hz, $\omega^1 = 100$ Hz, $L = 10$, filter transfer functions $1/(s+1)$, and triangle waveforms.

The simulation approach for PLL analysis and design, based on the obtained analytical results, is discussed in [22].

It should be noted that, instead of conditions (3) and (5) for simulations of real system, one has to consider the following conditions:

$$|\Delta\omega| \ll \omega_{\min} \quad |\lambda_A| \ll \omega_{\min}$$

where λ_A is the largest (in modulus) eigenvalue of the matix A. Also, for the correctness of transition from relation (8) to relation (12), one has to consider $T \ll \omega_{\min}$. Theoretical results are justified by simulation of PLL model in phase–frequency space and signal space (Fig. 4). Unlike the filter output for the phase–frequency model, the output of the filter for signal space PLL model contains additional high-frequcy oscillations. These high-frequency oscillations interfere with efficient qualitative analysis and simulation of PLL.

The passage to analysis of autonomous dynamical model of PLL (in place of the nonautonomous one) allows one to overcome the aforementioned difficulties, related with modeling PLL in time domain, which were noted in a survey lecture of well-known American specialist D. Abramovitch at American Control Conference, 2008: One has to simultaneously observe "*very fast time scale of the input signals*" and "*slow time scale of signal's phase*."

V. Conclusion

The approach, proposed in this brief, allows one (mathematically rigorously) to compute multiplier PD characteristics in the general case of signal waveforms and to proceed from analysis of classical PLL in time space to analysis and simulation in phase–frequency space. This allows one to effectively simulate classical PLL.

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PV

COMPUTATION OF PHASE DETECTOR CHARACTERISTICS IN SYNCHRONIZATION SYSTEMS

by

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CONTROL THEORY

Computation of Phase Detector Characteristics in Synchronization Systems

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Phase locked loops (PLLs) are widely used in radio engineering [10, 15] and computer architectures [1, 2, 4–9, 11–13]. Nowadays various software and hardware implementations of PPLs are used. An advantage of software implementations is that they are relatively easy to create. However, this limits the maximum possible speed of operation, which is due to the internal implementation of the software code [1, 3]. A shortcoming of hardware implementations is that they require a complex nonlinear analysis of PPL models [1, 11]. Below, we address one aspect of this analysis.

To construct an adequate nonlinear mathematical model of PPLs, we have to determine [1] the characteristic of a phase detector (PD), i.e., a nonlinear element whose input is fed with signals from a reference oscillator and a voltage controlled oscillator (VCO) and whose output contains a low-frequency correcting signal.

We consider a standard phase detector in the form of a signal multiplier [10, 15]. The approaches described in [7, 12] are used to develop a method for computing the phase detector characteristics for various classes of signals.

Consider the transmission of the product of high-frequency oscillations through a linear filter (Fig. 1). Here, $\otimes$ is the multiplier, $f^1(\theta^1(t))$ and $f^2(\theta^2(t))$ are high-frequency oscillations (signals produced by the reference and VCOs, respectively) with their product fed as input into the linear filter (low-frequency filter), and $g(t)$ is the output of the filter.

Assume that $f^1(\theta)$ and $f^2(\theta)$ are bounded 2π-periodic piecewise differentiable functions (i.e., functions with a finite number of jumps that are differentiable on their continuity intervals). Then, according to the Lipschitz criterion [14], the Fourier series corresponding to $f^1(\theta)$ and $f^2(\theta)$ converge to function values at continuity points and to the half-sum of the left and right limits at discontinuity points. Recall that functions different at a finite number of points are equivalent in $L^1_{[-\pi,\pi]}$. Therefore, $f^1(\theta)$ and $f^2(\theta)$ are considered with the values at discontinuity points indicated by the Lipschitz criterion; i.e.,

$$f^1(\theta) = c^1 + \sum_{i=1}^{\infty}(a_i^1\sin(i\theta) + b_i^1\cos(i\theta)),$$

$$f^2(\theta) = c^2 + \sum_{i=1}^{\infty}(a_i^2\sin(i\theta) + b_i^2\cos(i\theta)),$$

$$a_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)\sin(ix)dx, \qquad (1)$$

$$b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)\cos(ix)dx,$$

$$c^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)dx, \quad p \in \{1, 2\}, \quad i \in \mathbb{N}.$$

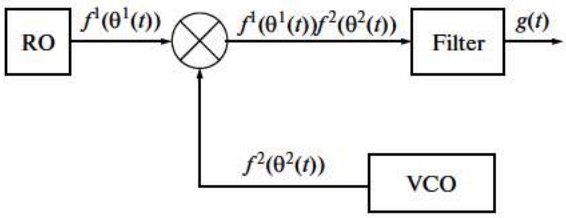

Fig. 1. Multiplier and the filter.

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The properties of Fourier coefficients for piecewise differentiable functions [14] imply the estimates

$$a_i^p = O\left(\frac{1}{i}\right), \quad b_i^p = O\left(\frac{1}{i}\right), \quad p \in \{1, 2\}. \qquad (2)$$

The input $\xi(t)$ and the output $\sigma(t)$ of the linear filter are related by the formula

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)d\tau, \qquad (3)$$

where $\alpha_0(t)$ is an exponentially decaying function depending linearly on the initial state of the filter at $t = 0$ and $\gamma(t)$ is the impulsive transition function of the linear filter ($\alpha_0(t)$ and $\gamma(t)$ are hereafter assumed to be differentiable functions with bounded derivatives). Then, according to (3), the function $g(t)$ has the form

$$g(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau) f^1(\theta^1(\tau)) f^2(\theta^2(\tau)) d\tau. \qquad (4)$$

Let $\theta^1(t)$ and $\theta^2(t)$ be given by

$$\theta^1(t) = \omega^1(t)t + \psi^1, \quad \theta^2(t) = \omega^2(t)t + \psi^2,$$

where $\omega^1(t)$ and $\omega^2(t)$ are positive differentiable functions, while ψ^1 and ψ^2 are constants. Based on the assumptions about $\omega^1(t)$ and $\omega^2(t)$ made above, $\omega^p(t)$ can be treated as the frequencies; $\theta^p(t)$, as the phases; and ψ^p, as the initial phase shifts of the reference oscillator and VCO at times corresponding to transient processes.

Let us formulate the high-frequency properties of the oscillators. On a fixed time interval $[0, T]$, which can be divided into small subintervals of the form $[\tau, \tau + \delta]$, we have

$$\begin{gathered} |\omega^p(\tau) - \omega^p(t)| \le C\delta, \quad p \in \{1, 2\} \\ \forall t \in [\tau, \tau+\delta], \quad \forall \tau \in [0, T-\delta], \end{gathered} \qquad (5)$$

where C is a constant independent of δ or τ. Without loss of generality, we assume that the boundedness of the derivative of $\gamma(t)$ implies the similar relation

$$\begin{gathered} |\gamma(\tau) - \gamma(t)| \le C\delta \quad \forall t \in [\tau, \tau+\delta], \\ \forall \tau \in [0, T-\delta]. \end{gathered} \qquad (6)$$

Suppose that there exists a number R such that

$$\begin{gathered} \omega^p(\tau) \ge R > 0, \quad p \in \{1, 2\}, \quad \forall \tau \in [0, T], \\ R = R(\delta) = O\left(\frac{1}{\delta^2}\right), \quad R > CT. \end{gathered} \qquad (7)$$

Assume that the frequency difference is uniformly bounded:

$$|\omega^1(\tau) - \omega^2(\tau)| \le C_1 \quad \forall \tau \in [0, T]. \qquad (8)$$

Here, C_1 is independent of δ.

It follows from (6) and (7) that, at short time intervals, $\omega^p(t)$ is "almost a constant," and its value is sufficiently large.

Now consider the 2π-periodic function

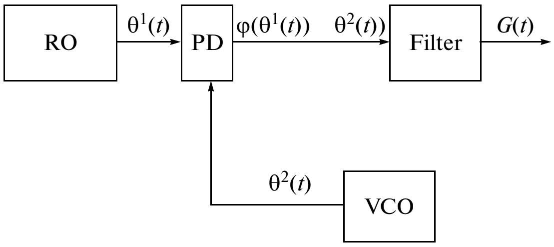

Fig. 2. Phase detector and the filter.

$$\varphi(\theta) = c^1c^2 + \frac{1}{2}\sum_{l=1}^{\infty}((a_l^1a_l^2 + b_l^1b_l^2)\cos(l\theta) + (a_l^1b_l^2 - b_l^1a_l^2)\sin(l\theta)). \qquad (9)$$

It follows from (2) that this series uniformly converges, while $\varphi(\theta)$ is continuous and bounded on $\mathbb{R}$.

Consider the block diagram shown in Fig. 2. Here, PD is a nonlinear phase detector with the output $\varphi(\theta^1(t) - \theta^2(t))$ and $G(t)$ is the output of the filter, which, according to (3), is given by

$$G(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi(\theta^1(\tau) - \theta^2(\tau)). \qquad (10)$$

Theorem 1. *If conditions* (5)–(8) *hold, then, in the same initial state of the filter, we have*

$$|G(t) - g(t)| \le C_2\delta \quad \forall t \in [0, T]. \qquad (11)$$

Proof. Let $t \in [0, T]$. Consider the difference

$$g(t) - G(t) = \int_0^t \gamma(t-s)[f^1(\theta^1(s))f^2(\theta^2(s)) - \varphi(\theta^1(s) - \theta^2(s))]ds. \qquad (12)$$

Let $K \in \mathbb{N}$ be the smallest positive integer such that $[0, T] \subset \bigcup_{k=0}^{K} [k\delta, (k+1)\delta]$. Let $m \in \mathbb{N}$ be such that $t \in [m\delta, (m+1)\delta]$. Without loss of generality, assume that $(m+1)\delta \le T$. Clearly, $m = m(\delta) = O\left(\frac{1}{\delta}\right)$. In what follows, let $k \le m$. The continuity condition implies that $\gamma(t)$ is bounded on $[0, T]$. Moreover, $f^1(\theta)$, $f^2(\theta)$, and $\varphi(\theta)$ are bounded on $\mathbb{R}$. Then

$$\begin{gathered} \int_t^{(m+1)\delta} \gamma(t-s)f^1(\theta^1(s))f^2(\theta^2(s))ds = O(\delta), \\ \int_t^{(m+1)\delta} \gamma(t-s)\varphi(\theta^1(s) - \theta^2(s))ds = O(\delta). \end{gathered} \qquad (13)$$

