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ON HOW LOGIC BECAME FIRST-ORDER

The logical systems within which Frege, Schröder, Russell, Zermelo and other early mathematical logicians worked were all higher-order. It was not until the 1910s that first-order logic was even distinguished as a subsystem of higher-order logic.[1] As late as in the 1920s, higher-order quantification was still quite generally allowed: in fact, it does not seem as if any major logician, among non-intuitionists, except Thoralf Skolem restricted himself to first-order logic.[2] Proofs were sometimes allowed to be infinite and infinitely long expressions were allowed in the languages that were used.

Today, however, first-order logic has gained considerable dominance. Neither higher-order quantification nor infinite expressions and proofs are standardly allowed within logic. In textbooks of logic, what is taught is standard first-order logic.[3] Zermelo-Fraenkel set theory and Peano arithmetic are almost always formalized in first-order languages.

In this paper I pose the question: how did it happen that first-order logic became so dominant? In particular, I am interested in why higher-order elements were excluded from logic.

Thoralf Skolem's work was undeniably of great importance for this development. Skolem presented the earliest first-order axiomatizations of set theory and arithmetic. Moreover, he first proved the Löwenheim-Skolem theorem for standard first-order logic: and it is in part by virtue of the fact that this theorem holds for first-order logic that first-order logic has a neat model theory. What I wish to do in this essay, however, is to refute the currently popularly held claim that Skolem also had quite a different kind of influence on the development toward

[1] See Moore 1988, pp. 113f. It was Hilbert who in a course he gave in the winter semester of 1917-1918 first introduced first-order logic as a distinct (sub)system of logic.

[2] See, e.g., Skolem 1922 and Skolem 1928.

[3] By standard first-order logic, I mean first-order logic without such extravaganzas as infinite expressions and infinite proofs.

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first-order logic: that Skolem's explicit arguments for adherence to first-order logic had a notable impact on this development.

It appears to be widely held today that arguments from Skolem and Kurt Gödel, both alleged proponents of the thesis that standard first-order logic is "*the* logic" (or, if you like, "the *true* logic" or "the *correct* logic"; below I will say something about just what preferring first-order logic as "the" logic might come to), had notable impact on the development toward first-order logic: that they, by means of rhetorically convincing arguments, helped effect the transition from higher-order systems to first-order systems being regarded as standard. For example, in Moore 1980 and Moore 1988, two standard works on the emergence and eventual dominance of first-order logic, and in Shapiro 1991, a very influential book on the first-order vs. second-order logic issue, the importance of Skolem and, to a somewhat lesser extent, Gödel is strongly emphasized. (There are certain differences between Moore and Shapiro and also between Moore's two works. For example, in Moore 1988, only Skolem's role is emphasized. Since I am primarily interested in the claim that there was an (influential) Skolem-Gödel proposal and not in Moore- and Shapiro-exegesis I will not here go into these differences.)

In Moore 1980 it is said that

> after 1930 mathematical logic became increasingly identified with first-order logic. The logicians (such as Gödel and Skolem) who had argued for a more restrictive logic had triumphed. (Moore 1980, p. 129)

And Stewart Shapiro says that

> From [the late 30s] the explicit controversy over higher-order languages subsided, and most logicians began to accept the Skolem-Gödel proposal that only the first-order languages are appropriate for their work. (Shapiro 1991, pp. 192f.)

Neither Moore nor Shapiro exclusively emphasize Gödel's and Skolem's arguments: but they both talk about a Gödel-Skolem proposal, a (heated) controversy over the first-order vs. second-order logic issue, and an eventual acceptance on the part of (the majority of) other logicians of the Gödel-Skolem proposal; and they emphasize this aspect of the development.

My main aim with this article is to show that there was no "Skolem-Gödel proposal" at all.[4] And, of course, still less did other logicians eventually accept such a proposal.

[4] *Later*, Skolem came to defend, vigorously, the hegemony of first-order logic; but in Skolem 1922, which is the paper to which Moore and Shapiro refer, he did not present anything resembling an argument for adherence to first-order logic. Indeed, you will not find even the thesis that first-order logic is to be adhered to.

If Skolem and Gödel meant to present arguments, these were—as Moore and Shapiro both recognize—quite clearly bad ones. Indeed, Shapiro prefers to say not that Skolem and Gödel *argued* for first-order logic, but that they *urged* that first-order logic was the full logic.[5] Hence, what I am doing here is in part defending the rationality of the community of logicians: I defend Skolem and Gödel from the charge that they presented obviously bad arguments and I defend other logicians from the charge that they were convinced by these obviously bad arguments.

The thesis that (standard) first-order logic is *the* logic (or "true" logic or "full" logic or whatever you like), I henceforth call the AFOL thesis (AFOL stands for Adherence to First-Order Logic). The historical triumph of first-order logic will be called the AFOL development. The view on AFOL development I wish to refute in this article is the one that arguments, or at any rate "urgings", from Skolem and Gödel caused the AFOL development.

1.

