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Revisiting the Anglo-Dutch Auction

Daniel Marszalec
The University of Tokyo

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Daniel Marszalec[a,1]

[a] *Graduate School of Economics, University of Tokyo 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*

Abstract

The Anglo-Dutch auction of Klemperer (1998) is the unit-demand precursor of the many two-stage hybrid auctions currently used for the allocation of high value goods such as mobile telephony licenses, bus routes, and public procurement. This breadth of practical applications has been largely matched by an absence of theoretical results regarding the performance of hybrid auctions relative to their simpler component counterparts: the ascending and first-price auctions. To address this imbalance, I analyze an asymmetric discrete-valuation model of the Anglo-Dutch auction and derive a complete revenue ranking between the Anglo-Dutch, ascending and first-price auctions. I find that the Anglo-Dutch auction can revenue-dominate for a small set of parameters, and ranks revenue-last in an even smaller number of cases. For most parameter values the Anglo-Dutch auction ranks as intermediate. I also show that the auction performs particularly well when bidders face small entry costs and almost-common values. Overall, the Anglo-Dutch auction is rarely "the best of both worlds", but even more rarely performs worst - for this reason, it may be a prudent policy choice if the auctioneer has imprecise information about the magnitude of asymmetries across bidders.

JEL Classification: D44, D47

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1. Introduction

Hybrid auctions, where a dynamic ascending or clock stage is followed by a single round of final sealed bids, have been recently used in many high-stakes auctions, including mobile spectrum licenses (Cramton, 2013), and public procurement (Lunander and Lundberg, 2013). Much of the intuitive appeal of hybrid auctions is that they are the "best of both worlds", outperforming both their pure ascending and first-price counterparts on both revenue and efficiency. Despite a breadth of practical applications, as surveyed by Ausubel and Baranov (2017), few economic models explicitly analyze the properties of hybrid auctions, due to their inherent complexity: models where the ascending phase is analytically solvable

Email address: dmarszalec@e.u-tokyo.ac.jp (Daniel Marszalec)

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result in a sealed-bid stage that is intractable, or vice versa. Though some experimental results, empirical case-studies, and theoretical conjectures are available for the general setting,[2] a general theoretical equilibrium analysis has not been feasible yet.

In this paper, I revisit one of the early formalizations of a hybrid auction: the Anglo-Dutch auction of Klemperer (1998), where each bidder has a unit demand. An Anglo-Dutch auction proceeds in two phases. First an ascending (i.e. English or Japanese) auction is run, until all but the last two bidders drop out. The price at which the last bidder drops out is noted, and set as the reserve price in a first-price auction among the remaining two bidders. After these 'best and final' offers have been submitted (the "Dutch" phase), the highest bidder wins and pays their bid.[3] ,[4]

Prior work on Anglo-Dutch auctions, such as Āzacis and Burguet (2008) and Bustos and Costinot (2003), have obtained only partial rankings of the Anglo-Dutch auction against the ascending auction. While Levin and Ye (2008) obtain a ranking relative to the first-price auction, their results are driven by risk-aversion and affiliation in bidders' values. My analysis complements these three papers by providing a full revenue ranking in a model with asymmetries, but no risk-aversion. This formulation allows me to highlight the significance of the information-revelation step, prior to the first-price phase of the Anglo-Dutch auction.

I present a one-item three-bidder model with discrete valuations and no entry, which admits a closed-form solution. Under these assumptions, I show that the Anglo-Dutch auction is revenue-dominant for a small range of parameters, but even more rarely ranks as revenue-worst. The auction also performs particularly well when bidders face small entry costs and almost-common values. While it may be unwise to put excessive trust in policy implications of simplified microeconomic models, the modest suggestion of my results is that a hybrid auction may be a good compromise when the auctioneer has imprecise information about relative valuations among the bidders: if the auctioneer is confident about bidders' characteristics, they may obtain superior revenue by picking the appropriate single-stage auction.

In Section 2 I survey the literature that analyzes the properties of Anglo-Dutch auctions. I introduce the discrete-valuation setup that underlies my analysis in Section 3. Equilibrium bidding strategies for the first-price, ascending and Anglo-Dutch auctions are derived in Sections 4, 5 and 6 respectively. I present analytical revenue rankings, for parameter values that admit such solutions, in Section 7, and proceed to present numerical results in Section 8. Section 9 covers the efficiency characteristics of the three auctions, and Section 10 concludes.