It follows that (12) can be represented as

$$g(t) - G(t) = \sum_{k=0}^{m} \int_{k\delta}^{(k+1)\delta} \gamma(t-s)$$

$$\times [f^1(\theta^1(s))f^2(\theta^2(s)) - \varphi(\theta^1(s) - \theta^2(s))]ds + O(\delta). \quad (14)$$

Conditions (6) imply that, on each of the intervals $[k\delta, (k+1)\delta]$, we have

$$\gamma(t-s) = \gamma(t-k\delta) + O(\delta), \quad t > s, \quad s \in [k\delta, (k+1)\delta], \quad (15)$$

which holds uniformly in t and $O(\delta)$ is independent of k. Then, using (14), (15), and the boundedness of $f^1(\theta)$, $f^2(\theta)$, and $\varphi(\theta)$, we obtain

$$g(t) - G(t) = \sum_{k=0}^{m} \gamma(t-k\delta)$$

$$\times \int_{k\delta}^{(k+1)\delta} [f^1(\theta^1(s))f^2(\theta^2(s)) - \varphi(\theta^1(s) - \theta^2(s))]ds + O(\delta). \quad (16)$$

Define

$$\theta^p_k(s) = \omega^p(k\delta)s + \psi^p, \quad p \in \{1, 2\}. \quad (17)$$

Since $\varphi(\theta)$ is continuous and bounded on $\mathbb{R}$, it is true that

$$\int_{k\delta}^{(k+1)\delta} |\varphi(\theta^1(s) - \theta^2(s)) - \varphi(\theta^1_k(s) - \theta^2_k(s))| ds = O(\delta^2). \quad (18)$$

By assumption, $f^1(\theta)$ and $f^2(\theta)$ are bounded on $\mathbb{R}$. If $f^1(\theta)$ and $f^2(\theta)$ are additionally continuous on $\mathbb{R}$, then an estimate similar to (18) also holds for $f^1(\theta^1(s))$ and $f^2(\theta^2(s))$.

Let us derive an estimate similar to (18) in the case when $f^1(\theta)$ and $f^2(\theta)$ have discontinuity points.

Since conditions (7) and (5) hold and a $\omega^p(s)$ is a continuous function, we can introduce a set W_k (the union of sufficiently small neighborhoods of the discontinuity points of $f^p(\theta^p(s))$ and $f^p(\theta^p_k(s))$, $p \in \{1, 2\}$, $s \in [k\delta, (k+1)\delta]$) such that

$$\int_{W_k} ds = O(\delta^2). \quad (19)$$

Combining this with the fact that $f^p(\theta)$ is piecewise continuous and bounded, we obtain

$$\int_{[k\delta, (k+1)\delta]} f^1(\theta^1(s))f^2(\theta^2(s))ds$$

$$= \int_{[k\delta, (k+1)\delta]\backslash W_k} f^1(\theta^1_k(s))f^2(\theta^2_k(s))ds + O(\delta^2). \quad (20)$$

Then (16) can be rewritten as

$$g(t) - G(t) = \sum_{k=0}^{m} \gamma(t-k\delta)$$

$$\times \int_{[k\delta, (k+1)\delta]\backslash W_k} [f^1(\theta^1_k(s))f^2(\theta^2_k(s)) - \varphi(\theta^1_k(s) - \theta^2_k(s))]ds$$

$$+ O(\delta) = \sum_{k=0}^{m} \gamma(t-k\delta)$$

$$\times \int_{[k\delta, (k+1)\delta]\backslash W_k} \left[\left(c^1 + \sum_{i=1}^{\infty} a^1_i \sin(i\theta^1_k(s)) + b^1_i \cos(i\theta^1_k(s))\right)\right.$$

$$\times \left(c^2 + \sum_{j=1}^{\infty} a^2_j \sin(j\theta^2_k(s)) + b^2_j \cos(j\theta^2_k(s))\right)$$

$$\left. - \varphi(\theta^1_k(s) - \theta^2_k(s))\right]ds + O(\delta). \quad (21)$$

By the Jordan test for the uniform convergence of Fourier series [14], on each of the intervals free of discontinuity points, the Fourier series of $f^1(\theta)$ and $f^2(\theta)$ converge uniformly. Then there exists a number $M = M(\delta) > 0$ such that the remainders of the series of $f^1(\theta)$ and $f^2(\theta)$ do not exceed δ outside the neighborhoods of the discontinuity points. A similar assertion holds for $\varphi(\theta)$. Then the boundedness of $f^1(\theta)$ and $f^2(\theta)$ implies

$$g(t) - G(t) = \sum_{k=0}^{m} \gamma(t-k\delta)$$

$$\times \int_{[k\delta, (k+1)\delta]\backslash W_{\varepsilon,k}} \left[c^1c^2 + \sum_{i=1}^{M}\sum_{j=1}^{M} \left\{\frac{c^1}{M}(a^2_j \sin(j\theta^2_k(s))\right.\right.$$

$$+ b^2_j \cos(j\theta^2_k(s))) + \frac{c^2}{M}(a^1_i \sin(i\theta^1_k(s)) + b^1_i \cos(i\theta^1_k(s)))$$

$$+ (a^1_i \sin(i\theta^1_k(s)) + b^1_i \cos(i\theta^1_k(s)))$$

$$\left.\times (a^2_j \sin(j\theta^2_k(s)) + b^2_j \cos(j\theta^2_k(s)))\right\}$$

$$\left. - \varphi(\theta^1_k(s) - \theta^2_k(s))\right]ds + O(\delta). \quad (22)$$

Using the formulas for the product of sines and cosines yields

$$(a^1_i \sin(i\theta^1) + b^1_i \cos(i\theta^1))(a^2_j \sin(j\theta^2) + b^2_j \cos(j\theta^2))$$

$$= \frac{1}{2}((a^1_i a^2_j + b^1_i b^2_j)\cos(i\theta^1 - j\theta^2) \quad (23)$$

$$+ (a^1_i b^2_j - b^1_i a^2_j)\sin(i\theta^1 - j\theta^2)) + \frac{1}{2}((b^1_i b^2_j - a^1_i a^2_j)$$

$$\times \cos(i\theta^1 + j\theta^2) + (a^1_i b^2_j + b^1_i a^2_j)\sin(i\theta^1 + j\theta^2)).$$

Define

$$\mu_{i,j}(s) = \frac{1}{2}((a_i^1 a_j^2 + b_i^1 b_j^2)\cos(i\theta_k^1(s) - j\theta_k^2(s)) + (a_i^1 b_j^2 - b_i^1 a_j^2)\sin(i\theta_k^1(s) - j\theta_k^2(s))) + \frac{1}{2}((b_i^1 b_j^2 - a_i^1 a_j^2)\cos(i\theta_k^1(s) + j\theta_k^2(s)) + (a_i^1 b_j^2 + b_i^1 a_j^2)\sin(i\theta_k^1(s) + j\theta_k^2(s))). \quad (24)$$

Then

$$g(t) - G(t) = \sum_{k=0}^{m} \gamma(t - k\delta) \times \int\limits_{[k\delta,(k+1)\delta]\setminus W_k} \Bigg[c^1 c^2 + \sum_{i=1}^{M}\sum_{j=1}^{M}\Bigg\{\frac{c^1}{M}(a_j^2\sin(j\theta_k^2(s)) + b_j^2\cos(j\theta_k^2(s))) + \frac{c^2}{M}(a_i^1\sin(i\theta_k^1(s)) + b_i^1\cos(i\theta_k^1(s))) + \mu_{i,j}(s)\Bigg\} - \varphi(\theta_k^1(s) - \theta_k^2(s))\Bigg]ds + O(\delta). \quad (25)$$

It follows from (7) that

$$\int\limits_{k\delta}^{(k+1)\delta} \cos(Rs + \psi)ds = O(\delta^2). \quad (26)$$

Taking into account (26), (7), and (2), we obtain the estimate

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{1}{M}\int\limits_{[k\delta,(k+1)\delta]} a_j^p\sin(j\theta_k^p(s)) + b_j^p\cos(j\theta_k^p(s))ds = O(\delta^2)\sum_{j=1}^{M} O\left(\frac{1}{j^2}\right) = O(\delta^2), \quad p \in \{1, 2\}. \quad (27)$$

Now we consider the term containing $\cos(i\theta_k^1(s) + j\theta_k^2(s))$ in $\mu_{i,j}(s)$. According to (7), $i\omega^1(k\delta) + j\omega^2(k\delta) \geq (i+j)R$. Then it follows from (26) that

$$\int\limits_{[k\delta,(k+1)\delta]\setminus W_k} \cos(i(\omega^1(k\delta)s + \psi^1) + j(\omega^2(k\delta)s + \psi^2))ds = O(\delta^2)O\left(\frac{1}{i+j}\right). \quad (28)$$

Conditions (2) imply

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\int\limits_{[k\delta,(k+1)\delta]\setminus W_k} \frac{b_i^1 b_j^2 - a_i^1 a_j^2}{2} \times \cos(i(\omega^1(k\delta)s + \psi^1) + j(\omega^2(k\delta)s + \psi^2))ds = O(\delta^2)\sum_{i=1}^{M}\sum_{j=1}^{M} O\left(\frac{1}{ij(i+j)}\right). \quad (29)$$

Since the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)}$ converges, we see that expression (29) is $O(\delta^2)$. An equality similar to (29) holds for the term $\sin(i\theta_k^1(s) + j\theta_k^2(s))$.

Thus, we find from (25) that

$$g(t) - G(t) = \sum_{k=0}^{m}\gamma(t - k\delta) \times \int\limits_{[k\delta,(k+1)\delta]\setminus W_k}\Bigg[\sum_{i=1}^{M}\sum_{j=1}^{M}\Bigg\{\frac{a_i^1 a_j^2 + b_i^1 b_j^2}{2}\cos(i\theta_k^1(s) - j\theta_k^2(s)) + \frac{a_i^1 b_j^2 - b_i^1 a_j^2}{2}\sin(i\theta_k^1(s) - j\theta_k^2(s))\Bigg\} - \varphi(\theta_k^1(s) - \theta_k^2(s))\Bigg]ds + O(\delta). \quad (30)$$

Note that the terms with indices $i = j$ in (30) sum to $\varphi(\theta_k^1(s) - \theta_k^2(s))$ up to $O(\delta)$.

Consider the terms with $i < j$ containing cos (similar relations hold for terms involving sin with indices $i > j$). In an analogous manner to (28), we have

$$\sum_{i=2}^{M}\sum_{j=1}^{i-1}\frac{a_i^1 a_j^2 + b_i^1 b_j^2}{2}\int\limits_{[k\delta,(k+1)\delta]\setminus W_k}[\cos(i(\omega^1(k\delta)s + \psi^1) - j(\omega^2(k\delta)s + \psi^2))]ds = O(\delta^2)\sum_{i=2}^{M}\sum_{j=1}^{i-1} O\left(\frac{1}{ij(i-j)}\right). \quad (31)$$

The convergence of the series

$$\sum_{i=2}^{\infty}\sum_{j=1}^{i-1}\frac{1}{ij(i-j)} \quad (32)$$

implies that (31) is $O(\delta^2)$.