Before I go on to criticize the received view on the AFOL development, I should say a few more general things about the first-order vs. second-order logic issue itself. There are lots and lots of differences between logical systems and lots and lots of different ways in which a logical system (or a certain aspect of a logical system) can be preferred over another. When, as in this essay, we only compare first-order systems with higher-order systems, we are only interested in differences with respect to expressive resources and semantics. Were we to compare, e.g., classical and intuitionistic systems, other differences would have to be taken into account.[6] I can think of at least three fairly distinct kinds of reasons for preferring one logical system over another, even when the logical systems differ only with respect to their order. First, you might have *pragmatic* reasons for preferring one system over another: you might think that more fruitful research can be conducted within the one system. Certainly, such considerations have been influential when first-order logic has been preferred over second-order logic: for in virtue of the fact that first-order languages equipped

[5] Shapiro 1991, p. 181.

[6] In this essay I ignore the existence of second-order languages with Henkin- or first-order semantics. This is because such systems were not taken into account at the time of the AFOL development. (See Shapiro 1991, section 4.3 for an account of such semantics for second-order languages.)

with the standard semantics, as opposed to second-order languages equipped with the standard semantics, are compact, complete and have the Löwenheim-Skolem property, they have a much nicer model theory. Another kind of consideration, which favours second-order systems over first-order systems, concerns *relative faithfulness* to *mathematics* and *mathematical practice*. A logical system which more closely captures ways of reasoning employed within mathematics and which has enough resources to capture concepts presupposed to be well understood within classical mathematics is by such considerations preferred over a logic which does not have these features, for it is then, it is thought, more useful for the study of (the foundations of) mathematics.[7] As concepts such as finitude, well-ordering, well-foundedness and so on—which are presupposed as well understood in mathematics—can be characterized within second-order logic but not within first-order logic, second-order logic is favoured by considerations of this kind.[8] A third broad category of reasons for preferring one logical system over the other is what I shall call *straightforwardly philosophical* or *conceptual* reasons. Such considerations may for example proceed from considerations of what logic, or logicality, really is. Given a characterization of logicality, the system which embodies all and only logical principles, in which all and only logical concepts can be characterized, and given which all and only logical truths are determined as true gets to be called the (full) logic.

It seems that someone who prefers, say, second-order logic for reasons having to do with relative faithfulness to mathematical practice and someone who prefers, say, first-order logic for conceptual reasons do not really have a genuine disagreement, but at best a verbal disagreement over the use of the word "logic": for there is no reason to

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[7] See Corcoran 1973, especially pp. 34ff. and Shapiro 1991, chapter V.

[8] Let us, by way of illustration, show that finitude is not characterizable in first-order logic. If finitude is characterizable in first-order logic, there is a set of first-order sentences whose models are precisely those with finite domains. We show that from the assumption that a set of sentences Γ has finite models of arbitrary finite cardinality it follows that it also has an infinite model and hence that finitude is not characterizable within first-order logic. Let ϕ_n be a sentence of pure first-order logic expressing that there are at least n objects. We show that $\Delta = \Gamma \cup \{\phi_n : n \in N\}$ has a model. Since first-order logic is compact it suffices to show that every finite subset of Δ has a model. Let Θ be a finite subset of Δ. From our definitions it immediately follows that there is an m such that Θ is a subset of $\Gamma \cup \{\phi_n : n \leq m\}$. Since Γ by assumption has models of arbitrary finite cardinality, Θ then has a model. Hence all finite subsets of Δ have models. Δ accordingly has a model, and this model is an infinite model of Γ.

On the other hand, finitude is characterizable in standard second-order logic by the single sentence $\mathrm{FIN}(X) = \neg\exists f[\forall x \forall y(fx = fy \rightarrow x = y) \wedge \forall x(Xx \rightarrow Xfx) \wedge \exists y(Xy \wedge \forall x(Xx \rightarrow fx \neq x))]$.

think that the system which most closely mirrors mathematical practice also happens to be the system which, for conceptual reasons, is to be called (full) logic. I am inclined to think that appearances are not deceiving. However, if the disagreement is not merely verbal, but there is a real disagreement here, it would seem to have to be about whether there is any real feature of logicality: the proponent of second-order logic might deny that and thus rob the proponent of first-order logic of the ground for his preferences; the proponent of first-order logic might try to establish the existence of a real feature of logicality and thus show that even though, for some purposes, second-order systems are preferable to first-order systems, they are not properly called logical.

There is also reason to distinguish between preference for a logic in, or as the underlying logic of, the metalanguage and in, or as the underlying logic of, the object language. There might be reasons for studying formalizations of theories in which rather weak logics are used while keeping the metalanguage rich in expressive resources and capable of characterizing notions like, say, cardinality and finitude. Ignacio Jané (1993) argues that set theory for the following reason should not be formalized in a higher-order language: when studying set theory we are interested in properties of the notion of *subset of*, but this notion is presupposed as well understood in the semantics of second-order logic, for example in that the second-order variables range over the powerset of the domain. Set theory should accordingly rather be formalized in first-order logic, where the notion of *subset of* is not supposed to be well-understood. This is a presumed reason for doing set theory, and, since virtually all of mathematics can be reduced to set theory, mathematics generally, within a first-order framework. But it seems that one can buy into Jané's argument while holding that the metalanguage should be strong enough to characterize central notions like cardinality and categoricity.