[2]See Kagel et al. (2014), National Audit Office (2014), and Kagel et al. (2010), respectively.

[3]The Anglo-Dutch auction can be directly generalized to $K > 1$ units, so long as all bidders have unit demands. See, for example, Āzacis and Burguet (2008).

[4]For clarity of exposition, I use the term "Anglo auction" to refer to the ascending phase of the Anglo-Dutch auction, and the "Dutch auction" to refer to the first-price sealed-bid phase. This is to keep the two phases of the hybrid auction more clearly separate from its single-phase component auctions.

2. Background and Literature review

Under standard independent private value (IPV) assumptions,[5] the Anglo-Dutch auction satisfies the conditions for applying the revenue equivalence theorem, and is therefore revenue-equivalent to the first-price and ascending auctions. For practical applications, this is a minor concern, as IPV assumptions are restrictive, and unlikely to ever hold in practice. To build theory models which do not collapse into revenue-equivalence, one or more of the IPV assumptions must be relaxed. In this vein, Levin and Ye (2008) relax independence and risk-aversion, while Āzacis and Burguet (2008) and Bustos and Costinot (2003) relax the assumption of bidder symmetry and change the assumption of "fixed number of bidders" to model entry endogenously. Against this backdrop, my model retains the assumption of risk-neutrality and independent private values, and focuses on the impacts of the asymmetry in bidder valuations only. In terms of outcome, in the models of Āzacis and Burguet (2008) and Bustos and Costinot (2003) the main explanation for revenue-differences is entry, while in Levin and Ye (2008) the results are driven by risk-aversion. In my model, the divergence follows from the different patterns of information-revelation across the auctions, stemming from the two-stage hybrid nature of the Anglo-Dutch auction.

When IPV assumptions do not hold, there are three main areas where the Anglo-Dutch auction aims to improve over its two components: firstly in encouraging entry and thus boosting seller revenue, secondly in furthering efficiency and in preventing collusion. Thirdly, the Anglo-Dutch auction uses the first-price phase to take advantage of risk-aversion.

The first-price stage at the end of the Anglo-Dutch auction encourages entry by the following logic. In a pure ascending auction advantaged bidders always get a chance to out-bid weaker rivals, hence weaker entrants can never win (Bulow et al., 1999). This is not true in the Anglo-Dutch auction, where the Dutch (first-price) phase gives the entrants a chance to out-bid a stronger incumbent in case the incumbent chooses to bid cautiously. More generally: weaker bidders have stronger incentives to participate in auctions with a first-price component (Marszalec et al., 2020).

Theoretical work by Āzacis and Burguet (2008) and Bustos and Costinot (2003) evaluates the importance of entry in the Anglo-Dutch auction in models with endogenous entry and asymmetric bidders. Both papers show that Anglo-Dutch auctions can revenue-dominate ascending auctions, and induce more entry. However, both papers use models of asymmetry which favor the first-price auction over the ascending auction; it is thus unsurprising that the Anglo-Dutch auction out-performs its ascending counterpart. The performance of the first-price auction relative to Anglo-Dutch remains an open question in both papers; it is plausible that if the Anglo-Dutch outperformed the ascending auction, first-price may have performed better still.

[5]IPV assumptions require that each bidder's value for the object is an independent draw from the same continuous, and atomless, distribution. Furthermore, bidders are assumed to be symmetric, risk-neutral, and not face budget constraints. The number of bidders is assumed to be fixed, and there is no additional entry at any point in the auction.

The results of Āzacis and Burguet (2008) and Bustos and Costinot (2003) are not robust to some small perturbations in the valuation model such as "almost-common values" in Klemperer (1998), or "toeholds" in Bulow et al. (1999). Here, when the incumbent has an arbitrarily small but certain value advantage, no entry occurs in any purely ascending auction, and revenue from such auctions is low.[6] The Anglo-Dutch auction would induce more entry, but it is likely that first-price could be most attractive to entrants.

