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On the non-isolation of the real projections of the zeros of exponential polynomials

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Abstract

This paper proves that the real projection of each zero of any function $P(z)$ in a large class of exponential polynomials is an interior point of the closure of the set of the real parts of the zeros of $P(z)$. In particular it is deduced that, for each integer value of $n \geq 17$, if $z_0 = x_0 + iy_0$ is an arbitrary zero of the nth partial sum of the Riemann zeta function $\zeta_n(z) = \sum_{j=1}^{n} \frac{1}{j^z}$, there exist two positive numbers ε_1 and ε_2 such that any point in the open interval $(x_0 - \varepsilon_1, x_0 + \varepsilon_2)$ is an accumulation point of the set defined by the real projections of the zeros of $\zeta_n(z)$.

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1. Introduction

Suppose that $\{g_1, g_2, \dots, g_k\}$ is a set of $k \geq 2$ positive real numbers which are linearly independent over the rationals. Let $n \geq 2$ be an integer number and take $w_j = \sum_{l=1}^{k} c_{j,l} g_l$, $j = 1, 2, \dots, n$, for some integer numbers $c_{j,l} \geq 0$. The purpose of this paper is to study the behavior of the set of the zeros, say Z_P, of the function

$$P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z} \tag{1}$$

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where the m_j's are non-null complex numbers (called coefficients) and the w_j's are positive and distinct (called frequencies).

For every natural number n, each exponential polynomial $P(z)$ is an entire function of order 1. It is easy to prove that the zeros of $P(z)$ are located on a vertical strip [10, Lemma 2.5], the critical strip, bounded by the real numbers a_P and b_P, where

$$a_P := \inf \{\operatorname{Re} z : P(z) = 0\} \tag{2}$$

and

$$b_P := \sup \{\operatorname{Re} z : P(z) = 0\}. \tag{3}$$

These bounds allow us to define an interval $I_P := [a_P, b_P]$ which contains the closure of the set of the real parts of the zeros, denoted by

$$R_P := \overline{\{\operatorname{Re} z : P(z) = 0\}}. \tag{4}$$

In this manner, when $a_P < b_P$, the existence of a real open interval J included in R_P would mean that $P(z)$ possesses zeros arbitrarily close to every vertical line contained in the strip $\{z \in \mathbb{C} : \operatorname{Re} z \in J\} \subset \{z \in \mathbb{C} : a_P < \operatorname{Re} z < b_P\}$.

As far as we know, the first theoretical work on location of open intervals in R_P, with $P(z)$ an exponential polynomial of the form (1), was made by Moreno by assuming that the frequencies of $P(z)$ are linearly independent over the rationals [11, Main Theorem] (see also [9, Theorem 1]). Later, through an auxiliary function and without specific conditions on the real frequencies of $P(z)$, Avellar and Hale [1, Theorem 3.1] obtained a criterion to decide whether a real number is in the set R_P.

On the other hand, about the special case of the partial sums of the Riemann zeta function

$$\zeta_n(z) = \sum_{j=1}^{n} \frac{1}{j^z},$$

there are more specific results along these lines. For example, from Montgomery and Vaughan's work [4, 5], we know that for large enough values of n the exponential polynomial $\zeta_n(z)$ has zeros in a strip of small width close to the line $x = 1 + \left(\frac{4}{\pi} - 1\right) \frac{\log(\log n)}{\log n}$ and the upper bound (3) verifies that $b_{\zeta_n} \leq 1 + \left(\frac{4}{\pi} - 1\right) \frac{\log(\log n)}{\log n}$ for all integer n greater than a certain N_0. Also, we deduce from [2, Theorem 4] that if $z_0 = x_0 + iy_0$ is a zero of order 1 of $\zeta_n(z)$, $n > 2$, there exist two non-negative numbers ε_1 and ε_2, with $\varepsilon_1 + \varepsilon_2 > 0$, such that $[x_0 - \varepsilon_1, x_0 + \varepsilon_2] \subset R_{\zeta_n}$. In this sense, Mora stated in 2013 that there exists an integer N such that $R_{\zeta_n} = [a_{\zeta_n}, b_{\zeta_n}]$ for every $n > N$ [7, Theorem 12], but the proof of this theorem is based on a result which is not true, such as will be shown in the last section of this paper.

In this way, the study of the density properties of the real projections of the zeros of the functions $P(z)$ of the form (1) is the main goal of this paper. In fact, the problem of the density of the zeros of $P(z)$ is related to the following question: are there strips $S_{(a,b)} = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$, contained in the critical strip $\{z \in \mathbb{C} : a_P \leq \operatorname{Re} z \leq b_P\}$, such that $P(z)$ has zeros near any line parallel to the imaginary axis inside $S_{(a,b)}$? We will prove that, under determined conditions, the answer is affirmative by demonstrating that the real part of each zero of $P(z)$ is an interior point of the set R_P.

If $\mathcal{P}_S$ denotes the class of exponential polynomials $P(z)$ of type (1) verifying the conditions of Definition 8, which contains all the sums $\zeta_n(-z) = \sum_{j=1}^{n} j^z$ for integer values of $n \geq 17$, the main results of this paper can be summarized as follows:

i) For each zero $z_0 = x_0 + iy_0$ of a exponential polynomial $P(z)$ in the class $\mathcal{P}_S$ there exist two positive numbers ε_1 and ε_2 such that $(x_0 - \varepsilon_1, x_0 + \varepsilon_2) \subset R_P$ (see Main Theorem, formulated in Theorem 13).

ii) From the result above, for each exponential polynomial $P(z)$ in the class $\mathcal{P}_S$, we deduce that $a_P < b_P$. More so, the real part of the infinitely many zeros of each function in the class $\mathcal{P}_S$ is never a_P or b_P (see Corollary 14).

iii) For every integer number $n \geq 17$ the real part of any zero of $\zeta_n(z)$, the nth partial sum of the Riemann zeta function, is an interior point in R_{ζ_n} (see Corollary 15).

iv) We show in the last section that for the function $G_{20}(z) = \zeta_{20}(-z)$, the condition $x_0 \in R_{G_{20}}$ is not sufficient to assure that there exists some $y \in \mathbb{R}$ such that $A_{G_{20}}(x_0, y) = 0$, where the function A_P is defined in (7). This contradicts [2, Theorem 2], [6, Theorem 2], [7, Theorem 5] and [8, Theorem 3.14] (see Theorem 0 in the present paper), which attaches more significance to the other main results of our work. In this respect, we also point out where the proof of Theorem 0 fails.