Taking into account the last argument yields

$$g(t) - G(t) = O(\delta), \quad (33)$$

as required.

Let us give examples of computing the characteristic of a phase detector (multiplier) with the use of formula (9) for basic types of signals, namely, sinusoidal and impulsive.

Corollary 1.

$$f^1(t) = A^1\sin(\theta^1(t)), \quad f^2(t) = A^2\sin(\theta^2(t)),$$
$$\varphi(\theta^1 - \theta^2) = \frac{A^1 A^2}{2}\cos(\theta^1 - \theta^2). \quad (34)$$

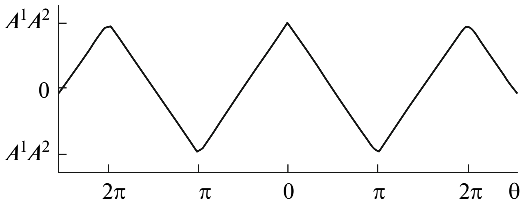

Fig. 3. Plot of $\varphi(\theta)$.

Corollary 2.

$$f^1(t) = A^1 \operatorname{sgn}\sin(\theta^1(t))$$
$$= \frac{4A^1}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin((2n+1)(\omega^1(t)t+\psi^1)),$$
$$f^2(t) = A^2 \operatorname{sgn}\sin(\theta^2(t))$$
$$= \frac{4A^2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin((2n+1)(\omega^2(t)t+\psi^2)), \quad (35)$$
$$\varphi(\theta^1-\theta^2)$$
$$= \frac{8A^1A^2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos((2n+1)(\theta^1-\theta^2)).$$

It is well known [14] that the resulting characteristic $\varphi(\cdot)$ coincides with the function plotted in Fig. 3.

Corollary 3.

$$f^1(t) = A^1\sin(\theta^1(t)),$$
$$f^2(t) = A^2 \operatorname{sgn}\sin(\theta^2(t))$$
$$= \frac{4A^2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin((2n+1)(\omega^2(t)t+\psi^2)), \quad (36)$$
$$\varphi(\theta^1-\theta^2) = \frac{2A^1A^2}{\pi}\cos(\theta^1-\theta^2).$$

ACKNOWLEDGMENTS

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PVI

HIGH-FREQUENCY ANALYSIS OF PHASE-LOCKED LOOP AND PHASE DETECTOR CHARACTERISTIC COMPUTATION

by

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HIGH-FREQUENCY ANALYSIS OF PHASE-LOCKED LOOP AND PHASE DETECTOR CHARACTERISTIC COMPUTATION

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Keywords: Nonlinear analysis, Phase-locked loop, Phase detector characteristic, Mathematical model.

Abstract: Problems of rigorous mathematical analysis of PLL are discussed. An analytical method for phase detector characteristics computation is suggested and new classes of phase detector characteristics are computed. Effective methods for nonlinear analysis of PLL are discussed.

1 INTRODUCTION

Phase-locked loop (PLL) systems were invented in the 1930s-1940s (De Bellescize, 1932; Wendt & Fredentall, 1943) and were widely used in radio and television (demodulation and recovery, synchronization and frequency synthesis). Nowadays PLL can be produced in the form of single integrated circuit and various modifications of PLL are used in a great amount of modern electronic applications (radio, telecommunications, computers, and others).

At present there are several types of PLL (classical PLL, ADPLL, DPLL, and others), intended for the operation with different types of signals (sinusoidal, impulse, and so on). In addition, it is also used different realizations of PLL, which are distinct from each other according to the principles of operation and realization of main blocks.

For the sake of convenience of description, in PLL the following main functional blocks are considered: phase detector (PD), low-pass filter (LPF), and voltage-controlled oscillator (VCO). Note that such a partition into functional blocks often turns out to be conditional, since in many cases in particular physical realization it is impossible to point out the strict boundaries between these blocks. However these blocks can be found in each PLL.

The general PLL operation consists in the generation of an electrical signal (voltage), a phase of which is automatically tuned to the phase of input (reference) signal, i.e. PLL eliminates misphasing (clock skew) between two signals. For this purpose the reference signal and the tunable signal of voltage-controlled oscillator are passed through a special nonlinear element — phase detector (PD). The phase detector produces an error correction signal, corresponding to phase difference of two input signals. For the discrimination of error correction signal, a signal at the output of phase detector is passed through low-pass filter (LPF). The error correction signal, obtained at the output of filter, is used for the frequency control of tunable oscillator, the output of which enters a phase detector, providing thus negative feedback.

The most important performance measure of PLL is the capture range (i.e. a maximal mistuning range of VCO, in which a closed contour of PLL stabilizes a frequency of VCO) and a locking speed (speed of frequency adjustment).

Thus, when designing PLL systems, an important task is to determine characteristics of system (involving parameters of main blocks) providing required characteristics of operation of PLL.

To solve this problem, it is used real experiments with concrete realization of PLL as well as the analytical and numerical methods of analysis of mathematical models of PLL. These tools are used for the obtaining of stability of required operating modes, the estimates of attraction domain of such modes, and the time estimates of transient processes.

Remark, however, that for the strict mathematical analysis of PLL it should be taken into account the fact that the above principles of operation of PLL result in the substantial requirements:

✓ *construction of adequate nonlinear mathemat-*

ical models (since PLL contains nonlinear elements) in signal space and phase-frequency space
and

✓ *justification of the passage between these models (since PLL translates the problem from signal response to phase response and back again).*

Despite this, as noted by well-known PLL expert Danny Abramovitch in his keynote talk at American Control Conference ACC'2002 (Abramovitch, 2002), the main tendency in a modern literature (see, e.g., (Egan, 2000; Best, 2003; Kroupa, 2003; Razavi, 2003)) on analysis of stability and design of PLL is the use of simplified linearized models and the application of the methods of linear analysis, a rule of thumb, and simulation.

However it is known that the application of linearization methods and linear analysis for control systems can lead to untrue results (e.g., Perron effects of Lyapunov exponent sign inversion (Leonov & Kuznetsov , 2007), counterexamples to Aizerman's conjuncture and Kalman's conjuncture on absolute stability, harmonic linearization and filter hypothesis (Leonov et al., 2010[2])) and requires special justifications. Also simple numerical analysis can not reveal nontrivial regimes (e.g., semi-stable or nested limit cycles, hidden oscillations and attractors (Gubar, 1961; Kuznetsov & Leonov, 2008; Leonov et al., 2010[2]; Leonov et. al., 2010[1]; Leonov et. al., 2011)).

2 NONLINEAR MATHEMATICAL MODELS OF PLL

Various methods for analysis of phase-locked loops are well developed by engineers and considered in many publications (see, e.g., (Viterbi, 1966; Gardner, 1966; Lindsey, 1972; Shakhgildyan & Lyakhovkin, 1972)), but the problems of construction of adequate nonlinear models and nonlinear analysis of such models are still far from being resolved turn out to be difficult. and require to use special methods of qualitative theory of differential, difference, integral, and integro-differential equations (Leonov et al., 1996; Suarez & Quere, 2003; Margaris, 2004; Leonov, 2006; Kudrewicz & Wasowicz, 2007; Leonov et al., 2009).

In the present paper some approaches to the nonlinear analysis of PLL are described. Nonlinear mathematical models of high-frequency oscillations are presented.

To construct an adequate nonlinear mathematical model of PLL in phase space it is necessary to find the characteristic of phase detector. The inputs of PD are high-frequency signals of reference and tunable oscillators and the output contains a low-frequency error correction signal, corresponding to a phase difference of input signals. For the suppression of high-frequency component of the output of PD (if such component exists) the low-pass filters are applied. The dependence of the signal at the output of PD (in phase space) on phase difference of signals at the input of PD is the characteristic of PD. This characteristic depends on the realization of PD and the types of signals at the input. Characteristics of the phase detector for standard types of signal are well-known to engineers (Viterbi, 1966; Shakhgildyan & Lyakhovkin, 1972; Abramovitch, 2002).

Further, on the examples of classical PLL with a phase detector in the form of multiplier, we consider general principles of computing phase detector characteristics for different types of signals based on a rigorous mathematical analysis of high-frequency oscillations (Leonov & Seledzhi , 2005a; Leonov, 2008; Kuznetsov et al., 2008; Kuznetsov et al., 2009[1]; Kuznetsov et al., 2009[2]; Leonov et al., 2010[3]).

2.1 Description of Classical PLL in the Signal Space

Consider classical PLL at the level of electronic realization (Fig. 1)

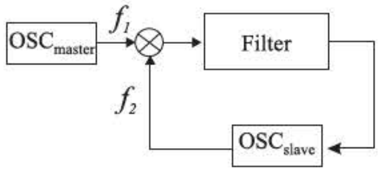

Figure 1: Block diagram of PLL at the level of electronic realization.

Here OSC_{master} is a master oscillator, OSC_{slave} is a slave (tunable voltage-controlled) oscillator, which generates oscillations $f_j(t)$ with high-frequencies $\omega_j(t)$.

Block $\otimes$ is a multiplier of oscillations of $f_1(t)$ and $f_2(t)$ and the signal $f_1(t)f_2(t)$ is its output. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter has the form

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,d\tau. \qquad (1)$$

Here $\gamma(t)$ is an impulse transient function of filter, $\alpha_0(t)$ is an exponentially damped function, depending on the initial data of filter at moment $t = 0$.

In the simplest ideal case, when

$f_1 = \sin(\omega_1), f_2 = \cos(\omega_2)$
$f_1 f_2 = [\sin(\omega_1 + \omega_2) + \sin(\omega_1 - \omega_2)]/2,$

standard engineering assumption is that the filter removes the upper sideband with frequency from the input but leaves the lower sideband without change. Thus it is assumed that the filter output is

$$\frac{1}{2}\sin(\omega_1-\omega_2).$$

Here to avoid these non-rigorous arguments we consider mathematical properties of high-frequency oscillations.