The reason why considerations such as this are relevant to the present issue is that the (supposed) evidence that Skolem adhered to first-order logic is that Skolem held that set theory and arithmetic should be given first-order axiomatizations, whereas at least for Shapiro the evidence that Gödel adhered to first-order logic is the (alleged) fact that Gödel insisted on a first-order metalanguage. Also, it is certainly not clear that Skolem (1922) held that the underlying logic of the metalanguage should be first-order. As Paul Benacerraf has argued,[9] Skolem argued that because of the Löwenheim-Skolem theorem, set-theoretic notions cannot be captured within any formal

[9] See Benacerraf 1985, p. 93.

system; but they are perfectly well understood within informal language. And since Skolem's metalanguage is informal it seems that in so far as one can speak of its underlying logic, its underlying logic must not have the Löwenheim-Skolem property. Needless to say, Skolem neither had the concept of a metalanguage nor that of the underlying logic of a language. Often the metalanguage is left informal also today, and I would think that it is an open question with what right one may speak of the underlying logic of an informal language.

I should note that quite probably, philosophers and mathematical logicians typically have different reasons for preferring one logical system over another. However, both philosophers and mathematical logicians have tended to adhere to first-order logic. That mathematical logicians have tended to adhere to first-order logic is for example seen from the fact that ZF and PA are almost always formalized as first-order theories. Another story needs to be told in order to establish that philosophers have adhered to first-order logic. There is no space here to tell this story but ingredients would be: Quine's influence, the fact that philosophers within the logical positivist tradition almost exclusively formalized scientific theories as first-order theories, the fact that Skolem's paradox (which we will talk about later) has been considered a genuine problem, and (as we also will touch upon later) the fact that acceptance of second-order logic as logic has been widely believed to entail a classification of certain clearly nonlogical truths as logical.

I believe, however, that we all like to think that within the *metalanguage*, notions like finitude and categoricity are not in the least relative. It seems then that we are committed to one of the theses (i) the underlying logic of the metalanguage is stronger than first-order logic; (ii) it is not legitimate to speak of the (or a) logic of the (informal) metalanguage; and (iii) there is something wrong with the Skolemite reasoning which leads us to think that because the underlying logic of the metalanguage is not stronger than first-order logic, we cannot with confidence speak of absolute concepts of, say, finitude and categoricity.

2.

The time has now come to discuss the idea that there was a Skolem-Gödel proposal.

Let us start by discussing what impact Skolem might have had on the AFOL development. Moore and Shapiro both appear to hold that

Skolem 1922 is the article by Skolem which has been of most importance for it.[10] Let us hence take a look at that article.

There are three things Skolem does there which need concern us. (i) Skolem presents the first ever first-order axiomatization of Zermelo set theory (which later has developed into *Zermelo-Fraenkel set theory*); (ii) Skolem proves a version of the Löwenheim-Skolem theorem downwards (in fact, Skolem for the first time proves the Löwenheim-Skolem theorem downwards for standard first-order logic); (iii) Skolem presents, for the first time, what later has become known as the Skolem paradox.

The Skolem paradox runs as follows. There is a theorem of ZF—and of all other standard set theories—which says that there is at least one nondenumerable set; more specifically, the set of real numbers, commonly called R, is such a set. At the same time, the Löwenheim-Skolem theorem downwards entails that ZF has a denumerable model. Hence, in a denumerable model it is true that there is a nondenumerable set. Paradox.

This paradox arises if we formalize set theory (and hold that set theory only can or should be formalized) in standard first-order logic: in standard second-order logic (where the second-order variables range over the whole power set of the domain) none of the Löwenheim-Skolem theorems holds. For us to obtain the paradox it is essential that we restrict ourselves to first-order logic or a logic with sufficiently similar features (i.e. a logic with the Löwenheim-Skolem downwards property).

Formally solving, or dissolving, the Skolem paradox is quite simple. What the theorem of ZF referred to above literally says is that there is no one-one relation from R into N, the set of natural numbers. Intuitively, we take the existential quantifier of the theorem[11] as ranging over all relations over $N \times R$, but within a given model the quantifier ranges only over relations in that model. We must distinguish between the N and the R of a given model and the "real" N and R. This is a useful way of putting it: from outside a particular denumerable model for ZF the set of real numbers of that model is denumerable, but from within that model it is nondenumerable since the conditions of the theorem we have referred to are satisfied: in the model there is no one-one relation from R into N.

[10] Shapiro 1991, pp. 181–185; Moore 1988, pp. 95 and 123. In the discussion of Skolem's role in Moore 1980 (pp. 113f. and 120ff.), Skolem 1922 is the paper which is discussed the most.

[11] The theorem may be formulated, schematically,

$$\neg\exists F(F \text{ is a one-one relation that holds between } N' \subset N \text{ and } R).$$

This "formal solution" to the paradox is the one presented by Skolem himself; but although Skolem knows the paradox to have this straightforward solution he thinks it has far-reaching philosophical implications. The conclusion Skolem (1922) draws from the paradox is that the notions of set theory are not capturable within any logical system. For example, nondenumerability is a set-theoretic notion, and Skolem thinks that since it is true in denumerable models of axiomatized set theory (for Skolem, set theory axiomatized in a first-order language) that there is a nondenumerable set, the notion of denumerability cannot be captured in axiomatized set theory. In later articles, Skolem draws even more radical conclusions from the Skolem paradox: he claims that the set-theoretic notions are *essentially relative*; that they have no absolute content for any formal system to capture. Now, since Skolem drew this conclusion he must either have been confused on the relationship between first- and second-order logic or have subscribed to the AFOL thesis. Second-order logic is otherwise a logic in which the set-theoretic notions, for all that has been shown, are capturable.[12]