2. First results

As we noted in the introduction, it is worth to show a first characterization of R_P, the closure of the set of real projections of the zeros of an exponential polynomial $P(z)$ of type (1), which was given by Avellar and Hale in 1980 [1, Theorem 3.1].

Theorem 1. *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial of type (1) and define the auxiliary function $F_P : \mathbb{R} \times \mathbb{R}^k \to \mathbb{C}$ associated to $P(z)$ as*

$$F_P(x, \mathbf{x}) := 1 + \sum_{j=1}^{n} m_j e^{w_j x} e^{i \sum_{l=1}^{k} c_{j,l} x_l},\ x \in \mathbb{R},\ \mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k. \tag{5}$$

Then $x_0 \in R_P$ if and only if $F_P(x_0, \mathbf{x}) = 0$ for some $\mathbf{x} \in \mathbb{R}^k$.

Now, let

$$P(z)=1+\sum_{j=1}^{n} m_j e^{w_j z},\ n\geq 2,\ m_j\in\mathbb{C}\setminus\{0\} \tag{6}$$

be an exponential polynomial of type (1) with increasing positive frequencies $w_1<w_2<\ldots<w_n$ such that the set of positive numbers $\{g_1,g_2,\ldots,g_k\}$, $k\geq 2$, is a basis of the group $W=\mathbb{Z}w_1+\ldots+\mathbb{Z}w_n$ (rank $W=k\geq 2$) and with the restrictions that $g_k=w_t$ for some $t\in\{1,2,\ldots,n\}$ and $g_k\notin\left\{\sum_{j=1,j\neq t}^{n} r_j w_j, r_j\in\mathbb{Q}\right\}$. In this manner, under these conditions, the group W can be written as a direct sum of the form

$$W=\left(\sum_{j=1,j\neq t}^{n}\mathbb{Z}w_j\right)\oplus\mathbb{Z}w_t.$$

It is worth pointing out that the set of $k\geq 2$ positive numbers $\{g_1,g_2,\ldots,g_k\}$ is not necessarily an ordered set.

Note that the sums $\zeta_n(-z)=\sum_{j=1}^{n} j^z$, $n\geq 3$, are exponential polynomials verifying the conditions above. Indeed, the set $\{\log p_1,\log p_2,\ldots,\log p_{k_n}\}$, where $\{p_1,p_2,\ldots,p_{k_n}\}$ is the set of all prime numbers less than or equal to n, can be used to obtain a basis of the group $W=\mathbb{Z}\log 2+\mathbb{Z}\log 3+\ldots+\mathbb{Z}\log n$. Furthermore, from Bertrand's postulate, it is known that for every integer number $m>1$ there is always at least one prime p such that $m<p<2m$ and, therefore, the frequency $w_t=\log p_{k_n}$ satisfies the required condition.

Let $P(z)$ be an exponential polynomial of type (6). Associated to $P(z)$, we define the complex function

$$P^*(z):=P(z)-m_t e^{g_k z},\ z=x+iy,$$

and the real function

$$A_P(x,y):=|P^*(x+iy)|-|m_t|e^{g_k x},\ x,y\in\mathbb{R}. \tag{7}$$

An important result which allows us to obtain points of R_P is given by means of the function $A_P(x,y)$.

Proposition 2. *Let $P(z)$ be an exponential polynomial of type (6) and x_0 a real number such that $A_P(x_0,y)=0$ for some $y\in\mathbb{R}$, then $x_0\in R_P$.*

Proof. If $A_P(x_0,y)=0$ for some $y\in\mathbb{R}$, then $|P^*(x_0+iy)|=|m_t|e^{g_k x_0}$ and there exists $\theta\in(-\pi,\pi]$ such that $P^*(x_0+iy)=|m_t|e^{g_k x_0}e^{i\theta}$. That is,

$$1+m_1e^{w_1(x_0+iy)}+\ldots+\overbrace{m_te^{g_k(x_0+iy)}}+\ldots+m_ne^{w_n(x_0+iy)}=|m_t|e^{g_k x_0}e^{i\theta}, \tag{8}$$

where $\widehat{m_t e^{g_k(x_0+iy)}}$ means that $m_t e^{g_k(x_0+iy)}$ is not on the left side of (8). Therefore,

$$1 + m_1 e^{w_1(x_0+iy)} + \ldots - |m_t| e^{g_k x_0} e^{i\theta} + \ldots + m_n e^{w_n(x_0+iy)} = 0$$

and, by taking the vector $\mathbf{x} = (g_1 y, g_2 y, \ldots, g_{k-1} y, -\operatorname{Arg}(m_t)+\theta+\pi)$, with $\operatorname{Arg}(m_t)$ the principal value in the range $(-\pi, \pi]$, we have $F_P(x_0, \mathbf{x}) = 0$ where F_P is defined in (5). Consequently, by Theorem 1, we conclude that $x_0 \in R_P$. ■

Remark 3. *In general the converse of Proposition 2 is not true, such as we show in the last section of the present paper.*

Proposition 4. *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial of type (6) with $m_j > 0$, $j = 1, 2, \ldots, n$, and $k = \operatorname{rank} W \geq 3$. Consider a real number x_0. Then*

$$\max \{|P^*(z)| : \operatorname{Re} z \leq x_0\} = P^*(x_0)$$

and this maximum is only attained at the point x_0.