The first-price phase of the Anglo-Dutch auction makes collusion more difficult by making the final bids unobservable, and hence non-punishable. Even if bidders can observe each other's bids in the Anglo phase and bid collusively in that phase, the final sealed-bid stage gives each bidder an incentive to renege on a collusive agreement, without the threat of punishment.[7] Brunner et al. (2010) find supporting evidence for this phenomenon in an experimental setting.[8]

Information revealed in the ascending phase of the Anglo-Dutch auction is also useful to bidders if their values are correlated, or risk-aversion is present. Here the points at which each bidder drops out in the ascending phase provides the remaining bidders with information about value of the object they are bidding for (in case of common values), or on the intensity of competition (if values are private, but affiliated). The influence of this information effect is compounded if bidders are risk averse. In the theoretical model of Levin and Ye (2008), where bidders are symmetric and risk averse, with affiliated private values, any hybrid auction with a first-price stage at the end will out-perform pure ascending auction. However, if bidders are sufficiently risk-averse, the first-price auction becomes revenue-dominant over any hybrid auction, including the Anglo-Dutch.

The empirical auction literature on two-stage auctions, relevant to this paper, has focused primarily on what van Bochove et al. (2012) call Anglo-Dutch premium auctions: auctions where the winner of the first open (ascending) phase is paid a premium for winning that round, before bidding proceeds to the second round, where sealed bids are submitted.[9] As the authors note, historical records show that such auctions have been in use since the 16th century for auctioning real estate in the Netherlands, and had since then also been used to sell timber, wine, and financial securities.

While the literature on premium auctions is broad and interesting in it's own right,[10]

[6] Āzacis and Burguet (2008) also introduce an Anglo-Anglo auction: a two-stage ascending auction with a high reserve in the first round, and entry at both stages. Due to multi-stage entry and the first-round reserve price, the Anglo-Anglo auction generates highest revenue and efficiency. In practice setting an appropriate reserve may not be feasible, and since my model does not include entry, I also exclude the Anglo-Anglo auction from my analysis. Furthermore, just like a standard ascending auction, the Anglo-Anglo auction induces no entry under a valuation model that features "almost-common values".

[7] The intuition here follows Robinson (1985), who shows that collusion is easiest in ascending auctions.

[8] Abbink et al. (2005), conversely, cannot find a significant difference between the performance of the Anglo-Dutch and multi-item ascending auctions in their experiment; the paper does not specifically look for collusion effects.

[9] Goeree and Offerman (2004) call a more general version of this auction the "Amsterdam auction": here, the sealed bid phase can consist of either second-price or first-price sealed bids.

[10] See, among others, Goeree and Offerman (2004); Hu et al. (2011); Brunner et al. (2014).

the emphasis there is on the influence of the premium on bidder behavior and auction performance. The existence of the premium distorts bidding incentives, relative to those in a standard Anglo-Dutch auction: since there is now a prize for wining the first stage irrespective of the result of the second stage, some bidders may aim to "win" the first phase, without actually wanting to win the item (and bid timidly in the second phase). Therefore results from premium auctions are not directly transferable to the context of Anglo-Dutch auctions in general.

3. The value model

My aim is to provide a ranking of the Anglo-Dutch auction relative to both the ascending and first-price auctions using a valuation model that is robust to small perturbations. I construct a discrete valuation model where a single object is sold to three risk-neutral bidders, two of whom are 'weak' and one of whom is 'strong'. This setting could be considered to model an auction with a single incumbent and two entrants, or a market that has one clearly advantaged bidder. I thus examine one of two chief motivations for the Anglo-Dutch auction discussed in Klemperer (1998): how revenues are affected by the presence of a (possibly advantaged) incumbent.

I assume there are two weak (W), and one strong (S) bidders. The realized value (or type) of a weak bidder, v^w, is either high (H) or low (L), with probability μ and $1-\mu$. The strong bidder's value, v, is common knowledge. Subsequent to the weak bidders' receiving their value signals, all three bidders participate in a one-off auction. I assume that bidders are risk-neutral, and do not face budget constraints. In what follows, I use b_i to denote bidder type i's bidding function.

While I assume that $v > L$, I do not insist that $v > H$, though for a non-degenerate first-price equilibrium to exist it is necessary that $v > \bar{w} = \mu H + (1-\mu) L$: for a non-trivial analysis, the strong bidder's value must exceed the expected value of the weak bidders' value.[11] There are no entry costs in the base model, though I discuss an extension with entry costs in Section 7.2.