2.2 Computation of Phase Detector Characteristic

A high-frequency property of signals can be reformulated as the following condition. Consider a large fixed time interval $[0,T]$, which can be partitioned into small intervals of the form

$$[\tau,\tau+\delta],\ \tau\in[0,T],$$

where the following relations

$$|\gamma(t)-\gamma(\tau)|\le C\delta,\ |\omega_j(t)-\omega_j(\tau)|\le C\delta,\quad \forall t\in[\tau,\tau+\delta],\ \forall\tau\in[0,T], \tag{2}$$

$$|\omega_1(\tau)-\omega_2(\tau)|\le C_1,\ \forall\tau\in[0,T], \tag{3}$$

$$\omega_j(t)\ge R,\ \forall t\in[0,T] \tag{4}$$

are satisfied.

We shall assume that δ is small enough relative to the fixed numbers T,C,C_1 and R is sufficiently large relative to the number $\delta: R^{-1}=O(\delta^2)$.

The latter means that on small intervals $[\tau,\tau+\delta]$ the functions $\gamma(t)$ and $\omega_j(t)$ are "almost constant" and the functions $f_j(t)$ on them are rapidly oscillating. Obviously, such a condition occurs for high-frequency oscillations.

Consider now harmonic oscillations

$$f_j(t)=A_j\sin(\omega_j(t)t+\psi_j),\quad j=1,2, \tag{5}$$

where A_j and ψ_j are certain numbers, $\omega_j(t)$ are differentiable functions.

Consider two block diagrams shown in Fig. 2 and Fig. 3.

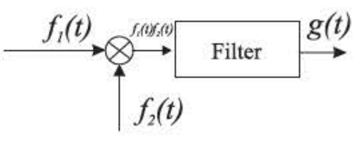

Figure 2: Multiplier and filter.

In Fig. 3 $\theta_j(t)=\omega_j(t)t+\psi_j$ are phases of oscillations $f_j(t)$, PD is a nonlinear block with the characteristic $\varphi(\theta)$. The phases $\theta_j(t)$ are the inputs of PD

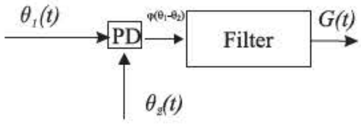

Figure 3: Phase detector and filter.

block and the output is the function $\varphi(\theta_1(t)-\theta_2(t))$. A shape of phase detector characteristic is based on a shape of input signals.

The signals $f_1(t)f_2(t)$ and $\varphi(\theta_1(t)-\theta_2(t))$ are inputs of the same filters with the same impulse transient function $\gamma(t)$. The filter outputs are the functions $g(t)$ and $G(t)$, respectively.

A classical PLL synthesis for the sinusoidal signals is based on the following result (Viterbi, 1966):

If conditions (2)–(4) are satisfied and

$$\varphi(\theta)=\frac{1}{2}A_1A_2\cos\theta,$$

then for the same initial data of filter, the following relation

$$|G(t)-g(t)|\le C_2\delta,\ \forall t\in[0,T]$$

is satisfied. Here C_2 is a certain number being independent of δ.

But what could be done for other types of signal?

Consider now signals in the following form of Fourier series

$$f_1(t)=\sum_{i=1}^{\infty}a_i\sin(i\theta_1(t)),\ f_2(t)=\sum_{j=1}^{\infty}b_j\sin(j\theta_2(t)), \tag{6}$$

where

$$a_k=O\left(\frac{1}{k}\right),\quad b_k=O\left(\frac{1}{k}\right),\qquad k=1,2,\ldots.$$

Let functions $f_1(t)$ and $f_2(t)$ are integrable and bounded on each of the intervals of length δ.

Then the following assertion is valid

Theorem 1 . *If conditions (2)–(4) are satisfied and*

$$\varphi(\theta_1-\theta_2)=\sum_{l=1}^{\infty}\frac{a_lb_l}{2}\cos(l(\theta_1-\theta_2)), \tag{7}$$

then for the same initial states of filter the following relation

$$|G(t)-g(t)|\le C_3\delta,\quad \forall t\in[0,T] \tag{8}$$

is valid.

Proof. Consider a decomposition of the interval $[0,T]$ into the δ length time intervals. Then using (2) we obtain

$$g(t)-G(t)=\sum_{k=0}^{m}\gamma(t-k\delta)\int_{k\delta}^{(k+1)\delta}\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big) -\varphi\big(\theta^1(s)-\theta^2(s)\big)\Big]ds+O(\delta). \tag{9}$$

Because the frequencies are almost constant in the δ-intervals (3), we could introduce $\theta_k^p(s)$

$$\theta_k^p(s)=\omega^p(k\delta)s+\psi^p,\ p\in\{1,2\}. \tag{10}$$

Lemma 1 *. Assuming conditions (2)–(4) the phases $\theta^p(t)$ could be replaced with $\theta_k^p(t)$*

$$\begin{aligned}
&\int_{k\delta}^{(k+1)\delta}\varphi\big(\theta^1(s)-\theta^2(s)\big)=\\
&\int_{k\delta}^{(k+1)\delta}\varphi\big(\theta_k^1(s)-\theta_k^2(s)\big)ds+O(\delta),\\
&\int_{k\delta}^{(k+1)\delta}f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big)=\\
&\int_{k\delta}^{(k+1)\delta}f^1\big(\theta_k^1(s)\big)f^2\big(\theta_k^2(s)\big)ds+O(\delta),
\end{aligned} \tag{11}$$

Then, usign Lemma 1, equation (9) can be rewritten

$$\begin{aligned}
&g(t)-G(t)=\\
&\sum_{k=0}^{m}\gamma(t-k\delta)\int_{[k\delta,(k+1)\delta)}\Big[f^1\big(\theta_k^1(s)\big)f^2\big(\theta_k^2(s)\big)\\
&-\varphi\big(\theta_k^1(s)-\theta_k^2(s)\big)\Big]ds+O(\delta)
\end{aligned} \tag{12}$$

Lemma 2 *. For the neighborhoods $W_{\varepsilon,k}$ of discontinuity points, there is a number M, such that*

$$\begin{aligned}
&g(t)-G(t)=\sum_{k=0}^{m}\gamma(t-k\delta)\\
&\int_{[k\delta,(k+1)\delta]\setminus W_{\varepsilon,k}}\Big[\Big(\sum_{i=1}^{M}a_i^1\sin\big(i\theta_k^1(s)\big)\Big)\Big)\\
&\Big(\sum_{j=1}^{M}a_j^2\sin\big(j\theta_k^1(s)\big)\Big)-\varphi\big(\theta_k^1(s)-\theta_k^2(s)\big)\Big]ds+O(\delta).
\end{aligned}$$

Lemma 2 implies

$$\begin{aligned}
&g(t)-G(t)=\sum_{k=0}^{m}\gamma(t-k\delta)\int_{[k\delta,(k+1)\delta]\setminus W_{\varepsilon,k}}\Big[\\
&\sum_{i=1}^{M}\sum_{j=1}^{M}\Big(a_i^1\sin\big(i\theta_k^1(s)\big)\Big)\Big(a_j^2\sin\big(j\theta_k^2(s)\big)\Big)-\\
&-\varphi\big(\theta_k^1(s)-\theta_k^2(s)\big)\Big]ds+O(\delta).
\end{aligned} \tag{13}$$

It's obvious, that

$$\begin{aligned}
&\sin\big(i\theta_k^1(s)\big)\sin\big(j\theta_k^2(s)\big)=\\
&\frac{1}{2}\Big(\cos\big(i\theta_k^1(s)-j\theta_k^2(s)\big)-\cos\big(i\theta_k^1(s)+j\theta_k^2(s)\big)\Big)
\end{aligned} \tag{14}$$

Lemma 3 *. Assuming conditions (2)–(4) the following equations can be obtained*

$$\int_{k\delta}^{(k+1)\delta}\frac{1}{q}\cos\big(p(Rs+\psi)\big)ds=\frac{O(\delta^2)}{pq}, \tag{15}$$

Using (13),(4),(14) and Lemma 3, the theorem statement can be obtained

$$g(t)-G(t)=O(\delta)\ . \tag{16}$$

■

This result could be easily extended to the case of full Fourier series and allows one to calculate the phase detector characteristic in the following standard cases of signals (Kuznetsov et al., 2010).

Example 1. Two sign signals

$$\begin{aligned}
&f_k(t)=A_k\mathrm{sign}\ \sin(\theta_k(t))=\\
&=\tfrac{4A_k}{\pi}\sum_{n=0}^{\infty}\tfrac{1}{2n+1}\sin((2n+1)(\omega_k(t)t+\psi_k)),\quad k=1,2
\end{aligned}$$

$$\varphi(\theta_1-\theta_2)=\tfrac{8A_1A_2}{\pi^2}\sum_{n=0}^{\infty}\tfrac{1}{(2n+1)^2}\cos(\theta_1-\theta_2)$$

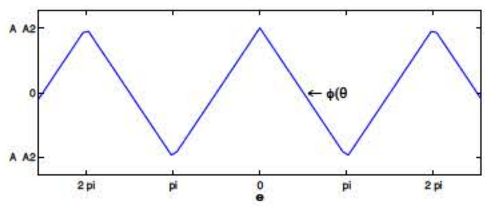

Figure 4: Phase detector characteristic $\varphi(\theta)$ for two sign signals.

Thus, here phase detector characteristic $\phi(\theta)$ corresponds to 2π-periodic function

$$A_1A_2\left(1-\frac{2|\theta|}{\pi}\right), \quad \text{for} \quad \theta\in(-\pi,\pi]. \tag{17}$$

Example 2. Sin signal and sign signal

$$f_1(t)=A_1\sin(\theta_1(t))$$
$$f_2(t)=A_2\text{sign}\,\sin(\theta_2(t))=$$
$$=\tfrac{4A_2}{\pi}\sum_{n=0}^{\infty}\tfrac{1}{2n+1}\sin((2n+1)(\omega_2(t)t+\psi_2))$$
$$\varphi(\theta_1-\theta_2)=\tfrac{2A_1A_2}{\pi}\cos(\theta_1-\theta_2)$$

Example 3. Triangle wave signals.

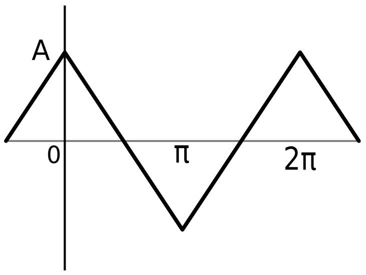

Figure 5: Triangle-wave signal.

$$f_k(t)=A_k\sum_{i=0}^{\infty}\frac{1}{(2i-1)^2}\sin\big((2i-1)\theta_k(t)\big) \tag{18}$$

$$\varphi(\theta_1-\theta_2)=A_1A_2\sum_{l=1}^{\infty}\frac{1}{(2l-1)^4}\cos\big((2l-1)(\theta_1-\theta_2)\big) \tag{19}$$

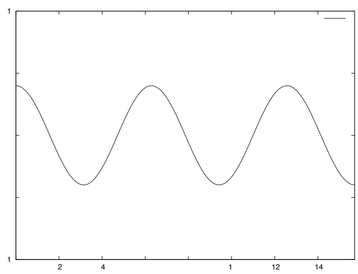

Figure 6: Phase detector characteristic $\varphi(\theta)$ for triangle signals.