Proof. Since $m_j, w_j > 0$ for each $j = 1, 2, \ldots, n$, for any $z = x + iy$ such that $x \leq x_0$ we have

$$|P^*(z)| = \left| 1 + m_1 e^{w_1 z} + \ldots + \widehat{m_t e^{g_k z}} + \ldots + m_n e^{w_n z} \right| \leq$$

$$1 + m_1 e^{w_1 x} + \ldots + \widehat{m_t e^{g_k x}} + \ldots + m_n e^{w_n x} \leq$$

$$1 + m_1 e^{w_1 x_0} + \ldots + \widehat{m_t e^{g_k x_0}} + \ldots + m_n e^{w_n x_0} = P^*(x_0),$$

which proves that

$$\max \{|P^*(z)| : \operatorname{Re} z \leq x_0\} = P^*(x_0).$$

We next prove that this maximum is only attained at the point x_0. For this, let $z = x + iy$ be so that $x \leq x_0$ and $|P^*(z)| = P^*(x_0)$. Note first that we immediately deduce from above that $x = x_0$. Suppose now $y \neq 0$. For complex numbers $z, w \neq 0$, remark that $|z + w| = |z| + |w|$ if and only if there is some $\lambda > 0$ such that $w = \lambda z$. Thus, from above, and since $k = \operatorname{rank} W \geq 3$, there exist positive real numbers λ and μ such that $m_r e^{w_r(x_0+iy)} = \lambda$ and $m_s e^{w_s(x_0+iy)} = \mu$, where r and s are different from t and so that w_r and w_s are linearly independent over the rationals. Hence there exist two non-null integer numbers k_1 and k_2 such that $w_r y = k_1 \pi$ and $w_s y = k_2 \pi$, which means that $k_2 w_r = k_1 w_s$. This clearly represents a contradiction, which implies that $y = 0$ and the proposition holds. ■

Before to prove the lemmas which will lead us to the main theorem of this paper, we first analyse the particular case when the largest frequency w_n coincides with the element g_k of the basis of W or, equivalently, $W = \left(\sum_{j=1}^{n-1} \mathbb{Z} w_j\right) \oplus \mathbb{Z} w_n$.

Proposition 5. *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial of type (6) with $m_j > 0$, $j = 1, 2, \ldots, n$, and $w_n = g_k$. Then $b_P = x_P^1$, where x_P^1 denotes the unique real solution of the real equation*

$$P^*(x) = m_n e^{g_k x}.$$

Proof. We next justify that x_P^1 is the unique real solution of the real equation $P^*(x) = m_n e^{g_k x}$. Let $f(x) = m_n e^{g_k x} - P^*(x)$. As $g_k = w_n$, observe that in this case we have $P^*(x) = 1 + \sum_{j=1}^{n-1} m_j e^{w_j x}$. Since $\lim_{x\to-\infty} f(x) = -1$ and $\lim_{x\to\infty} f(x) = \infty$, there exists at least a real number α such that $f(\alpha) = 0$. As the derivative

$$f'(\alpha) = w_n m_n e^{w_n \alpha} - \sum_{j=1}^{n-1} w_j m_j e^{w_j \alpha} > w_{n-1} m_n e^{w_n \alpha} - \sum_{j=1}^{n-1} w_j m_j e^{w_j \alpha} =$$

$$w_{n-1} P^*(\alpha) - \sum_{j=1}^{n-1} w_j m_j e^{w_j \alpha} = w_{n-1} + w_{n-1} \sum_{j=1}^{n-1} m_j e^{w_j \alpha} - \sum_{j=1}^{n-1} w_j m_j e^{w_j \alpha} =$$

$$w_{n-1} + \sum_{j=1}^{n-1} (w_{n-1} - w_j) m_j e^{w_j \alpha} \geq w_{n-1} > 0,$$

the function $f(x)$ is strictly increasing at the point α and then the equation $f(x) = 0$ has only the solution $x = x_P^1$. On the other hand,

$$|P(x+iy)| = |P^*(x+iy) + m_n e^{g_k(x+iy)}| \geq m_n e^{g_k x} - |P^*(x+iy)| \geq m_n e^{g_k x} - |P^*(x)|$$

and then $|P(z)| > 0$ for any $z = x + iy$ with $x > x_P^1$. Hence $b_P \leq x_P^1$. Furthermore, if $A_P(x, y)$ is the real function defined in (7), then

$$A_P(x_P^1, 0) = P^*(x_P^1) - m_n e^{g_k x_P^1} = 0$$

and, by Proposition 2, we have $x_P^1 \in R_P$. Hence $x_P^1 \leq b_P$ and consequently $b_P = x_P^1$. ■

Corollary 6. *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial of type (6) with $m_j > 0$, $j = 1, 2, \ldots, n$, $k = \operatorname{rank} W \geq 3$ and $w_n = g_k$. Then $\operatorname{Re} z < b_P$ for all $z \in Z_P$. That is, $b_P = \sup\{\operatorname{Re} z : P(z) = 0\}$ is not attained.*

Proof. If we suppose the existence of some $z_0 \in Z_P$ such that $\operatorname{Re} z_0 = b_P$, on the one hand it is clear that $\operatorname{Im} z_0 \neq 0$ because $P(x) > 0$ for any real x and, on the other hand, we have

$$0 = |P(z_0)| = |P^*(z_0) + m_n e^{w_n z_0}| \geq m_n e^{w_n b_P} - |P^*(z_0)|. \tag{9}$$

Now, in view of Proposition 4, we have that $P^*(b_P) > |P^*(z_0)|$ and thus, we deduce from (9) and Proposition 5 that

$$0 = |P(z_0)| > m_n e^{w_n b_P} - P^*(b_P) = 0,$$

which is a contradiction. Hence the result holds. ■

3. The main theorem

From now on, let ∂R_P denote the set determined by all the boundary points of the set R_P. In the first part of this section we are going to study the consequences of the theoretical existence of the zeros of $P(z)$ whose real part belongs to ∂R_P.