Similarly to Āzacis and Burguet (2008) solution concept I use is perfect Bayesian equilibrium (PBE) in undominated strategies. Since the S-type's value is common knowledge, only the beliefs over the W-types' valuations may get updated in the course of the auction. I will use $\gamma_i(v_j = H)$ to denote type $i's$ prior belief over the probability that the j type's valuation is H, and $m_i(v_j = H|b_j)$ to denote the posterior belief that bidder type i assigns to bidder j's valuation being v_j, after having observed the bid b_j. Furthermore, since in the first-price auction all bids are submitted before any additional information is revealed, no updating of beliefs happens in that auction.

Due to the discreteness of the valuation setup, multiple bidders may submit the same bid with positive probability, and thus a tie-breaking rule is necessary. To break ties, I

[11] See Online Appendix C, Section 13.1.

assume that when two bidders with different values submit the same bid, the bidder with the higher value wins; if both bidders have the same value, ties are broken randomly.[12]

4. The first-price sealed bid auction

Given the nature of asymmetry used in my model, both a pure strategy and a hybrid equilibrium is possible.[13] Depending on where the value of v is relative to an interval $[v_\alpha, v_\beta]$, with $v_\alpha = H - \frac{(1-\mu)^2}{1+(1-\mu)^2}(H-L)$ and $v_\beta = H + \frac{(1-\mu)^2}{1-(1-\mu)^2}(H-L)$, one of three cases applies; they are summarized in Propositions 1.1, 1.2, and 1.3.

When $v > v_\beta$, then v is much higher than $\bar{w}$ and only a pure strategy equilibrium exists. Here the strong bidder's value is so high that he doesn't risk competing with the weak bidders at all, and prefers to win for sure by bidding $b_S = H$ always; the weak bidders then bid their value, and never win.[14]

Proposition 1.1. *When $v > v_\beta$, the equilibrium of the first-price auction is characterized as follows:*

- *Type L weak bidders bid L, Type H weak bidders bid H; the strong bidder bids H.*
- *The expected revenue is H.*

Proof. See Online Appendix C, Section 13. □

When $v \in [v_\alpha, v_\beta]$, the strong bidder's value is moderate, and a hybrid equilibrium exists. Here the L-type weak bidder always bids L, while the H-type weak bidders and the strong bidder mix over a common interval.[15] By standard arguments, I can rule out the presence of atoms at the supremum or on the interior of the mixing interval. Similarly, I can rule out

[12]This tie-breaking rule is one way of eliminating open-set problems with payoffs in a discrete-valuation setup like mine. It is standard within this literature, including Āzacis and Burguet (2008) who also use a discrete-valuation model of auctions.

Furthermore, there are two other economic intuitions for justifying this tie-breaking rule. Firstly, this type of tie-breaking rule is also common in price-setting Bertrand games with asymmetric costs. For example, in a two-player version of this game, the equilibrium price is the marginal cost of the higher-cost player, and the low-cost player attracts all the buyers. What is usually meant here is that the low-cost player actually undercuts the other player by some arbitrarily small ε, without specifying the size of ε.

Secondly, the situation can be viewed in terms of pricing on a discrete grid. In this model the tie-breaking rule is particularly important since the bidder whose valuation is common knowledge will have an atom in his bidding distribution at L. The tie-breaking rule is constructed in such a way that if prices were set on a discrete grid, the strong bidder would have an atom at L, and mix thereafter, while the H-type weak bidder would mix over a range which starts from L + (1 increment).

[13]My use of the terminology here parallels the literature on signaling games: in a hybrid equilibrium, some bidder types play a pure strategy, while others mix. This is in contrast to a fully mixed equilibrium, where every type plays a mixed strategy.

[14]In the pure strategy equilibrium the strong bidder wins the auction due to the tie-breaking rule. The dependence on tie-break rule can be removed by specifying that the strong bidder bids $H + \varepsilon$ instead.