2.3 PLL Equations in Phase-frequency Space

From Theorem 1 it follows that block-scheme of PLL in signal space (Fig. 1) can be asymptotically changed (for high-frequency generators) to a block-scheme at the level of frequency and phase relations (Fig. 7).

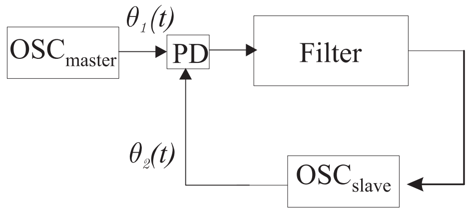

Figure 7: Phase-locked loop with phase detector.

Here PD is a phase detector with corresponding characteristics. Thus, here on basis of asymptotical analysis of high-frequency pulse oscillations characteristics of phase detector can be computed.

Characteristic $\varphi(\theta)$, computed in Examples 1 and 2, tends to zero if $\theta=(\theta_1-\theta_2)$ tends to $\pi/2$, so one can proceed to stability analysis (Leonov, 2006; Leonov et al., 2009) of differential (or difference) equations depend on misphasing θ.

Let us make a remark necessary for derivation of differential equations of PLL.

Consider a quantity

$$\dot\theta_j(t)=\omega_j(t)+\dot\omega_j(t)t.$$

For the well-synthesized PLL such that it possesses the property of global stability, we have exponential damping of the quantity $\dot\omega_j(t)$:

$$|\dot\omega_j(t)|\le Ce^{-\alpha t}.$$

Here C and α are certain positive numbers independent of t. Therefore, the quantity $\dot\omega_j(t)t$ is, as a rule, small enough with respect to the number R (see conditions (3)– (4)). From the above we can conclude that the following approximate relation $\dot\theta_j(t)\approx\omega_j(t)$ is valid. In deriving the differential equations of this PLL, we make use of a block diagram in Fig. 7 and exact equality

$$\dot\theta_j(t)=\omega_j(t). \tag{20}$$

Note that, by assumption, the control law of tunable oscillators is linear:

$$\omega_2(t)=\omega_2(0)+LG(t). \tag{21}$$

Here $\omega_2(0)$ is initial frequency of tunable oscillator, L is a certain number, and $G(t)$ is a control signal, which is a filter output (Fig. 3). Thus, the equation of PLL is as follows

$$\dot\theta_2(t)=\omega_2(0)+L\Big(\alpha_0(t)+\int_0^t\gamma(t-\tau)\varphi\big(\theta_1(\tau)-\theta_2(\tau)\big)d\tau\Big).$$

Assuming that the master oscillator is such that $\omega_1(t)\equiv\omega_1(0)$, we obtain the following relations for

PLL

$$\big(\theta_1(t)-\theta_2(t)\big)'+L\Big(\alpha_0(t)+\int_0^t\gamma(t-\tau)\varphi\big(\theta_1(\tau)-\theta_2(\tau)\big)d\tau\Big)$$
$$=\omega_1(0)-\omega_2(0). \tag{22}$$

This is an equation of standard PLL. Note, that if filter (1) is an integrating filter with the transfer function $(p+\alpha)^{-1}$

$$\dot\sigma+\alpha\sigma=\varphi(\theta)$$

then for $\phi(\theta)=\cos(\theta)$ in place of of equation (22) from (20) and (21) we have pendulum-like equation (Leonov & Smirnova, 1996; Leonov et al., 1996)

$$\ddot{\tilde\theta}+\alpha\dot{\tilde\theta}+L\sin\tilde\theta=\alpha\big(\omega_1(0)-\omega_2(0)\big) \tag{23}$$

with $\tilde\theta=\theta_1-\theta_2+\frac{\pi}{2}$. Thus, if here phases of the input and output signals mutually shifted by $\pi/2$, then the control signal $G(t)$ equals zero.

Arguing as above, we can conclude that in PLL it can be used the filters with transfer functions of more general form $K(p)=a+W(p)$, where a is a certain number, $W(p)$ is a proper fractional rational function. In this case in place of equation (22) we have

$$\big(\theta_1(t)-\theta_2(t)\big)'+L\Big[a\varphi\big(\theta_1(t)-\theta_2(t)\big)+\alpha_0(t)+$$
$$+\int_0^t\gamma(t-\tau)\varphi\big(\theta_1(\tau)-\theta_2(\tau)\big)d\tau\Big]=\omega_1(0)-\omega_2(0). \tag{24}$$

In the case when the transfer function of the filter $a+W(p)$ is non-degenerate, i.e. its numerator and denominator do not have common roots, equation (24) is equivalent to the following system of differential equations

$$\dot z=Az+b\psi(\sigma),\quad \dot\sigma=c^*z+\rho\psi(\sigma). \tag{25}$$

Here $\sigma=\theta_1-\theta_2$, A is a constant $(n\times n)$-matrix, b and c are constant (n)-vectors, ρ is a number, and $\psi(\sigma)$ is 2π-periodic function, satisfying the relations:

$$\rho=-aL,\quad W(p)=L^{-1}c^*(A-pI)^{-1}b,$$
$$\psi(\sigma)=\varphi(\sigma)-\frac{\omega_1(0)-\omega_2(0)}{L(a+W(0))}.$$

The discrete phase-locked loops obey similar equations

$$\begin{aligned} z(t+1)&=Az(t)+b\psi(\sigma(t))\\ \sigma(t+1)&=\sigma(t)+c^*z(t)+\rho\psi(\sigma(t)),\end{aligned} \tag{26}$$

where $t\in Z$, Z is the set of integers. Equations (25) and (26) describe the so-called standard PLLs (Shakhgildyan & Lyakhovkin, 1972).

For analysis of the above mathematical models of PLL is applied in the theory of phase synchronization, which was developed in the second half of the last century on the basis of three applied theories: the theory of synchronous and induction electrical motors, the theory of auto-synchronization of the unbalanced rotors, and the theory of phase-locked loops. Modification of direct Lyapunov method with the construction of periodic Lyapunov-like functions, the method of positively invariant cone grids, and the method of nonlocal reduction turned out to be most effective (Leonov et al., 1996; Leonov, 2006; Leonov et al., 2009). The last method, which combines the elements of direct Lyapunov method and bifurcation theory, allows one to extend the classical results of F. Tricomi (Tricomi , 1933) and his progenies (Kudrewicz & Wasowicz, 2007) to the multidimensional dynamical systems.

3 CONCLUSIONS

Considered above methods for high-frequency analysis of PLL allow one to construct adequate nonlinear dynamical model of PLL and to apply special methods of qualitative theory of differential, difference, integral, and integro-differential equations for PLL design.

ACKNOWLEDGEMENTS

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PVII

ANALYTICAL METHODS FOR COMPUTATION OF PHASE-DETECTOR CHARACTERISTICS AND PLL DESIGN

by

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Analytical Methods for Computation of Phase-Detector Characteristics and PLL Design

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***Abstract*— An effective analytical methods for computation of phase detector characteristics are suggested. For high-frequency oscillators new classes of such characteristics are described. Approaches to a rigorous nonlinear analysis of PLL are discussed.**

I. Introduction

Various methods for analysis of phase-locked loops are well developed by engineers and considered in many publications (see, e.g., [1]–[4]), but the problems of construction of adequate nonlinear models and nonlinear analysis of such models are still far from being resolved and require using special methods of qualitative theory of differential, difference, integral, and integro-differential equations [5]–[10]. So for a strict mathematical analysis of the model should be taken into account the fact that the main principles of operation of PLL result in the substantial requirements:

✓ *construction of adequate nonlinear mathematical models in signal space and in phase-frequency space* (since main purpose of PLL is to eliminate misphasing and PLL contains nonlinear elements) and

✓ *justification of the passage between these models* (since PLL translates the problem from signal response to phase response and back again).

Despite this, as noted by D. Abramovitch in his keynote talk at American Control Conference [11], the main tendency in a modern literatures on analysis of stability and design of PLL [12]–[15] is the use of simplified linearized models and the application of the methods of linear analysis, a rule of thumb, and simulation.

However it is known that the application of linearization methods and linear analysis for control systems can lead to untrue results (e.g., Perron effects of Lyapunov exponent sign inversion [16], counterexamples to Aizerman's conjuncture and Kalman's conjuncture on absolute stability, harmonic linearization and filter hypothesis [17]) and requires special justifications. Also simple numerical analysis can not reveal nontrivial regimes (e.g., semi-stable or nested limit cycles, hidden oscillations and attractors [17]–[21]).

In the present work an approach to nonlinear analysis and design of PLL is described and its application to classical PLL is considered. This approach is based on the construction of adequate nonlinear mathematical models and on applying the methods of nonlinear analysis of high-frequency oscillations.

II. Nonlinear Analysis of PLL

For the analysis of PLL it is necessary to consider the models of PLL in signal space and phase space [1], [2], [4]). In this case for constructing of an adequate nonlinear mathematical model of PLL in phase space it is necessary to find the characteristic of phase detector (PD — a nonlinear element, used in PLL for matching tunable signals). The inputs of PD are high-frequency signals of reference and tunable oscillators and the output contains a low-frequency error correction signal, corresponding to a phase difference of input signals. For the suppression of high-frequency component of the output of PD (if such component exists) the low-pass filter can be applied. The characteristic of PD is the dependence of the signal at the output of PD (in the phase space) on the phase difference of signals at the input of PD. This characteristic depends on the realization of PD and the types of signals at the input. Characteristics of the phase detector for standard types of signal are well-known to engineers [1], [4], [11].

Further following [22], on the examples of classical PLL with a phase detector in the form of multiplier, we consider the general principles of computation of phase detector characteristics for different types of signals based on a rigorous mathematical analysis of high-frequency oscillations [23]–[27].

III. Description of the Classical PLL in the Signal Space

Consider the classical PLL on the level of electronic realization (Fig. 1)

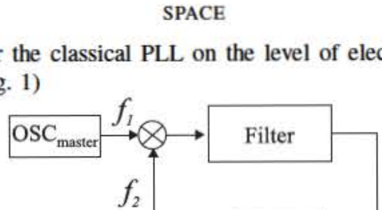

Fig. 1. Block diagram of PLL on the level of electronic realization.