Lemma 7. *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial of type (6) with $m_j > 0$, $j = 1, 2, \ldots, n$, and $k = \operatorname{rank} W \geq 3$. If $z_0 = x_0 + iy_0 \in Z_P$ with $x_0 \in \partial R_P$, then*

$$\min\{|P^*(z)| : \operatorname{Re} z = x_0\} = |P^*(z_0)|.$$

Proof. Observe that if x_0 is a boundary point of the set R_P, every neighbourhood of x_0 intersects R_P and its complementary set. Hence, according to Proposition 2, for any $\epsilon > 0$ there exists either $x_1 \in (x_0 - \epsilon, x_0)$ or $x_2 \in (x_0, x_0 + \epsilon)$ such that

$$A_P(x_1, y) \neq 0 \text{ or } A_P(x_2, y) \neq 0 \text{ for all } y \in \mathbb{R}. \tag{10}$$

Let $z_0 = x_0 + iy_0 \in Z_P$ and $A_P(x, y) = |P^*(x + iy)| - m_t e^{g_k x}$. If we suppose that there exists some $z_1 = x_0 + iy_1$ verifying $|P^*(z_1)| < |P^*(z_0)| = m_t e^{g_k x_0}$, we have

$$A_P(x_0, y_1) = |P^*(x_0 + iy_1)| - m_t e^{g_k x_0} < 0, \tag{11}$$

which means, by Proposition 4, that $y_1 \neq 0$. Furthermore, since $z_0 = x_0 + iy_0$ is a zero of $P(z)$, it clearly satisfies $y_0 \neq 0$ and $|P^*(z_0)| = m_t e^{g_k x_0}$, which means that $A_P(x_0, y_0) = 0$. At this point, also by Proposition 4, we have

$$A_P(x_0, 0) = P^*(x_0) - m_t e^{g_k x_0} > |P^*(z_0)| - m_t e^{g_k x_0} = A_P(x_0, y_0) = 0. \tag{12}$$

In this manner, the continuity of $A_P(x, y)$ and the inequalities (11) and (12) assure the existence of some $\delta > 0$ such that $A_P(x, y_1) < 0$ and $A_P(x, 0) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Hence, again from the continuity of A_P, for any $x \in (x_0 - \delta, x_0)$ there is a point y_x^-, and for any $x \in (x_0, x_0 + \delta)$ there is a point y_x^+ so that

$$A_P(x, y_x^-) = 0 \text{ and } A_P(x, y_x^+) = 0,$$

which contradicts (10). Now the proof is completed. ■

The subsequent results that we are going to prove in this section, and specifically the main theorem of this paper, are valid for the exponential polynomials of type (1) which are in the following large class of functions.

Definition 8. *Let $n \geq 3$ be an integer number. We will say that an exponential polynomial $P(z) = 1 + \displaystyle\sum_{j=1}^{n} m_j e^{w_j z}$ is in the class $\mathcal{P}_S$ when it verifies the following properties:*

i) $m_j > 0$ *for each* $j = 1, 2, \ldots, n$;

ii) $0 < w_1 < w_2 < \ldots < w_n$;

iii) *the set of positive numbers* $\{g_1, g_2, \ldots, g_k\}$, $k \geq 3$, *is a basis of* $W = \mathbb{Z}w_1 + \ldots + \mathbb{Z}w_n$ *(observe that* $n \geq \operatorname{rank} W = k \geq 3$*) such that*

a) $g_k = w_{t_0}$, $g_{k-1} = w_{t_1}$ *and* $g_{k-2} = w_{t_2}$ *for some distinct indexes* $t_0, t_1, t_2 \in \{1, 2, \ldots, n\}$;

b) W *is the direct sum* $\left(\sum_{j=1, j\notin I}^{n} \mathbb{Z}w_j\right) \oplus \mathbb{Z}w_{t_0} \oplus \mathbb{Z}w_{t_1} \oplus \mathbb{Z}w_{t_2}$, *with* $I = \{t_0, t_1, t_2\}$.

It is worth pointing out that the set of $k \geq 3$ *positive numbers* $\{g_1, g_2, \ldots, g_k\}$ *is not necessarily an ordered set. Under the conditions above, note that if* $k = 3$ *then* $W = \mathbb{Z}w_{t_0} \oplus \mathbb{Z}w_{t_1} \oplus \mathbb{Z}w_{t_2}$. *Moreover, if* $k > 3$ *and* $j \in \{1, 2, \ldots, n\} \setminus I$ *then* $w_j = \displaystyle\sum_{l=1}^{k-3} c_{j,l} g_l$ *for some integer numbers* $c_{j,l} \geq 0$.

Remark 9. *The class* $\mathcal{P}_S$ *contains all exponential polynomials*

$$P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z},\ n \geq 3,\ m_j > 0$$

with positive real frequencies $w_1 < w_2 < \ldots < w_n$ *linearly independent over the rationals. Also, since* 17 *is the smallest positive integer* N_3 *such that for all* $x \geq N_3$ *the interval* $\left(\frac{x}{2}, x\right)$ *contains at least* 3 *primes (i.e. the third Ramanujan prime is* 17 *[12, 13]), then for every* $n \geq 17$ *it is verified that* $2p_{k_n-2} > n$ *where* $p_{k_n-2} < p_{k_n-1} < p_{k_n}$ *are the last three prime numbers less than or equal to* n. *Hence the class* $\mathcal{P}_S$ *contains all the sums*

$$\zeta_n(-z) = \sum_{j=1}^{n} j^z,\ n \geq 17.$$

Furthermore, note that Lemma 7 is not vacuous when $P(z) = \zeta_n(-z)$, $n \geq 17$.

Although it is not used in the proof of the main theorem, we next provide a variation of Proposition 2 for exponential polynomials in the class $\mathcal{P}_S$. Associated to $P(z)$ we consider the real function

$$B_P(x, y) := |P^{***}(x + iy)| - m_{t_0} e^{w_{t_0} x} - m_{t_1} e^{w_{t_1} x} - m_{t_2} e^{w_{t_2} x},\ x, y \in \mathbb{R}, \tag{13}$$

where $P^{***}(z) = P(z) - m_{t_0} e^{w_{t_0} z} - m_{t_1} e^{w_{t_1} z} - m_{t_2} e^{w_{t_2} z}$.