[15]Suppose one of the bidders bids over a closed interval. Then the rival has no incentive to bid above the supremum of that interval (since that only decreases his expected surplus when he wins). Submitting a bid below the infimum of such an interval, conversely, would never win. Hereby the mixing interval must be the same for both types.

the case where both types' distributions have an atom at the infimum of the interval, but I cannot exclude the case in which at most one type has such an atom.[16]

Proposition 1.2. *When $v \in [v_\alpha, v_\beta]$, the equilibrium of the first-price auction is characterized as follows:*

- *Type L weak bidders bid L. Type H weak bidders and the strong bidder mix over an interval $\left[L, \bar{b}\right]$, following the distributions G_H and G_S:*

$$\begin{aligned} & G_H(b) = \tfrac{1-\mu}{\mu}\left(\tfrac{\sqrt{v-L}-\sqrt{v-b}}{\sqrt{v-b}}\right) \qquad G_S(b) = \tfrac{1}{1-\mu}\tfrac{\left(H-\bar{b}\right)}{\sqrt{v-L}}\tfrac{\sqrt{v-b}}{(H-b)} \\ & \text{where } \bar{b} = v - (1-\mu)^2(v-L) \end{aligned} \tag{1}$$

- *The expected revenue is:*

$$R_F = v - (1-\mu)^2(v-L) - (1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \int_L^{\bar{b}} \frac{1}{(H-t)(v-t)^{\frac{1}{2}}} dt \tag{2}$$

Proof. See Online Appendix C, Section 13. ∎

When $v < v_\alpha$ the strong bidder's value is very low, and he will not compete with the H-type bidders at all. Instead, the strong bidder bids $b_S = L$, hoping to win in case both weak bidders have a value of L. The H-type bidders then bid according to a mixing distribution over a common interval.

Proposition 1.3. *When $v \in [L, v_\alpha)$, the equilibrium of the first-price auction is characterized as follows:*

- *Type L weak bidders bid L, and the strong bidder also bids L.*

- *Type H weak bidders bid according to the distribution $G_{H\alpha}(b)$, over the interval $[L, L+\mu(H-L)]$:*

$$G_{H\alpha}(b) = \frac{1-\mu}{\mu}\frac{b-L}{H-b}$$

- *The expected revenue is :*

$$R_{F\alpha} = (1-\mu)^2\left(L + 2\int_L^{L+\mu(H-L)} \frac{(H-L)^2}{(H-t)^3} t dt\right)$$

[16] If both players did have an atom at the infimum, one of them could deviate such that his mixing distribution starts just an ε above the opponent's atom. This reduces the expected surplus only by an arbitrarily small ε, but increases winning probability by a discrete amount.

Proof. See Online Appendix C, Section 13. □

In the case where $v \in [v_\alpha, v_\beta]$ the strong bidder has an atom at L. Its derivative with respect to μ is given by:

$$\frac{d\left(G_S\left(L\right)\right)}{d\mu} = \frac{1}{\left(1-\mu\right)^2\left(H-L\right)}\left(H+L-2v-\left(2-\mu\right)\mu\left(L-v\right)\right)$$

This is always negative when $v > H$: the weight of the atom decreases with μ when the strong bidder's valuation is high. An increase in μ makes it more likely that the strong bidder is bidding against H-type weak bidders, and makes the strong bidder more aggressive. When $v \in (v_\alpha, H)$, the mass of the atom varies non-monotonically with μ : the derivative is negative for small μ, and positive for large μ. When μ is high, the competition from the H-type weak bidders is so intense that the strong bidder prefers to not compete with them: even if the strong bidder would win, the expected surplus would be low. Thus the strong bidder instead plays L with a higher probability and hopes for a large surplus when both weak bidders turn out to be type L. In the limit, when $v < v_\alpha$, it becomes equilibrium behavior for the strong bidder to always bid L.

5. The ascending auction

In an ascending auction, it is a weakly dominant strategy for each bidder is to bid up to their value.[17] The auction stops once all but one bidders have dropped out, and the object is sold to the highest value bidder, at a price equal to the second highest value. Depending on the position of v relative to H, there are two possible outcomes.

When $v \in (L, H]$, if both weak bidders are of type L, the auction stops at price L, with the strong bidder winning. If one weak bidder is of L type, then the auction proceeds up to the strong bidder's valuation, v, and terminates there, with the H-type weak bidder winning. Finally, if both weak bidders are of type H, the auction terminates at H.