As we have seen, Moore and Shapiro both hold that arguments—or "urgings"—from Skolem were of importance for the AFOL development. Furthermore, they hold that Skolem 1922 in particular was important. We have just seen that a case can be made that Skolem subscribed to the AFOL thesis at the time of writing the article. Still, I wish to hold that Moore and Shapiro are wrong. I have five reasons for this. First, neither Moore nor Shapiro even tries to show that Skolem 1922 had a notable *impact* on the subsequent development. And those who have investigated the matter have concluded that the article in fact was not very widely read. Thus, van Heijenoort writes,

> *Skolem 1922* seems to have had few readers. It called forth, so far as I know, only two responses, a review by Fraenkel (1927) and a mention by von Neumann (1925, p. 232)[13]

Second, Skolem (1922) does not *stress* that he formalizes set theory in a first-order language. It is indeed clear that the only sentences that

[12] Löwenheim (1915) had shown that the Löwenheim-Skolem theorem downwards does not hold for second-order logic.

[13] Van Heijenoort 1981, p. 112. See also Goldfarb 1979, p. 357 and van Heijenoort 1967, p. 291. Paul Benacerraf (1985, p. 113) claims that as late as 1929, Zermelo had not read Skolem 1922. This might be some indication of just how unnoticed Skolem 1922 went. I should also point out that the evidence that Skolem's paper went unnoticed *ipso facto* is evidence that Skolem's ideas were not very well known: for the evidence that Skolem 1922 was not very widely read is precisely that other authors seemed ignorant of, and hardly ever commented on, Skolem's ideas.

can be formulated in the formal language Skolem defines in the article are first-order sentences; but there is no emphasis on this fact. Neither does Skolem point out that a global restriction to first-order logic (or, rather, logics with the Löwenheim-Skolem property) is necessary for the potentially philosophically significant consequences of the Skolem paradox to follow. Third, Skolem seems unaware of the novelty of regarding first-order as *the* logic. After having presented what he takes to be the philosophically important consequences of the Skolem paradox, Skolem goes on to say:

> So far as I know, no one has called attention to this paradoxical state of affairs. ... Even the notions of 'finite', 'infinite', 'simply infinite sequence' and so forth turn out to be merely relative within axiomatic set theory. (Skolem 1922, p. 295)

Had Skolem been aware of the novelty of formalizing set theory in a first-order language, or, generally, of the difference between first- and second-order logic, he would not have expressed himself like that. Skolem's way of axiomatizing set theory was *new*. There was no paradoxical state of affairs to call attention to before Skolem formalized set theory in a first-order language. It really does seem as if Skolem was confused on the relation between first- and second-order logic. Maybe it seems strange that a logician could be confused concerning what holds within a logic of one order and what holds within a logic of another order; but Gregory Moore (1988) actually argues forcefully that both Fraenkel and von Neumann in the 1920s were confused on the very same issue.[14] So maybe it should not strike us as that odd that the same went for Skolem. We should note that Skolem does not really make explicit in his 1922 paper that his system is first-order. What he does when characterizing the language of his system is that he says propositions are built from simple propositions of the form $a = b$ and $a \in b$ by means of the five operations conjunction, disjunction, negation, universal quantification and existential quantification. It is clear that the system is first-order in virtue of (i) Skolem's reasoning in his proof of the Löwenheim-Skolem theorem; (ii) Skolem's use of the term "Zählaussage" to denote the propositions formed in his system: when introducing the term "Zählaussage", Skolem explicitly refers to Löwenheim's term "Zählausdruck", which unambiguously meant first-order expression; (iii) the fact that in a proposition of the form $a \in b$ in Skolem's system, b is of the same logical type as a. But then Skolem also says that propositions are formed by means of the five operations mentioned "in the sense of Schröder (1895)", and this indicates that

[14] Moore 1988, pp. 124f.

Skolem's system is precisely that of Schröder: but Schröder's system is higher-order.[15] Fourth, in Skolem 1922 there is not even a hint of an argument (not even an "urging") as to why first-order logic should be regarded as *the* logic. Fifth, why would logicians after having studied Skolem 1922 go over to working within the framework of first-order logic when in that article attention was put to a potential major drawback of doing so: the Skolem paradox? If they had completely grasped the difference between first- and second-order logic, they should have seen from Skolem's article that important set-theoretic notions cannot be captured within first-order logic.

The result proved in the article—the Löwenheim-Skolem theorem downwards for (standard) first-order logic—of course provides an incentive to work within a first-order framework: it is for example in virtue of the fact that first-order logic has the Löwenheim-Skolem property that it has a "nice" model theory. But the fact that Skolem 1922 provided this kind of incentive does nothing to support Moore's and Shapiro's case: for Skolem's alleged arguments for first-order logic had nothing to do with the nice model theory for first-order logic.

I conclude that we have good reason to believe that Skolem 1922 did not have the kind of influence on the AFOL development that Moore and Shapiro take it to have had.