Proposition 10. *Let* $P(z)$ *be an exponential polynomial in the class* $\mathcal{P}_S$ *and* x_0 *a real number such that* $B_P(x_0, y) = 0$ *for some* $y \in \mathbb{R}$, *then* $x_0 \in R_P$.

Proof. If $B_P(x_0, y) = 0$ for some $y \in \mathbb{R}$, where B_P is defined in (13), then

$$|P^{***}(x_0 + iy)| = m_{t_0} e^{w_{t_0} x_0} + m_{t_1} e^{w_{t_1} x_0} + m_{t_2} e^{w_{t_2} x_0}$$

and there exists $\theta \in (-\pi, \pi]$ such that

$$P^{***}(x_0 + iy) = (m_{t_0} e^{w_{t_0} x_0} + m_{t_1} e^{w_{t_1} x_0} + m_{t_2} e^{w_{t_2} x_0})\, e^{i\theta}.$$

Therefore, by taking the vector $\mathbf{x} = (g_1 y, g_2 y, \ldots, g_{k-3} y, \theta + \pi, \theta + \pi, \theta + \pi)$ and since $P(z)$ is in the class $\mathcal{P}_S$, we have $F_P(x_0, \mathbf{x}) = 0$ where F_P is defined in (5). Consequently, by Theorem 1, we conclude that $x_0 \in R_P$. ■

The next lemma plays an important role in the proof of the main theorem of this paper. It is proved by using Kronecker's theorem [3, Th.444, p.382] on simultaneous diophantine approximation, which states that if $\{a_j\}$ is any finite collection of rationally independent real numbers, then given any sequence of real numbers $\{b_j\}$ and $\varepsilon, T > 0$ there are integers $\{n_j\}$ and $t > T$ such that $|ta_j - n_j - b_j| < \varepsilon$ for all j. In this respect, fixed an exponential polynomial $P(z)$ of type (6), the idea behind the next lemma is that the alignment of all the terms of the form $e^{w_{t_s} z}$ and $P(z) - e^{w_{t_s} z}$, with w_{t_s} linearly independent from the remaining frequencies of $P(z)$, is a necessary condition to get a non-null minimum of the modulus of $P(z)$ on a certain straight line $\operatorname{Re} z = x_0$.

Lemma 11. *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial of type (6) and $z_0 = x_0 + iy_0$ a complex number such that*

$$\min\{|P(z)| : \operatorname{Re} z = x_0\} = |P(z_0)| \neq 0.$$

Thus for each frequency w_{t_s} such that $W = \left(\sum_{j=1, j\neq t_s}^{n} \mathbb{Z} w_j\right) \oplus \mathbb{Z} w_{t_s}$ there exists a real number $\mu_s \neq 0$ such that

$$P(z_0) = \mu_s m_{t_s} e^{w_{t_s} z_0}. \tag{14}$$

Proof. Take w_{t_s} so that $W = \mathbb{Z} w_1 + \ldots + \mathbb{Z} w_n$ is the direct sum $\left(\sum_{j=1, j\neq t_s}^{n} \mathbb{Z} w_j\right) \oplus \mathbb{Z} w_{t_s}$, then we can suppose without loss of generality that $w_{t_s} = g_{k-s}$ for some $s \in \{0, 1, \ldots, k-1\}$, where $\{g_1, \ldots, g_{k-s}, \ldots, g_k\}$ is a basis of W and $k = \operatorname{rank} W$. Let $z_0 = x_0 + iy_0$ be a complex number such that

$$\min\{|P(z)| : \operatorname{Re} z = x_0\} = |P(z_0)| \neq 0.$$

This means that if (14) is true, then μ_s will be different from 0. Also, under the assumption that $P(z_0) = m_{t_s} e^{g_{k-s} z_0}$, we have $\mu_s = 1$ and (14) is true. Hence, suppose that $P(z_0) \neq m_{t_s} e^{g_{k-s} z_0}$. Now, if β_s denotes the principal argument of $-m_{t_s} e^{g_{k-s} z_0}$, then

$$y_0 g_{k-s} = \beta_s \pm \pi + 2\pi l, \text{ for some } l \in \mathbb{Z}. \tag{15}$$

Let α_s be the principal argument of $P(z_0)-m_{t_s}e^{g_{k-s}z_0}$ and suppose that $\alpha_s<\beta_s$, then $\beta_s=\alpha_s+\alpha$ for some $0<\alpha<2\pi$ and (15) becomes

$$y_0 g_{k-s}=\alpha_s+\alpha+\pi(2l\pm 1). \tag{16}$$

Herein we next adopt the nomenclature of Kronecker's theorem [3, Th.444, p.382] and we define the numbers

$$a_j=\frac{g_j}{2\pi},\ j=1,2,\ldots,k;$$

$$b_j=\frac{y_0g_j}{2\pi},\ j\in\{1,2,\ldots,k\}\setminus\{k-s\};\ b_{k-s}=\frac{\alpha_s\pm\pi}{2\pi}.$$

Then, since the a_j's are linearly independent over the rationals, by applying Kronecker's theorem [3, Th.444, p.382], given $T=|y_0|$ and $\varepsilon_q=\dfrac{1}{2\pi q}$ with $q=1,2,\ldots$, there exist integers $d_{j,q}$, $j=1,2,\ldots,k$, and a real number $y_{\varepsilon_q}>y_0$ such that

$$\left|y_{\varepsilon_q}a_j-d_{j,q}-b_j\right|=\left|y_{\varepsilon_q}\frac{g_j}{2\pi}-d_{j,q}-\frac{y_0g_j}{2\pi}\right|<\varepsilon_q,\ j\in\{1,2,\ldots,k\}\setminus\{k-s\}$$

and

$$\left|y_{\varepsilon_q}a_{k-s}-d_{k-s,q}-b_{k-s}\right|=\left|y_{\varepsilon_q}\frac{g_{k-s}}{2\pi}-d_{k-s,q}-\frac{\alpha_s\pm\pi}{2\pi}\right|<\varepsilon_q.$$