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Here $\mathrm{OSC}_{\mathrm{master}}$ is a master oscillator, $\mathrm{OSC}_{\mathrm{slave}}$ is a slave (tunable voltage-controlled) oscillator, which generates oscillations $f_j(t)$ with high-frequencies $\omega_j(t)$. Block $\otimes$ is a multiplier (used as PD) of oscillations of $f_1(t)$ and $f_2(t)$ and the signal $f_1(t)f_2(t)$ is its output. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter has the form

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,\mathrm{d}\tau. \tag{1}$$

Here $\gamma(t)$ is an impulse transient function of filter, $\alpha_0(t)$ is an exponentially damped function, depending on the initial data of filter at the moment $t = 0$.

A. *High-frequency property of signals*

In the simplest ideal case, when two high-frequency sinusoidal signals

$$f_1 = \sin(\omega_1), f_2 = \cos(\omega_2),$$
$$f_1 f_2 = [\sin(\omega_1 + \omega_2) + \sin(\omega_1 - \omega_2)]/2,$$

are considered, standard engineering assumption is that the low-pass filter has to remove the upper sideband with frequency from the input but leaves the lower sideband without change. Thus it is assumed that the filter output is $\frac{1}{2}\sin(\omega_1 - \omega_2)$.

But how to prove this assumption in the general case of signals?

Here to avoid the above non-rigorous arguments we consider mathematical properties of high-frequency oscillations.

Suppose that $f^1(\theta), f^2(\theta)$ — bounded 2π-periodic piecewise differentiable functions. Then, Fourier series, corresponding to the functions $f^1(\theta)$ and $f^2(\theta)$, converge to the function values at points of continuity and to half the sum of left and right limits at the discontinuity points.

Further, since in $L^1_{[-\pi,\pi]}$ functions that differ in a finite number of points are equivalent, we consider $f^1(\theta)$ and $f^2(\theta)$ with the above values at the points of discontinuity, i.e.,

$$\begin{aligned} f^1(\theta) &= c^1 + \sum_{i=1}^{\infty}\left(a_i^1\sin(i\theta) + b_i^1\cos(i\theta)\right),\\ f^2(\theta) &= c^2 + \sum_{i=1}^{\infty}\left(a_i^2\sin(i\theta) + b_i^2\cos(i\theta)\right), \end{aligned} \tag{2}$$

$$a_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)\sin(ix)dx, \quad b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)\cos(ix)dx,$$

$$c^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(x)dx, \quad p \in \{1,2\}, \quad i \in \mathbb{N}.$$

According to the properties of Fourier coefficients for piecewise differentiable functions the following estimates

$$a_i^p = O\left(\frac{1}{i}\right), b_i^p = O\left(\frac{1}{i}\right), \; p \in \{1,2\}. \tag{3}$$

are valid.

A high-frequency property of signals can be reformulated as the following condition. Consider a large fixed time interval $[0,T]$, which can be partitioned into small intervals of the form

$$[\tau, \tau+\delta], \; \tau \in [0,T],$$

where the following relations

$$\begin{aligned} &|\gamma(t) - \gamma(\tau)| \le C\delta, \;\; |\omega_j(t) - \omega_j(\tau)| \le C\delta,\\ &\forall\, t \in [\tau, \tau+\delta], \;\; \forall\, \tau \in [0,T], \end{aligned} \tag{4}$$

$$|\omega_1(\tau) - \omega_2(\tau)| \le C_1, \;\; \forall \tau \in [0,T], \tag{5}$$

$$\omega_j(t) \ge R, \;\; \forall\, t \in [0,T] \tag{6}$$

are satisfied.

We shall assume that δ is small enough relative to the fixed numbers T, C, C_1 and R is sufficiently large relative to the number $\delta : R^{-1} = O(\delta^2)$.

The latter means that on small intervals $[\tau, \tau+\delta]$ the functions $\gamma(t)$ and $\omega_j(t)$ are almost constant and the functions $f_j(t)$ on them are rapidly oscillating. Obviously, such a condition occurs for high-frequency oscillations.

IV. Phase-detector characteristic computation

Consider two block diagrams On Fig. 2 $\theta_j(t) = \omega_j(t)t + \psi_j$

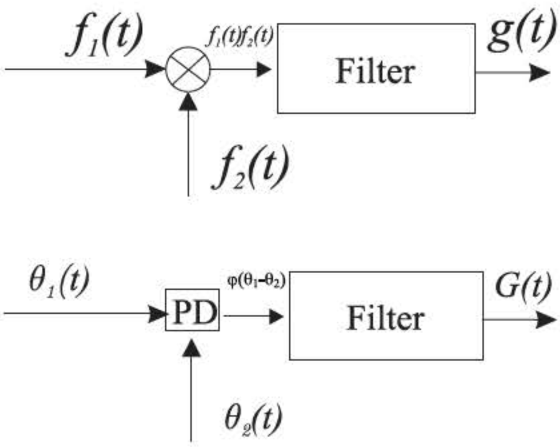


Fig. 2. Multiplier and filter. Phase detector and filter.

are phases of the oscillations $f_j(t)$, PD is a nonlinear block with the characteristic $\varphi(\theta)$. The phases $\theta_j(t)$ are the inputs of PD block and the output is the function $\varphi(\theta_1(t) - \theta_2(t))$. The shape of the phase detector characteristic is based on the shape of input signals. The signals $f_1(t)f_2(t)$ and $\varphi(\theta_1(t) - \theta_2(t))$ are inputs of the same filters with the same impulse transient function $\gamma(t)$. The filter outputs are the functions $g(t)$ and $G(t)$, respectively.

Then, using the approaches outlined in [22] the following result can be proved.

Theorem 1: If conditions (4)–(6) of high-frequency of signals are satisfied and

$$\varphi(\theta) = c^1 c^2 + \\ + \frac{1}{2}\sum_{l=1}^{\infty}\Big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + (a_l^1 b_l^2 - b_l^1 a_l^2)\sin(l\theta)\Big). \tag{7}$$

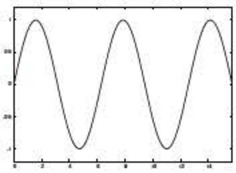
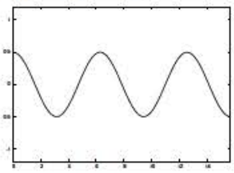
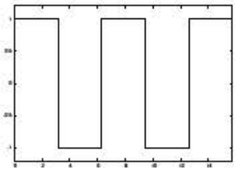
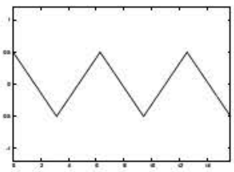
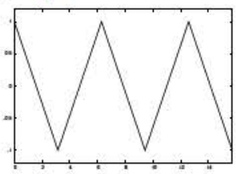
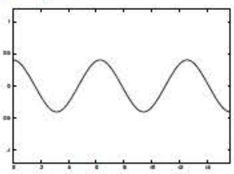
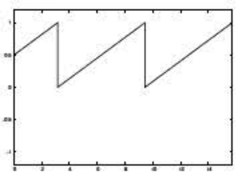
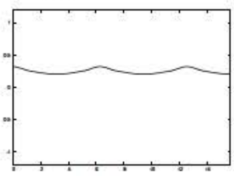
then for the same initial states of filter the following relation

$$|G(t) - g(t)| \le C_2\delta, \quad \forall t \in [0, T] \tag{8}$$

is valid.

The proof of lemmas is omitted here because of its cumbersome. The correctness of this result is confirmed by obtaining as a corollary to well-known formulas for the typical signals.

V. Example of phase detector characteristics computation

$f^{1,2}(\theta) = \sin(\theta)$ 	$\varphi(\theta) = \frac{1}{2}\cos(\theta)$ 
$f^{1,2}(\theta) = \mathrm{sign}\,(\sin(\theta))$ 	$\varphi(\theta) = \frac{8}{\pi^2}\sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big)$ 
$f^{1,2}(\theta) = \frac{8}{\pi^2}\sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big)$ 	$\varphi(\theta) = \frac{32}{\pi^4}\sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^4}\cos\big((2n-1)\theta\big)$ 
$f^{1,2}(\theta) = \frac{1}{2} + \frac{1}{\pi}\sum\limits_{n=1}^{\infty}\frac{1}{n}\sin(n\theta)$ 	$\varphi(\theta) = \frac{1}{4} + \frac{1}{2\pi^2}\sum\limits_{n=1}^{\infty}\frac{1}{n^2}\cos(n\theta)$ 

VI. PLL equations in phase-frequency space

From Theorem 1 it follows that block-scheme of PLL in signal space (Fig. 1) can be asymptotically changed (for high-frequency generators) to a block-scheme on the level of frequency and phase relations (Fig. 3).

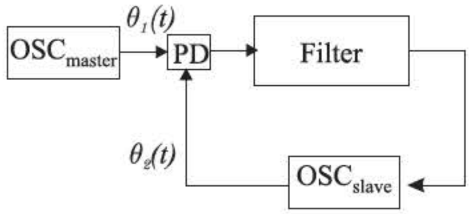


Fig. 3. Phase-locked loop with phase detector

Here PD is a phase detector with corresponding characteristics. Thus, here on basis of asymptotical analysis of high-frequency pulse oscillations a characteristics of phase detector can be computed.

Let us make a remark necessary for derivation of differential equations of PLL.

Consider a quantity

$$\dot\theta_j(t) = \omega_j(t) + \dot\omega_j(t)t.$$

For the well-synthesized PLL such that it possesses the property of global stability, we have exponential damping of the quantity $\dot\omega_j(t)$:

$$|\dot\omega_j(t)| \le Ce^{-\alpha t}.$$

Here C and α are certain positive numbers being independent of t. Therefore, the quantity $\dot\omega_j(t)t$ is, as a rule, sufficiently small with respect to the number R (see conditions (4)–(6)). From the above we can conclude that the following approximate relation $\dot\theta_j(t) \approx \omega_j(t)$ is valid. In deriving the differential equations of this PLL, we make use of a block diagram in Fig. 3 and exact equality

$$\dot\theta_j(t) = \omega_j(t). \tag{9}$$

Note that, by assumption, the control law of tunable oscillators is linear:

$$\omega_2(t) = \omega_2(0) + LG(t). \tag{10}$$

Here $\omega_2(0)$ is the initial frequency of tunable oscillator, L is a certain number, and $G(t)$ is a control signal, which is a filter output (Fig. 2). Thus, the equation of PLL is as follows

$$\dot\theta_2(t) = \omega_2(0) + L\Big(\alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\big(\theta_1(\tau) - \theta_2(\tau)\big)\mathrm{d}\tau\Big).$$

Assuming that the master oscillator is such that $\omega_1(t) \equiv \omega_1(0)$, we obtain the following relations for PLL

$$\big(\theta_1(t) - \theta_2(t)\big)' + L\Big(\alpha_0(t) + \int_0^t \gamma(t-\tau)\varphi\big(\theta_1(\tau) - \theta_2(\tau)\big)\mathrm{d}\tau\Big) \\ = \omega_1(0) - \omega_2(0). \tag{11}$$

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This is an equation of standard PLL.