When $v \in (H, \infty)$, the strong bidder can always out-bid the weak bidders. The weak bidders never win, but the winning price is determined by the highest realized valuation held by a weak bidder. With probability $(1-\mu)^2$, both weak bidders have valuation L, and with probability $(1-(1-\mu)^2)$ at least one of them has value H. Hence:

Proposition 2. *The revenue from the ascending auction is:*

$$R_{ASC} = \left(1-\mu\right)^2 L + 2\mu(1-\mu)\min(v, H) + \mu^2 H \tag{3}$$

[17] As Li (2017) shows, "staying in" until the price reaches a bidder's own value is an "obviously dominant" strategy; and Pycia and Troyan (2019) further show that this strategy is also One-Step-Foresight dominant. Therefore, I restrict myself to this equilibrium.

6. The Anglo-Dutch auction

Modeling the Anglo-Dutch auction has two main differences from the first-price auction: firstly, only two of three bidders are present in the final bidding stage, and secondly, the remaining bidders have more information since they will have seen at what price one of the bidders dropped out. This observed drop-out point also serves as a reserve price in the Dutch phase of the auction.

I assume that if two bidders drop out simultaneously in the Anglo phase, one of these two is selected at random to play in the Dutch phase. Furthermore, I assume that the remaining bidder does not know whether or not his opponent in the Dutch phase previously tried to drop out of the Anglo phase, or not.[18]

The L-type bidder always bids up to L in the Anglo phase of the Anglo-Dutch auction, and provided he is allowed into the second stage, he also submits a bid of L. In case an L-type does make it to the Dutch stage, they know for sure that the other weak bidder was also an L-type, implying $\gamma_L(H|L) = 0$. Since the strong bidder always has a value above L, the L-type never wins.

Next, observe that the Anglo phase of the auction will never terminate above $\min(v, H)$. If an H-type weak bidder is still present in the Anglo-Dutch auction when the Anglo-phase terminates at $\min(v, H)$, he knows for sure that the rival he is facing has a valuation of at least H.[19] In such a situation the H-type bidder must bid H in the Dutch phase of the auction. Similar reasoning applies to the strong bidder. If the auction terminates at $\min(v, H)$, then provided that the strong bidder is admitted to the Dutch stage, he submits a sealed bid of $\min(v, H)$.[20] The posterior beliefs then are $\gamma_H(H|\min(v, H)) = \gamma_S(H|\min(v, H)) = 1$: if the ascending phase terminates at v, and an H-type made it to the Dutch phase, then $v < H$, and the H-type knows they are facing another H-type for sure. If, conversely, the Anglo phase stopped at H, then it must the case that $v > H$, and it will an H-type and the strong bidder that remain in the Dutch phase.

[18] Āzacis and Burguet (2008) make the same assumption regarding simultaneous drop-outs at the end of the Anglo stage.

While it may seem counter-intuitive that the remaining bidder does not know whether their opponent previously tried to drop out, it is in line with most current implementations of Anglo-Dutch auctions that I am aware if. For example, rather than having all bidders physically present in the same room "holding up paddles" to indicate that they are still in the auction, the Anglo phase is frequently implemented using an electronic clock, with bidders indicating their activity in the Anglo stage remotely, without observing each other directly.

The auctioneer may indicate at each clock step how many bidders are still active, but when the Anglo phase ends, typically that is the only information that is communicated to the bidders who make it to the Dutch phase. The bidder who did not try to drop out at the final step therefore does not know whether one, or two, of their rivals in fact tried to exit at the final step.

[19] If the Anglo phase terminates at v, it must be the case that $v < H$, and both remaining bidders have a valuation of H. When the Anglo phase terminates at H, then $v \geq H$for sure, and so the H-type bidder will be facing a strong bidder with valuation $v \geq H$ in the second round.

[20] If the ascending phase terminates at v, it must be the case that $v < H$, and the strong bidder is not admitted to the second round. If the ascending phase terminates at H, the strong bidder knows that his opponent in the second round has valuation H. Due to the tie-breaking rule, the strong bidder can then ensure winning by bidding $H = \min(H, v)$.