That is,

$$y_{\varepsilon_q}g_j=2\pi d_{j,q}+y_0g_j+\eta_{j,q},\ j\in\{1,2,\ldots,k\}\setminus\{k-s\} \tag{17}$$

and

$$y_{\varepsilon_q}g_{k-s}=2\pi d_{k-s,q}+\alpha_s\pm\pi+\eta_{k-s,q}, \tag{18}$$

with $\eta_{j,q}$ real numbers satisfying

$$|\eta_{j,q}|<2\pi\varepsilon_q=\frac{1}{q},\ j=1,\ldots,k. \tag{19}$$

Now, we define the sequence $z_{\varepsilon_q}:=x_0+iy_{\varepsilon_q}$, $q=1,2,\ldots$, and we claim that

$$\lim_{q\to\infty}\left(P(z_{\varepsilon_q})-m_{t_s}e^{g_{k-s}z_{\varepsilon_q}}\right)=P(z_0)-m_{t_s}e^{g_{k-s}z_0}. \tag{20}$$

Indeed, if $j\in\{1,2,\ldots,n\}$ then $w_j=\sum_{l=1}^{k}c_{j,l}g_l$ for some integer numbers $c_{j,l}\geq 0$, with $w_{t_s}=g_{k-s}$ and $c_{r,l}=0$ for $l=k-s$ and $r\in\{1,2,\ldots,n\}\setminus\{t_s\}$. Therefore, from (17) we have for each w_j, $j\in\{1,2,\ldots,n\}\setminus\{t_s\}$, that

$$w_jz_{\varepsilon_q}=w_j(x_0+iy_{\varepsilon_q})=w_jx_0+iy_{\varepsilon_q}\sum_{\substack{l=1\\ l\neq k-s}}^{k}c_{j,l}g_l=$$

$$w_j x_0 + i \sum_{\substack{l=1 \\ l \neq k-s}}^{k} c_{j,l}(y_0 g_l + \eta_{j,q} + 2\pi d_{j,q}) = w_j z_0 + i \sum_{\substack{l=1 \\ l \neq k-s}}^{k} c_{j,l}\eta_{j,q} + 2\pi i \sum_{\substack{l=1 \\ l \neq k-s}}^{k} c_{j,l} d_{j,q}.$$

In this manner,

$$e^{w_j z_{\varepsilon_q}} = e^{w_j z_0} e^{i \sum_{l=1, l\neq k-s}^{k} c_{j,l}\eta_{j,q}}$$

and, by taking the limit as $q \to \infty$, we deduce from (19) that

$$\lim_{q\to\infty} e^{w_j z_{\varepsilon_q}} = e^{w_j z_0},\ j \in \{1, 2, \ldots, n\} \setminus \{t_s\},$$

which proves (20). On the other hand, by noting $R_s = |P(z_0) - m_{t_s} e^{g_{k-s} z_0}| \neq 0$ and using (16), we have

$$|P(z_0)| = |P(z_0) - m_{t_s} e^{g_{k-s} z_0} + m_{t_s} e^{g_{k-s} z_0}| = |R_s e^{i\alpha_s} + m_{t_s} e^{g_{k-s} x_0} e^{i g_{k-s} y_0}| =$$
$$|R_s e^{i\alpha_s} + m_{t_s} e^{g_{k-s} x_0} e^{i(\alpha_s + \alpha + \pi(2l \pm 1))}| = |R_s e^{i\alpha_s} - m_{t_s} e^{g_{k-s} x_0} e^{i(\alpha_s + \alpha)}| =$$
$$|R_s - m_{t_s} e^{g_{k-s} x_0} e^{i\alpha}|.$$

Moreover, by noting $R_{\varepsilon_q} = \left|P(z_{\varepsilon_q}) - m_{t_s} e^{g_{k-s} z_{\varepsilon_q}}\right|$, which from (20) is also different from 0 for all q greater than a certain q_0, α_{ε_q} the principal argument of $P(z_{\varepsilon_q}) - m_{t_s} e^{g_{k-s} z_{\varepsilon_q}}$, $q \geq q_0$, and taking (18) into account, we have

$$|P(z_{\varepsilon_q})| = |P(z_{\varepsilon_q}) - m_{t_s} e^{g_{k-s} z_{\varepsilon_q}} + m_{t_s} e^{g_{k-s} z_{\varepsilon_q}}| =$$
$$|R_{\varepsilon_q} e^{i\alpha_{\varepsilon_q}} + m_{t_s} e^{g_{k-s} x_0} e^{i g_{k-s} y_{\varepsilon_q}}| = |R_{\varepsilon_q} e^{i\alpha_{\varepsilon_q}} - m_{t_s} e^{g_{k-s} x_0} e^{i(\alpha_s + \eta_{k-s,q})}| =$$
$$|R_{\varepsilon_q} - m_{t_s} e^{g_{k-s} x_0} e^{i(\alpha_s - \alpha_{\varepsilon_q} + \eta_{k-s,q})}|.$$