The closest you ever get, in Skolem's writings, to an argument for adherence to first-order logic is in Skolem 1928. There, Skolem re-

[15]Skolem 1922, p. 293. The whole passage reads as follows: "By a *first-order proposition* [*Zählaussage*] (Löwenheim says "first-order expression" ["Zählausdruck"]) is meant a finite expression constructed from class and relative coefficients in the sense of Schröder (1895) by means of the five logical operations mentioned above." Thus, Skolem talks about the systems of Löwenheim, whose system is first-order, and of Schröder, whose system is higher-order, as though they were identical. This could be some indication that there was confusion on Skolem's part. Löwenheim, as opposed to Skolem, was clearly not confused on this issue. When Löwenheim defines first-order expressions ("Zählausdrücke") he does it as follows: "A relative expression in which every Σ and Π ranges over the subscripts, that is, over the individuals of 1^1 (in other words, none ranges over the relatives), will be called a *first-order expression*" (1915, p. 233). Löwenheim also states that the Löwenheim-Skolem theorem does not hold for Schröder's system. Oddly, Skolem is in Skolem 1920 as careful as Löwenheim to make clear that the only quantification allowed is first-order. There he says "A first-order proposition [Zählaussage] is a proposition constructed from relative coefficients in the sense of Schröder by means of the five operations mentioned above, with productations [universal quantifiers] and summations [existential quantifiers] ranging over individuals only" (p. 254). It would be strange if Skolem in his 1922 paper was confused on the relation between first-order and second-order logic when in Skolem 1920 he seems to have appreciated this distinction.

gards first-order quantification as unproblematic, whereas he regards second-order quantification, or, as Skolem puts it, *quantification over propositional functions*, as at least potentially problematic. Skolem thinks that it is not clear just what the totality of propositional functions is, and that for quantification over propositional functions to be admissible it has to be *made* clear what this totality is. So he asks how the class of all propositional functions can be defined.[16] He finds two "scientifically tenable" answers.[17] One is that one could develop a theory of propositional functions analogous to the theory of sets that was already developed. (Skolem adds that the concept of a propositional function pretty well corresponds to the concept of a set.[18]) Such a theory would naturally have to be in a first-order language, whence this suggestion in reality has the immediate consequence that second-order logic is abolished.

The other "scientifically tenable" answer is that we should first form all first-order propositional functions. These form a well-defined totality. Then we may introduce second-order quantification as quantification over first-order propositional functions and so on. It is not entirely clear just what this suggestion comes to; but I cannot see but that under neither plausible interpretation of Skolem's suggestion is Skolem relativity avoided, and under neither interpretation does Skolem's suggestion amount to a justification for full second-order logic. One possible interpretation is that Skolem means that we should first form the first-order propositional functions as *syntactical* entities and then introduce *substitutional* second-order quantification with the second-order variables ranging over these countably many first-order propositional functions. Another possibility is that Skolem thinks that we should first give a domain for the first-order variables, independently of any assignment of values to second-order variables, and then let the second-order variables range over the powerset of the domain for the first-order variables.[19]

Either way, only a very mutilated second-order logic is introduced. If we only allow substitutional second-order quantification, the second-order variables range over at most denumerably many entities, given the usual restriction to a countable language. Neither the full strength of the axiom of induction of PA nor the full strength of the replacement

[16] Skolem 1928, p. 516.

[17] *Ibid.*

[18] Skolem belonged to Schröder's algebraic school, whose formal calculus could be interpreted, and was interpreted, both as a calculus of classes and as a calculus of propositions.

[19] Skolem 1928, p. 516f.

axiom of ZF can be captured. And if we first separately assign values to the first-order variables, we have Skolem relativity at hand. For in the class of models for a consistent set of first-order sentences (in a countable language), there are, according to the Löwenheim-Skolem theorem downwards, denumerable models. And the cardinality of the domain does not change when we assign values to the second-order variables.

It should be clear that this argument of Skolem's rests upon the *presupposition* that there is something problematic about second-order quantification. Hence there is no reason to believe that Skolem 1928 had decisive impact on the AFOL development: either Skolem's suspicion toward second-order quantification was not widely shared, in which case his argument should not be very convincing, or there was widespread suspicion toward second-order logic already when Skolem published his 1928 paper, in which case we should look toward the sources of this suspicion.

Let us now examine Moore's and Shapiro's contention that arguments (or, as Shapiro prefers to call them, urgings) from Gödel were important for the AFOL development. The only text they refer to when arguing Gödel's importance for that development is the Gödel-Zermelo correspondence of 1931.[20] Shapiro alleges that Gödel there "insisted on a finitary and first-order metalanguage".[21] Moore says that "The question of which logic was appropriate for set theory—first-order logic, second-order logic, or an infinitary logic—culminated in a vigorous exchange between Zermelo and Gödel around 1930".[22] However, neither Moore nor Shapiro provides us with any quotation from the Gödel-Zermelo correspondence showing it to be the case that Gödel there did insist on a first-order metalanguage.

We may also wonder how what Gödel wrote in private correspondence can be enough for Shapiro to talk about a "Skolem-Gödel proposal", eventually accepted by the majority of logicians: it would certainly have to be shown that any arguments for a first-order logic Gödel presented in his correspondence with Zermelo were known by (prominent persons in) the community of logicians for Shapiro's claim that the community of logicians eventually accepted a proposal from Gödel to be made good.