Characteristic $\varphi(\theta)$, computed in examples 1 and 2, tends to zero if $\theta = (\theta_1 - \theta_2)$ tends to $\pi/2$, so one can proceed to stability analysis [8], [10], [28] of differential (or difference) equations depend on the misphasing θ.

Note, that if the filter (1) is integrated with the transfer function $(p+\alpha)^{-1}$

$$\dot{\sigma} + \alpha\sigma = \varphi(\theta)$$

then for $\phi(\theta) = \cos(\theta)$ instead of equation (11) from (9) and (10) we have the following pendulum-like equation [5], [29]

$$\ddot{\tilde{\theta}} + \alpha\dot{\tilde{\theta}} + L\sin\tilde{\theta} = \alpha\big(\omega_1(0) - \omega_2(0)\big) \tag{12}$$

with $\tilde{\theta} = \theta_1 - \theta_2 + \frac{\pi}{2}$. So, if here phases of the input and output signals mutually shifted by $\pi/2$ then the control signal $G(t)$ equals zero.

In the case when the transfer function of the filter $W(p)$ is non-degenerate (its numerator and denominator do not have common roots) equation (11) is equivalent to the following system of differential equations

$$\dot{z} = Az + b\psi(\sigma), \quad \dot{\sigma} = c^* z. \tag{13}$$

Here $\sigma = \theta_1 - \theta_2$, A is a constant $(n \times n)$-matrix, b and c are constant (n)-vectors, and $\psi(\sigma)$ is 2π-periodic function, satisfying the relations:

$$\rho = -aL, \quad W(p) = L^{-1}c^*(A - pI)^{-1}b,$$
$$\psi(\sigma) = \varphi(\sigma) - \frac{\omega_1(0) - \omega_2(0)}{LW(0)}.$$

The discrete phase-locked loops obey similar equations

$$\begin{aligned} z(t+1) &= Az(t) + b\psi(\sigma(t)) \\ \sigma(t+1) &= \sigma(t) + c^* z(t), \end{aligned} \tag{14}$$

where $t \in Z$, Z is a set of integers. Equations (13) and (14) describe the so-called standard PLLs [4].

For analysis of the above mathematical models PLL is applied in the theory of phase synchronization, which was developed in the second half of the last century on the basis of three applied theories: theory of synchronous and induction electrical motors, theory of auto-synchronization of the unbalanced rotors, theory of phase-locked loops. Modification of direct Lyapunov method with the construction of periodic Lyapunov-like functions, the method of positively invariant cone grids, and the method of nonlocal reduction turned out to be most effective [5], [8], [10]. The last method, which combines the elements of direct Lyapunov method and bifurcation theory, allows one to extend the classical results of F. Tricomi [30] and his progenies to the multidimensional dynamical systems [5], [9].

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PVIII

NONLINEAR ANALYSIS OF PHASE-LOCKED LOOP

by

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Nonlinear Analysis of Phase-Locked Loop ⋆

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Abstract: New method for the rigorous mathematical nonlinear analysis of PLL systems is suggested. This method allows to calculate the characteristics of phase detectors and carry out a rigorous mathematical analysis of transient process and stability of the system.

Keywords: Phase-locked loops, nonlinear analysis, phase detector characteristic, misphasing, clock skew elimination, justification of linearization

1. INTRODUCTION

Phase-locked loop (PLL) systems were invented in the 1930s-1940s [De Bellescize (1932); Wendt & Fredentall (1943)] and were widely used in radio and television (demodulation and recovery, synchronization and frequency synthesis). Nowadays PLL can be produced in the form of single integrated circuit and various modifications of PLL are used in a great amount of modern electronic applications (radio, telecommunications, computers and others).

At present there are several types of PLL (classical PLL, ADPLL, DPLL, and others), intended for the operation with different types of signals (sinusoidal, impulse and so on). In addition it is also used different realizations of PLL, which are distinct from each other according to the principles of operation and realization of main blocks.

For the sake of convenience of description, the following main functional blocks are considered in PLL: Phase detector (PD), Low-Pass Filter (LPF), and Voltage-Controlled Oscillator (VCO). Note that such a partition into functional blocks often turns out conditional, since in many cases in particular physical realization it is impossible to point out the strict boundaries between these blocks. However these blocks can be found in each PLL.

The general PLL operation consists in the generation of an electrical signal (voltage), a phase of which is automatically tuned to the phase of input "reference" signal, i.e. PLL eliminates misphasing (clock skew) between two signals. For this purpose the reference signal of reference oscillator and the tunable signal of voltage-controlled oscillator enter a phase detector. The phase detector is a nonlinear element permitting one to obtain an error correction signal, corresponding to phase difference of two input signals. For the discrimination of error correction element, a signal at the output of phase detector is passed through low-pass filter (LPF). The error correction signal, obtained at the output of filter, is used for the frequency control of tunable oscillator, the output of which enters a phase detector, providing thus negative feedback.

The most important performance measure (characteristics) of PLL is the capture range i.e. a maximal mistuning range of VCO, in which a closed contour of PLL stabilizes a frequency of VCO, and a speed of locking under frequency jumping.

Thus, when designing PLL systems, an important task is to determine characteristics of system (involving parameters of main blocks) providing required characteristics of operation of PLL.

To solve this problem, it is used real experiments with concrete realization of PLL as well as the analytical and numerical methods of analysis of mathematical models of PLL. These tools are used for the obtaining of stability of required operating modes, the estimates of domain of attraction of such modes, and the time estimates of transient processes.

Remark, however, that for the strict mathematical analysis of PLL it should be taken into account the fact that the above principles of operation of PLL result in the substantial requirement:

> *construction of adequate nonlinear mathematical models in signal space and in phase-frequency space (since PLL contains nonlinear elements) and justification of the passage between these models (since PLL translates the problem from signal response to phase response and back again)*

Despite this, as noted by Danny Abramovitch in his keynote talk [Abramovitch (2002)], the main tendency in a modern literature on analysis of stability and design of PLL is the use of simplified "linearized" models and the application of the methods of linear analysis, a rule of thumb, and simulation [Egan (2000); Best (2003); Kroupa (2003);

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Razavi (2003)]. However it is well-known that the application of linearization methods and linear analysis for control systems can come to untrue results (e.g., Perron effects of Lyapunov exponent sign inversion [Leonov & Kuznetsov (2007)], Aizerman's conjuncture and Kalman's conjuncture in absolute stability problem, harmonic linearization and filter hypothesis [Leonov et al. (2010)]) and requires special justifications.

2. MATHEMATICAL METHODS OF NONLINEAR ANALYSIS AND DESIGN OF PLL

Various methods for analysis of phase-locked loops are well developed by engineers and considered in many publications (see, e.g., Viterbi (1966); Gardner (1966); Lindsey (1972); Shakhgildyan & Lyakhovkin (1972)), but the problems of construction of adequate nonlinear models and nonlinear analysis of such models are still far from being resolved and require using special methods of qualitative theory of differential, difference, integral, and integro-differential equations [Gelig et al. (1978); Leonov et al. (1992, 1996); Suarez & Quere (2003); Margaris (2004); Leonov (2006); Kudrewicz & Wasowicz (2007); Kuznetsov (2008); Leonov et al. (2009)].

In the present work, on the examples of classical PLL, it is described the approaches to nonlinear analysis and design of PLL, which are based on the construction of adequate nonlinear mathematical models and applying the methods of nonlinear analysis of high-frequency oscillations.

For the analysis of PLL it is usually considered the models of PLL in signal space and phase space [Viterbi (1966); Gardner (1966); Shakhgildyan & Lyakhovkin (1972))]. In this case for constructing of an adequate nonlinear mathematical model of PLL in phase space it is necessary to find the characteristic of phase detector (i.e. a nonlinear element, used in PLL for matching tunable signals). The inputs of PD are high-frequency signals of reference and tunable oscillators and the output contains a low-frequency error correction signal, corresponding to a phase difference of input signals. For the suppression of high-frequency component of the output of PD (if such component exists) the low-pass filters are applied. The characteristic of PD is the dependence of the signal at the output of PD (in the phase space) on the phase difference of signals at the input of PD. This characteristic depends on the realization of PD and the types of signals at the input. Characteristics of the phase detector for standard types of signal are well-known to engineers [Viterbi (1966); Shakhgildyan & Lyakhovkin (1972); Abramovitch (2002)].

Further following [Leonov (2008)], on the examples of classical PLL with a phase detector in the form of multiplier, we consider the general principles of calculating of phase detector characteristics for different types of signals based on a rigorous mathematical analysis of high-frequency oscillations [Leonov & Seledzhi (2005a,b); Kuznetsov et al. (2008, 2009a,b); Kudryashova et al. (2010); Kuznetsov et al. (2010); Leonov et al. (2010)].

2.1 Description of the classical PLL in the signal space

Consider the classical PLL on the level of electronic realization (Fig. 1)

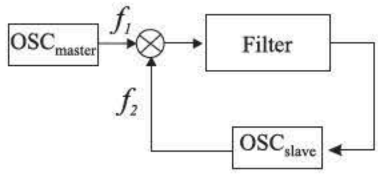

Fig. 1. Block diagram of PLL on the level of electronic realization.

Here OSC_{master} is a master oscillator, OSC_{slave} is a slave (tunable voltage-controlled) oscillator, which generates oscillations $f_j(t)$ with high-frequencies $\omega_j(t)$.

Block $\otimes$ is a multiplier of oscillations of $f_1(t)$ and $f_2(t)$ and the signal $f_1(t)f_2(t)$ is its output. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter has the form

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,d\tau. \tag{1}$$

Here $\gamma(t)$ is an impulse transient function of filter, $\alpha_0(t)$ is an exponentially damped function, depending on the initial data of filter at the moment $t = 0$.

In the simplest "ideal" case, when

$$f_1 = \sin(\omega_1),\ f_2 = \cos(\omega_2)$$
$$f_1 f_2 = [\sin(\omega_1+\omega_1) + \sin(\omega_1-\omega_1]/2,$$

standard engineering assumption is that the filter removes the upper sideband with frequency $\omega_1(t) + \omega_2(t)$ from the input but leaves the lower sideband $\omega_1(t) - \omega_2(t)$ without change. Here to avoid this "trick" we consider mathematical properties of high-frequency oscillations.