Now, since $\min\{|P(z)| : \operatorname{Re} z = x_0\} = |P(z_0)|$, we have $|P(z_0)| \leq |P(z_{\varepsilon_q})|$. Thus

$$|R_s - m_{t_s} e^{g_{k-s} x_0} e^{i\alpha}|^2 \leq |R_{\varepsilon_q} - m_{t_s} e^{g_{k-s} x_0} e^{i(\alpha_s - \alpha_{\varepsilon_q} + \eta_{k-s,q})}|^2$$

and, by taking the limit when $q \to \infty$, we get

$$|R_s - m_{t_s} e^{g_{k-s} x_0} e^{i\alpha}|^2 \leq |R_s - m_{t_s} e^{g_{k-s} x_0}|^2,$$

which means that $\cos\alpha = 1$ and this contradicts $\alpha_s < \beta_s = \alpha_s + \alpha$ with $0 < \alpha < 2\pi$. Thus $\alpha_s \geq \beta_s$ and, by supposing $\beta_s = \alpha_s + \alpha$ with $-2\pi < \alpha < 0$, we are analogously led to a contradiction (note that if $\alpha_s = \pi$ then either $\alpha_{\varepsilon_q} \to -\pi$ or $\alpha_{\varepsilon_q} \to \pi$, but the conclusion is the same). Consequently, we have $\alpha_s = \beta_s$ and there exists $\lambda_s > 0$ such that

$$P(z_0) - m_{t_s} e^{g_{k-s} z_0} = \lambda_s(-m_{t_s} e^{g_{k-s} z_0}).$$

Now, by taking $\mu_s = 1 - \lambda_s$, the lemma is proved. ■

Remark 12. *Let $P(z)$ be in the class $\mathcal{P}_S$. Since $P^*(z)$ is clearly an exponential polynomial of type (6), Lemma 11 could be used with $P^*(z)$ and, with the notation of Definition 8, it is possible to consider the frequencies $w_{t_1} = g_{k-1}$ and $w_{t_2} = g_{k-2}$.*

We next prove the main theorem which states that the real part of any zero of $P(z)$ in the class $\mathcal{P}_S$ does not belong to ∂R_P.

Theorem 13 (The main theorem). *Let $P(z) = 1 + \sum_{j=1}^{n} m_j e^{w_j z}$ be an exponential polynomial in the class $\mathcal{P}_S$ and $z_0 = x_0 + iy_0 \in Z_P$. Thus there exists an open interval $J_{x_0} \subset R_P$ such that $x_0 \in J_{x_0}$.*

Proof. If $z_0 = x_0 + iy_0$ is a zero of $P(z)$, it is clear that $x_0 \in R_P$, $y_0 \neq 0$ and

$$P^*(z_0) = -m_{t_0} e^{g_k z_0}. \tag{21}$$

Hence $|P^*(z_0)| = m_{t_0} e^{g_k x_0} \neq 0$. By reductio ad absurdum, assume that x_0 is a boundary point of the set $R_P = \operatorname{Int} R_P \cup \partial R_P$, then by Lemma 7 we have $\min\{|P^*(z)| : \operatorname{Re} z = x_0\} = |P^*(z_0)|$. Now by Lemma 11 (see also Remark 12), applied to $P^*(z)$, there exist two real numbers μ_1 and μ_2, different from 0, verifying

$$P^*(z_0) = \mu_1 m_{t_1} e^{g_{k-1} z_0} \tag{22}$$

and

$$P^*(z_0) = \mu_2 m_{t_2} e^{g_{k-2} z_0}. \tag{23}$$

Hence, we have from (22) and (23) that

$$\mu_1 m_{t_1} e^{g_{k-1} z_0} = \mu_2 m_{t_2} e^{g_{k-2} z_0}$$

and thus

$$e^{(g_{k-1} - g_{k-2}) z_0} = \frac{\mu_2}{\mu_1} \frac{m_{t_2}}{m_{t_1}} \in \mathbb{R} \setminus \{0\}.$$

Therefore there exists a non-null integer u_1 such that

$$(g_{k-1} - g_{k-2}) y_0 = u_1 \pi. \tag{24}$$

Analogously, we have from (21) and (22) that $-m_{t_0} e^{g_k z_0} = \mu_1 m_{t_1} e^{g_{k-1} z_0}$. That is

$$e^{(g_k - g_{k-1}) z_0} = -\mu_1 \frac{m_{t_1}}{m_{t_0}} \in \mathbb{R} \setminus \{0\}.$$

Therefore there exists a non-null integer u_2 such that

$$(g_k - g_{k-1}) y_0 = u_2 \pi. \tag{25}$$

Dividing (24) and (25), we get

$$\frac{g_{k-1}-g_{k-2}}{g_k-g_{k-1}}=\frac{u_1}{u_2},$$

which means that $\{g_k, g_{k-1}, g_{k-2}\}$ are linearly dependent over the rationals and this is a contradiction. Hence x_0 is an interior point of R_P and there exists an open interval $J_{x_0}\subset R_P$ such that $x_0\in J_{x_0}$. ■

It is clear that $a_P, b_P\in\partial R_P$ and R_P is a closed set in $[a_P, b_P]$. Furthermore, according to the main theorem, the real part of a zero of an exponential polynomial $P(z)$ in the class $\mathcal{P}_S$ is not a boundary point of R_P. Thus, in the class $\mathcal{P}_S$, the bounds a_P and b_P are not real parts of zeros of $P(z)$ and therefore $P(z)$ possesses infinitely many zeros having real part between a_P and b_P.

Corollary 14. *Let $P(z)=1+\sum_{j=1}^{n}m_je^{w_jz}$ be an exponential polynomial in the class $\mathcal{P}_S$. Then $a_P<\operatorname{Re}z<b_P$ for all $z\in Z_P$; that is, $a_P=\inf\{\operatorname{Re}z : P(z)=0\}$ and $b_P=\sup\{\operatorname{Re}z : P(z)=0\}$ are not attained.*

4. Final considerations

Since the class $\mathcal{P}_S$ contains all the sums

$$G_n(z):=\zeta_n(-z)=\sum_{j=1}^{n}j^z,\ n\geq 17,$$

(see Remark 9), we are led to formulate the following corollary.

Corollary 15. *For every $n\geq 17$ the real part of any zero of $\zeta_n(z)$ is an interior point in R_{ζ_n}.*

Proof. The result follows directly from Remark 9, Theorem 13 and the fact that $Z_{\zeta_n}=-Z_{G_n}$. ■

We next show the existence of exponential polynomials of type (6) (also in the class $\mathcal{P}_S$) which have at least a point $x_0\in R_P$ such that $A_P(x_0,y)\neq 0\ \forall y\in\mathbb{R}$, where A_P is defined in (7). This means that the sufficiency of Proposition 2 is not valid.