[20]The Gödel-Zermelo correspondence is published in Grattan-Guinness 1979 and Dawson 1985.

[21]Shapiro 1991, p. 191.

[22]Moore 1980, p. 95. As I said above, Moore no longer emphasizes Gödel's role in Moore 1988.

Furthermore, if we study Gödel's letter to Zermelo, we find that Gödel there, though he does insist on an—in certain respects—finitary metalanguage (Gödel did insist, as against Zermelo, that proofs are, by their very nature, finite[23]) does not explicitly insist on a first-order language.

In Moore 1980, we find the claim that Gödel in the famous article "Über formal unentscheidbare Sätze der *Principia mathematica* und verwandter Systeme I" (1931), in which Gödel presented his incompleteness theorem, Gödel worked within a system of first-order logic:

> Shortly afterward [after 1930, when Gödel published his completeness theorem for first-order logic] Gödel published an abstract of his incompleteness theorem, which made a positive solution to the *Entscheidungsproblem* highly unlikely. The theorem stated that Peano arithmetic contains undecidable propositions if formulated in the (first-order) logic of *Principia mathematica.* (Moore 1980, p. 125)[24]

Presumably, if Gödel (1931) worked within the framework of first-order logic, this would lend some support, if not much, to the claim that Gödel only thought first-order languages appropriate. I guess this is the reason why Moore (1980) claims that Gödel did restrict himself to such a language.

Moore's claim is, of course, false. In his 1931 article, Gödel worked within a version of type theory:

> The primitive signs for the system P [the formal system Gödel proved incomplete (1931)] are the following: (I) Constants ... (II) Variables of type 1 (for individuals, that is, for natural numbers including 0): "x_1", "y_1", "z_1", ... Variables of type 2 (for classes of individuals): "x_2", "y_2", "z_2", ... Variables of type 3 (for classes of classes of individuals): "x_3", "y_3", "z_3", ... And so on, for every natural number as a type. (Gödel 1931, pp. 151, 153)

The axiom of induction is in its full second-order form:

> The following formulas are called *axioms* ...: I. ... (3) $x_2(0).x_1 \prod (x_2(x_1) \supset x_2(fx_1)) \supset x_1 \prod (x_2(x_1))$ [or, in more modern notation, $(x_2(0) \wedge \forall x_1(x_2(x_1) \rightarrow x_2(sx_1))) \rightarrow \forall x_1(x_2(x_1))$; M.E.]. (Gödel 1931, p. 155)

Gödel also makes use of the axioms of extensionality and of comprehension:

> ... (IV) Every formula that results from 1. $(Eu)(v \prod (u(v) \equiv a))$ [in more modern notation, $\exists u \forall v(u(v) \leftrightarrow a)$; M.E.] when for v we substitute any variable of type n, for u one of type $n+1$, and for a any formula that does not

[23] This insistence amounts neither to an insistence that logic (somehow) is finitary nor that the metalanguage should be finitary, as Shapiro intimates; it is merely a reflection on the nature of *proof*.

[24] This claim is, I should add, no longer made in Moore 1988.

contain u free ... (V) Every formula that results from 1. $x_1 \prod (x_2(x_1) \equiv y_2(x_1)) \supset x_2 = y_2$ [$\forall x_1(x_2(x_1) \leftrightarrow y_2(x_1)) \rightarrow x_2 = y_2$] by type elevation. (*Ibid.*)

This is not first-order logic; this is (a version of) type theory.[25] It is somewhat misleading to speak of today's Peano arithmetic as the system which Gödel proved incomplete, since our first-order Peano arithmetic in several respects differs from that system.

There is, indeed, a well-known interpretation of type theory in many-sorted first-order logic, by virtue of the existence of which the system of *Principia mathematica* may be regarded as first-order, albeit many-sorted. But there is no hint in Gödel 1931 that Gödel viewed type theory that way. Also, *today* we would regard classes as being in the range of the first-order variables, but that was emphatically *not* how Russell and most of those who used the system of *Principia mathematica*—for example Gödel—viewed the matter.

Moore also alleges that "Influenced by Hilbert and Skolem, Gödel operated within a ... finitistic tradition of logic. Thus he confined his researches to first-order logic".[26] The latter statement appears to be quite far from the truth. Gödel worked within many different areas and systems. By no means did he confine his researches to standard first-order logic. If we for example take a look at Volume I of Gödel's Collected Works, which contains all of Gödel's published works between 1929 and 1936, we find that Gödel occasionally worked within standard first-order logic (most famously, of course, in the papers in which Gödel presents his proof of the completeness of first-order logic, or, as Gödel calls first-order logic, *der engere Funktionenkalkül*—the narrower functional calculus), in other places worked within type theory (for example in Gödel 1931), and in several other places discussed intuitionistic logic and the propositional calculus. If we also see to Gödel's reviews of the works of other authors, we find that Gödel worked within still other systems.

Similar things can be said concerning Gödel's role in the AFOL development as can be said about Skolem's role in this development. Gödel, like Skolem, undeniably had an important role to play in this development. He proved first-order logic complete and he contributed to the development of von Neumann-Bernays-Gödel (NBG) set theory, which, unlike ZF, is finitely axiomatizable in first-order logic.