A high-frequency property of signals can be reformulated as the following condition. Consider a large fixed time interval $[0, T]$, which can be partitioned into small intervals of the form

$$[\tau, \tau+\delta],\ \tau \in [0, T],$$

where the following relations

$$|\gamma(t)-\gamma(\tau)| \le C\delta,\ |\omega_j(t)-\omega_j(\tau)| \le C\delta,$$
$$\forall t \in [\tau, \tau+\delta],\ \ \forall \tau \in [0, T], \tag{2}$$

$$|\omega_1(\tau)-\omega_2(\tau)| \le C_1,\ \forall \tau \in [0, T], \tag{3}$$

$$\omega_j(t) \ge R,\ \forall t \in [0, T] \tag{4}$$

are satisfied.

We shall assume that δ is small enough relative to the fixed numbers T, C, C_1 and R is sufficiently large relative to the number δ : $R^{-1} = O(\delta^2)$.

The latter means that on small intervals $[\tau, \tau+\delta]$ the functions $\gamma(t)$ and $\omega_j(t)$ are "almost constant" and the functions $f_j(t)$ on them are rapidly oscillating. Obviously, such a condition occurs for high-frequency oscillations.

2.2 Calculation of phase detector characteristic for sinusoidal signals

Consider now harmonic oscillations

$$f_j(t) = A_j \sin(\omega_j(t)t + \psi_j),\quad j = 1, 2, \tag{5}$$

where A_j and ψ_j are certain numbers, $\omega_j(t)$ are differentiable functions.

Consider two block diagrams shown in Fig. 2 and Fig. 3.

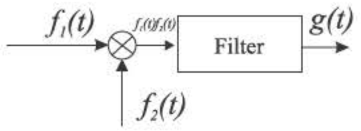

Fig. 2. Multiplier and filter.

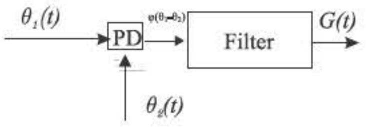

Fig. 3. Phase detector and filter.

On Fig. 3 $\theta_j(t) = \omega_j(t)t + \psi_j$ are phases of the oscillations $f_j(t)$, PD is a nonlinear block with the characteristic $\varphi(\theta)$ (being called a phase detector or discriminator). The phases $\theta_j(t)$ are the inputs of PD block and the output is the function $\varphi(\theta_1(t) - \theta_2(t))$. The shape of the phase detector characteristic is based on the shape of input signals.

The signals $f_1(t)f_2(t)$ and $\varphi(\theta_1(t) - \theta_2(t))$ are inputs of the same filters with the same impulse transient function $\gamma(t)$. The filter outputs are the functions $g(t)$ and $G(t)$, respectively.

A classical PLL synthesis for the sinusoidal signals is based on the following result:

Theorem 1. (Viterbi, 1966) If conditions (2)–(4) are satisfied and we have

$$\varphi(\theta) = \frac{1}{2}A_1A_2\cos\theta,$$

then for the same initial data of filter, the following relation

$$|G(t) - g(t)| \le C_2\delta, \ \forall t \in [0,T]$$

is satisfied. Here C_2 is a certain number being independent of δ.

2.3 Mixed signal

Consider one sinusoidal signal and one impulse signal

$$f_1(t) = A_1\sin(\omega_1(t)t + \psi_1) \tag{6}$$

$$f_2(t) = A_2\,\mathrm{sign}\sin(\omega_2(t)t + \psi_2) \tag{7}$$

Theorem 2. If conditions (2)–(4) are satisfied and we have

$$\varphi(\theta_1 - \theta_2) = \frac{2A_1A_2}{\pi}\cos(\theta_1 - \theta_2) \tag{8}$$

then for the same initial data of filter, the following relation

$$|G(t) - g(t)| \le C_3\delta, \quad \forall t \in [0,T] \tag{9}$$

is satisfied. Here C_3 is a certain number being independent of δ.

2.4 Impulse signals

Consider impulse signals

$$f_j(t) = A_j\,\mathrm{sign}\sin(\omega_j(t)t + \psi_j). \tag{10}$$

Theorem 3. If conditions (2)–(4) are satisfied and we have

$$\varphi(\theta) = A_1A_2\left(1 - \frac{2|\theta|}{\pi}\right), \quad \theta \in [-\pi,\pi], \tag{11}$$

then for the same initial data of filter, the following relation

$$|G(t) - g(t)| \le C_4\delta, \ \forall t \in [0,T]$$

is satisfied. Here C_4 is a certain number being independent of δ.

Proof.

Consider a differentiable 2π-periodic function $g(x)$, having two and only two extremums on $[0,2\pi]: g^- < g^+$, and the following properties.

For any number $\alpha \in (g^-, g^+)$ there exist two and only two roots of the equation $g(x) = -\alpha$:

$$0 < \beta_1(\alpha) < \beta_2(\alpha) < 2\pi.$$

Consider the function

$$F(\alpha) = 1 - \frac{\beta_2(\alpha) - \beta_1(\alpha)}{\pi} \tag{12}$$

if $g(x) < -\alpha$ on $(\beta_1(\alpha), \beta_2(\alpha))$ and the function

$$F(\alpha) = -\left(1 - \frac{\beta_2(\alpha) - \beta_1(\alpha)}{\pi}\right)$$

if $g(x) > -\alpha$ on $(\beta_1(\alpha), \beta_2(\alpha))$.

Suppose, $a < b$, ω is sufficiently large relative to the numbers a, b, α, π.

Then from (12) we have

$$\int_a^b \mathrm{sign}[\alpha + g(\omega t)]dt = F(\alpha)(b-a) + O(\frac{1}{\omega}). \tag{13}$$

It is readily seen that

$$g(t) - \alpha_0(t) = \int_0^t \gamma(t-s)A_1A_2\,\mathrm{sign}[\cos((\omega_1(s) -$$
$$-\omega_2(s))s + \psi_1 - \psi_2) - \cos((\omega_1(s) + \omega_2(s))s +$$
$$+\psi_1 + \psi_2)]ds =$$
$$= A_1A_2\sum_{k=0}^{m}\gamma(t-k\delta)\left[\int_{k\delta}^{(k+1)\delta}\mathrm{sign}[\cos((\omega_1(k\delta) -\right.$$
$$-\omega_2(k\delta))k\delta + \psi_1 - \psi_2) - \cos((\omega_1(k\delta) + \omega_2(k\delta))s +$$
$$\left. +\psi_1 + \psi_2)]ds + O(\delta^2)\ \right],\ t \in [0,T].$$

Here the number m is such that

$$t \in [m\delta, (m+1)\delta].$$

By (13) this implies the estimate

$$g(t) = \alpha_0(t) + A_1A_2\left(\sum_{k=0}^{m}\gamma(t-k\delta)\varphi(\theta_1(k\delta) -\right.$$
$$\left. -\ \theta_2(k\delta))\delta\right) + O(\delta) = G(t) + O(\delta).$$

This relation proves the assertion of Theorem 3.

Theorems 1 and 2 can be proved in a similar way.

2.5 Description of PLL in phase-frequency space

From Theorem 1, 2, and 3 it follows that block-scheme of PLL in signal space (Fig. 1) can be asymptotically changed

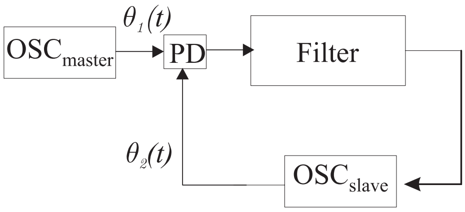

Fig. 4. Phase-locked loop with phase detector

(for high-frequency generators) to a block-scheme on the level of frequency and phase relations (Fig. 4)

Here PD is a phase detector with corresponding characteristics. Thus, here on basis of asymptotical analysis of high-frequency pulse oscillations a characteristics of phase detector can be computed.

Characteristic $\varphi(\theta)$, computed here, tends to zero if $\theta = (\theta_1 - \theta_2)$ tend to $\pi/2$, so one can proceed to stability analysis [Leonov (2006); Kuznetsov (2008); Leonov et al. (2009)] of differential (or difference) equations depend on the misphasing θ.

Finally it may be remarked that for modern processors a transient process time in PLL is less than 10 sec. and a frequency of clock oscillators attains 10Ghz. Given $\delta = 10^{-4}$ (i.e. partitioning each second into thousand time intervals), we obtain an expedient condition

$$\omega^{-1} = 10^{-10} = 10^{-2}(\delta^2) = O(\delta^2).$$

for the proposed here asymptotical computation of phase detectors characteristics.

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Vol. 1

Renat V. Yuldashev

Nonlinear Analysis and Synthesis of Phase-Locked Loops

In modern science and technology, devices that automatically adjust the frequency of quasiperiodic processes to achieve certain phase relationships between them are very important. Examples of such devices are electric generators and motors, artificial cardiac pacemakers, and computer architectures. One of the technical solutions used in the context of these problems is to utilise a phase-locked loop (PLL). In the present study, several conditions that enable high-frequency signals have been formulated. Also methods of asymptotic analysis for the signals with discontinuous waveforms have been developed. The main result of the study is the development of formulas for the characteristics of the phase detector (PD) for a PLL and for a PLL system with a squarer that allows one to derive differential equations for the circuits of that kind.

Vol. 2

Marat V. Yuldashev

Nonlinear Mathematical Models of Costas Loops

This work is devoted to the development of nonlinear mathematical models of Costas loops. In this work, nonlinear mathematical models of the classic Costas loop and he Quadrature Phase Shift Keying (QPSK) Costas loop have been developed. All theoretical results have been rigorously proved. An effective numerical procedure for the simulation of Costas loops based on the phase-detector characteristics is proposed. The results of the study have been published in 22 papers (8 of which are indexed in Scopus).

Vol. 3

Maria A. Kiseleva

Oscillations and Stability of Drilling Systems:
Analytical and Numerical Methods

In this work two mathematical models of drilling systems actuated by an induction motor have been developed. For the first model of a drilling system with absolutely rigid drill string, a limit load problem has been solved. An appropriate load characteristic represented as a non-symmetrical dry friction is introduced. It is shown that, under certain conditions, the limiting value of the permissible rapidly alternating load can be defined as the maximum constant load under which the system has stable state. The second model is a double-mass mathematical model of a drilling system actuated by an induction motor. This model has been analysed in the Matlab package. Through this analysis, hidden oscillations have been found.

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