Proposition 16. *Let $P(z)=\zeta_{20}(-z)=\sum_{j=1}^{20}j^z$, then $A_P(a_P,y)>0\ \forall y\in\mathbb{R}$, where a_P is defined in (2).*

Proof. Observe first that $P(z)$ is clearly in the class $\mathcal{P}_S$. In fact, the set $\{g_1, g_2, \ldots, g_8\}$ with $g_1=\log 2$, $g_2=\log 3$ $g_3=\log 5$, $g_4=\log 7$, $g_5=\log 11$, $g_6=\log 13$, $g_7=\log 17$ and $g_8=\log 19$, is a basis of $W = \mathbb{Z}\log 2 + \ldots + \mathbb{Z}\log 20$ which satisfies the required conditions. Also, by taking the vector $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (0, 0, \pi, \pi, \pi, \pi, \pi, \pi)$, note that

$$\begin{aligned} F_P(0,\mathbf{x}) = 1 &+ e^{ix_1} + e^{ix_2} + e^{i2x_1} + e^{ix_3} + e^{i(x_1+x_2)} + e^{ix_4} + e^{i3x_1} + e^{i2x_2} + \\ &+ e^{i(x_1+x_3)} + e^{ix_5} + e^{i(2x_1+x_2)} + e^{ix_6} + e^{i(x_1+x_4)} + e^{i(x_2+x_3)} + \\ &+ e^{i4x_1} + e^{ix_7} + e^{i(x_1+2x_2)} + e^{ix_8} + e^{i(2x_1+x_3)} = 0 \end{aligned}$$

where F_P is the function defined in (5). Consequently, by Theorem 1, we conclude that $0 \in R_P$. Therefore $a_P \leq 0$ and thus

$$A_P(a_P, 0) = 1 + 2^{a_P} + 3^{a_P} + \ldots + 18^{a_P} + 20^{a_P} - 19^{a_P} > 0. \tag{26}$$

Assume that there exists a real number y_1 such that

$$A_P(a_P, y_1) < 0. \tag{27}$$

Thus the continuity of $A_P(x, y)$ and the inequalities (26) and (27) assure the existence of some $\delta > 0$ such that $A_P(x, 0) > 0$ and $A_P(x, y_1) < 0$ for all $x \in (a_P - \delta, a_P + \delta)$. Therefore, again from the continuity of $A_P(x, y)$, for each $x \in (a_P - \delta, a_P)$ there is a point y_x such that $A_P(x, y_x) = 0$, which implies by Proposition 2 that $(a_P - \delta, a_P) \subset R_P$, contradicting the definition of a_P. Hence $A_P(a_P, y) \geq 0$ for all $y \in \mathbb{R}$ or, equivalently, $|P^*(a_P + iy)| \geq 19^{a_P}$ for all $y \in \mathbb{R}$. Now, assume that y_0 is a real number such that $A_P(a_P, y_0) = 0$. Thus $y_0 \neq 0$ and the point $z_0 = a_P + iy_0$ verifies

$$\min\{|P^*(z)| : \operatorname{Re} z = a_P\} = |P^*(z_0)| = 19^{a_P}.$$

Therefore, from Lemma 11 we deduce the existence of three non-null real numbers μ_1, μ_2 and μ_3 such that

$$P^*(z_0) = \mu_1 17^{a_P + iy_0},$$

$$P^*(z_0) = \mu_2 15^{a_P + iy_0}$$

and

$$P^*(z_0) = \mu_3 13^{a_P + iy_0}.$$

Consequently, $\mu_1 17^{a_P+iy_0} = \mu_2 15^{a_P+iy_0}$ and $\mu_2 15^{a_P+iy_0} = \mu_3 13^{a_P+iy_0}$, which implies that

$$\left(\frac{17}{15}\right)^{a_P+iy_0} = \frac{\mu_2}{\mu_1}$$

and

$$\left(\frac{15}{13}\right)^{a_P+iy_0} = \frac{\mu_3}{\mu_2}.$$

That is, there exist non-null integers u_1 and u_2 such that

$$y_0(\log 17 - \log 15) = u_1\pi \tag{28}$$

and

$$y_0(\log 15 - \log 13) = u_2\pi. \tag{29}$$

Dividing (28) and (29), we get

$$\frac{\log 17 - \log 15}{\log 15 - \log 13} = \frac{u_1}{u_2},$$

which means that $\{\log 13, \log 15, \log 17\}$ are linearly dependent over the rationals and this is a contradiction. Now the proposition is proved. ■

Nevertheless, in [2, Theorem 2], [6, Theorem 2], [7, Theorem 5] and [8, Theorem 3.14] one states the following result for the functions $G_n(z) = \zeta_n(-z)$:

Theorem 0. *(Mora, 2013, 2014; Dubon et al., 2014) For every integer $n > 2$, a real number $x \in R_{G_n}$ if and only if $A_{G_n}(x, y) = 0$ for some $y \in \mathbb{R}$.*

It can be easily checked that this false equivalency does not affect any subsequent result of the paper [2] because it is only used in the right sense of Proposition 2 of the present paper. However, Theorem 0 is essential to prove the main results in [6, 7, 8], particularly [7, Theorem 12] and [6, Theorem 3.18]. The mistake in that proof consists of supposing that $\lim_{n\to\infty} e^{iCy_n} = e^{iC\lambda}$ when $y_1, y_2, \ldots, y_n, \ldots$ are real numbers so that $\lim_{n\to\infty} e^{iy_n} = e^{i\lambda}$ for $\lambda \in [0, 2\pi)$ and C is a fixed real number (see [7, p.123, l.6] or [2, p.328, l.5]). For example, if $y_n = \frac{3\pi n}{n+1}$, $n \geq 1$, and $C = \frac{1}{3}$, then $\lim_{n\to\infty} e^{iy_n} = e^{i\pi}$ (observe that $\lambda = \pi$) and $\lim_{n\to\infty} e^{iCy_n} = e^{i\pi} \neq e^{iC\lambda}$.

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