But Gödel did not contribute to the AFOL development by presenting arguments for adherence to first-order logic.

[25] See Boolos 1993, p. xx; Dawson 1991, p. 98; Kleene 1986, p. 129.

[26] Moore 1980, p. 129.

3.

If arguments (or "urgings") from Skolem and Gödel did not play a major role in the AFOL development, what did? Of course, there were several causes of the AFOL development. A (most certainly incomplete) list of probable causes will be presented below. What I will do in the remainder of this essay is to add some pieces to the answer to the question of the AFOL development. First I will sketch an account of how the analysis of quantification in the 1920s might have helped cause the AFOL development, and below I will present the (even more tentative) suggestion that there might be a connection between Tarski's model-theoretic analyses of the notions of logical truth and logical consequence (and, quite generally, the emergence of model theory as a mathematical discipline) and the emergence of first-order logic as the *de facto* standard in logic.

Because of the paradoxes that had been discovered (e.g. Russell's and other paradoxes) and to some extent because of the intuitionistic challenge, several logicians in the 1920s felt induced to embrace (Hilbertian) finitism. The idea was to secure (the consistency of) classical logic and classical mathematics by "finitary", and hence epistemologically innocuous, methods. Even logicians not directly belonging to Hilbert's school, like Thoralf Skolem, were clearly influenced by this development.

The *quantifiers* constituted a major obstacle for any finitistic analysis of logic: they brought in the possibly *in*finitary. It was thus important, for Hilbert and those who were influenced by his finitism, to find an acceptable finitary analysis of the quantifiers, so that also sentences with unrestricted quantifiers were, as it were, secured epistemologically.

Several analyses of the quantifiers, intended to make them more innocuous, were offered. In my presentation of such analyses here, I will closely follow Goldfarb 1979. In this paper, Goldfarb argues that these analyses were not only meant as technical devices, but also to capture the very meaning of quantification and to explain how finite intelligences like us can grasp quantification over infinite totalities.

In his proof of the Löwenheim-Skolem theorem downwards (1920), Skolem passes from satisfaction of a formula in a model to the existence of what we today would call *Skolem functions* for the existential quantifiers of that formula. According to Goldfarb, Skolem saw a close connection between existential variables and *choice functions*, and this view was typical for how one in the 1920s came to regard quantifiers.[27] Goldfarb claims that Skolem was of the opinion that the most that could be done by means of existential quantification was to demand there to be a suitable value to choose for the variable existentially quantified over for every assignment of values to the universally quantified variables. An important consequence was that the at most denumerably many Skolem functions cannot lead us "out of the denumerable".[28]

In his 1922 proof of (a slightly different version of) the Löwenheim-Skolem theorem downwards (Skolem 1922), Skolem starts out by defining effective functions over natural numbers, and then shows that these functions can function as an interpretation of the existential variables.[29] Goldfarb claims that this even more accentuates the fact that Skolem viewed choice functions as the most that existential quantifiers can imply the existence of:

> This gives an even sharper form to the idea that choice functions provide an upper bound on the power of quantification. With respect to satisfiability, first-order propositions cannot distinguish their intended objects from the integers; consequently we may take the quantifiers to represent effective functions on the integers. (*Ibid.*)

Influenced by Hilbertian ideas, *Jacques Herbrand* wanted to analyse quantification theory in a finitistically acceptable way. For this purpose he attempted to analyse the provability of quantified sentences in terms of truth-functional validity. In order to do this he introduced the concept of *expansion*.[30] An expansion of an existentially quantified formula is a disjunction of quantifier-free instances of the formula obtained according to certain *instantiation rules*. Herbrand sought to show that a formula is derivable in his axiomatic system if and only if the expansion of its negation is truth-functionally inconsistent. It is the left-to-right implication which is problematic.[31]

Let us now turn to *Hilbert* himself. Hilbert used his ϵ-*operator* to obtain a finitistically acceptable analysis of quantification theory. If $A(x)$ contains no free variables other than x, the ϵ-term $\epsilon_x A(x)$ denotes an object c such that $A(c)$ holds, given that there is such an object; otherwise it denotes an arbitrary object. If $A(x)$ contains free variables other than x, $\epsilon_x A(x)$ represents a function of these variables;

[27] Goldfarb 1979, p. 357. By no means does Goldfarb claim that Skolem *caused* this development.

[28] *Ibid.*

[29] Goldfarb 1979, p. 358.

[30] In the French original, Herbrand uses the term *rèduite*, which literally means reduction.

[31] Goldfarb 1979, p. 364.

this function gives a value for x such that $A(x)$ holds, if there is such a value, given assignments of values to the other free variables. Hilbert provides an axiom schema for the ϵ-operator, $A(t) \rightarrow A(\epsilon_x A(x))$, such that we can define the quantifiers in terms of this operator. $\exists x A(x)$ is defined to be equivalent to $A(\epsilon_x A(x))$, and $\forall x A(x)$ is defined to be equivalent to $A(\epsilon_x(\neg A(x)))$.

ϵ-analysis of quantification is for several reasons very attractive to the finitist: (i) Since every proof is finite, the ϵ-operator is used there only finitely many times, so only finitely many choices of values are ever made; only finitely many instances of the axiom schema for the ϵ-operator are used in a proof; (ii) Only finitely many elements are needed as ϵ-terms. Thus, only finitely many elements are ever taken account of in a proof. According to Hilbert, we can, through the use of ϵ-analysis of quantification, prove the consistency of a logical system by finitary means through repeated assignments of values to the ϵ-terms, which assignments can be made effectively. This bears a marked resemblance to Herbrand's ideas. In a given proof, there are only finitely many quantified sentences, each with only finitely many quantifiers. The quantifiers are analysed by Hilbert with the aid of the ϵ-operator, and by Herbrand by means of the so-called expansions. Goldfarb claims that Herbrand's work was of considerable importance for obtaining the Hilbert-Bernays ϵ-theorems.[32]

Hilbert's ϵ-operator is in effect only a device for analysis of first-order quantification, as I will now argue. Hilbert certainly cannot have thought so himself; on the contrary, ϵ-analysis was supposed to be applicable to quantification in general, and Hilbert allowed also higher-order quantification. But the values of the ϵ-function are explicitly said to be elements of the domain: and only the elements in the range of the first-order variables are elements of the domain. Hence it is natural that even in contexts where Hilbert eventually introduces second-order quantification, the ϵ-operator is used only for analysis of first-order quantification.[33] Hilbert's official view is that all quantification should be analysed by means of the ϵ-operator, but in practice this only goes for first-order quantification. The step to regard only first-order quantification as kosher is not far. Moreover, if infinite objects such as functions defined for infinitely many values or properties such as being a natural number can be values of variables, as is the case when

[32] Goldfarb 1979, p. 365.

[33] See Hilbert 1927, p. 466. I do not mean to say that ϵ-analysis was never used to analyse higher-order quantification. There are papers in which quantification over functions is allowed and where the quantifiers ranging over functions are analysed by means of the ϵ-operator; a prominent example is Ackermann 1924.

higher-order quantification is allowed, nothing is achieved by ϵ-analysis of quantification. For then the question how we can grasp the infinite objects that are being quantified over remains: it has not been explained how finite intelligences can grasp the infinite. We have indeed two distinct problems: one concerns grasp of quantification over infinite domains; the other concerns grasp of infinite objects. But it seems that an answer to the problem of our grasp of infinite objects *ipso facto* should be an answer to the question of how we can grasp quantification over infinite domains: for it would show how, in general, we can grasp infinite totalities. The ϵ-analysis of quantification does not provide us with an answer to the question how we can grasp infinite objects—on the contrary it provides a means for evading that problem—and it seems that any distinct solution to this as yet unsolved problem would render ϵ-analysis of quantification obsolete. Hence there is a conflict between the ϵ-analysis of quantification and allowing infinite objects in the range of the variables.

These remarks on the importance of finitist analyses of quantification for the AFOL development are of course only meant to *add* something to the complex picture of the development: as everyone seems agreed on, the development had several distinct (though presumably related) causes. Among these are (a) the exclusion of set theory from the realm of logic, which exclusion was probably partly caused by the discovery of the set-theoretic paradoxes and the axiomatic treatment of set theory, (b) the then only recently clearly made distinction between syntax and semantics, (c) the presentation of first-order logic as a distinct subsystem of logic in Hilbert and Ackermann's influential textbook (1928) and Gödel's proof of the completeness of this logical system, (d) finitist and nominalist qualms about the objects over which second-order variables were supposed to range, (e) Skolem's first-order formulations of Zermelo set theory and Peano arithmetic and Bernays' and Gödel's first-order formulations of NBG set-class theory. Needless to say, (a)–(e) are not independent of each other; there are connections between them.

I would also like to make another suggestion concerning the AFOL development, one that is even more tentative than the previous one.

In his book *The Concept of Logical Consequence* (1990), John Etchemendy discusses Tarski's analyses of the concepts of logical truth and logical consequence very critically. Etchemendy's contention is that the analyses are faulty: they are not even extensionally correct.

Today, an influential criticism of second-order logic is that within second-order logic, sentences which should not come out logically true or false come out that way. For example, the sentence of pure second-order logic which expresses the continuum hypothesis itself comes out logically true or false, respectively, depending on its truth or falsity in the underlying set theory. So if we hold that the continuum hypothesis is not *logically* true or false, second-order logic is, *qua* logic, seriously deficient. But this line of reasoning depends crucially on our presupposing an understanding of logical truth as truth in all models: Tarski's conception of logical truth. If Etchemendy is right, it is Tarski's analysis and not second-order logic that is deficient.

The relevance of all this to the question of what caused the AFOL development is that if Tarski's analysis of logical truth and logical consequence turns out to give correct results only with regard to first-order logic, this faulty analysis, favouring first-order logic, might seem to coincide too well in time with the development toward first-order for it to be a mere coincidence. There seems then to be a very interesting answer to the question of what the connection is between the acceptance of first-order logic and the acceptance of Tarski's model-theoretic analyses of the notions of logical truth and logical consequence. A plausible suggestion is that the acceptance of first-order logic as standard as well as the acceptance of Tarski's analyses had to do with the emerging interest in *model theory*: as is well known, standard first-order logic has a much nicer model theory than standard second-order logic; and the Tarskian analyses of logical truth and logical consequence are model-theoretical.

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