In the only remaining case, the Anglo phase terminates at L. This can occur either because one of the weak bidders is an L-type, or both of them are. As in the first-price auction, there are now two possible equilibria, depending on the strong bidder's value. If v is very high, a pure strategy equilibrium prevails in the Dutch stage: the strong bidder will bid H, and win for sure, while the H-type weak bidder also bids H. If the value of v is not extremely high, both the strong bidder and the H-type weak bidder will mix. Conditional on the Anglo phase terminating at L, the remaining weak bidder will learn that the other weak bidder must have been an L-type, so $\gamma_L(H|L) = \gamma_H(H|L) = 0$. The strong bidder learns at this point that not both weak bidders are H-types, whereby $\gamma_S(L|L) = \frac{(1-\mu)^2}{1-\mu^2}$, and $\gamma_S(H|L) = \frac{2\mu(1-\mu)}{1-\mu^2}$.

Proposition 3 summarizes these equilibria fully.

Proposition 3. *Define the boundary value v_γ as: $v_\gamma = H + \frac{1-\mu}{2\mu}(H-L)$, and the corresponding upper bound of a bidding distribution $b^* = v - \frac{(1-\mu)^2}{1-\mu^2}(v-L)$. Then the equilibrium strategies in an Anglo-Dutch auction are:*

When $v \in (L, v_\gamma)$ (i.e. $b^* \leq H$)

- *The L-type weak bidder bids L in both the Anglo and Dutch phases.*
- *The strong bidder bids up to v in the Anglo phase. If the Anglo phase terminates at* $\min(v,H)$,*the strong bidder submits a bid of* $\min(v,H)$ *in the Dutch phase. If the Anglo phase terminates at L, the strong bidder submits bids in the Dutch phase according to the distribution:*
$$G_S^*(b) = \frac{H-b^*}{H-b} \tag{4}$$
- *The H-type weak bidder bids up to H in the Anglo phase. If the Anglo phase terminates at* $\min(v,H)$, *the H-type bidder submits a bid of H in the Dutch phase. If the Anglo phase terminates at L, the H-type bidder submits bids in the Dutch phase according to the distribution:*
$$G_H^*(b) = \frac{1-\mu}{2\mu}\left(\frac{b-L}{v-b}\right) \tag{5}$$
- *The expected revenue is:*
$$\begin{aligned} R_{AD} \;=\;& \mu^2 H + \left(1-\mu^2\right)v - (1-\mu)^2(v-L) \\ & -(1-\mu)^2(H-b^*)(v-L)\int_L^{b^*}\frac{1}{(H-t)(v-t)}dt \end{aligned} \tag{6}$$

When $v > v_\gamma$ (i.e. $b^* > H$) :

- *The L-type weak bidder bids L in both the Anglo and Dutch phases.*
- *The H-type weak bidder bids H in both the Anglo and Dutch phases.*

- *Strong bidder bids v in the Anglo phase, and bids H in the Dutch phase*
- *The expected revenue is H.*

In both cases, the updated beliefs are:

- *If the Anglo phase terminates at* $\min(v, H)$*, then* $\gamma_H(H|\min(v,H)) = 1$*, and* $\gamma_S(H|\min(v,H)) = 1$.
- *If the Anglo phase terminates at L, then* $\gamma_L(H|L) = \gamma_H(H|L) = 0$*, while* $\gamma_S(L|L) = \frac{(1-\mu)^2}{1-\mu^2}$*, and* $\gamma_S(H|L) = \frac{2\mu(1-\mu)}{1-\mu^2}$.

Proof. See Online Appendix D, Section 14. □

Comparing the first-price and Anglo-Dutch auction equilibria, two corollaries follow.

Corollary 1. *If for given parameter values an equilibrium in mixed strategies exists in the first-price auction, then it also exists in the Anglo-Dutch auction.*

Corollary 2. *There exist parameters for which an equilibrium in mixed strategies exists in the Anglo-Dutch auction, but not in the first-price auction. If $v > v_\alpha$, and an equilibrium in mixed strategies does not exist in the first-price auction, the strong bidder always bids H in that auction.*

These corollaries capture the intuition that, in expectation, the strong bidder faces stricter competition in an outright first-price auction, than in Anglo-Dutch. In the first-price, the strong bidder knows that he can be facing up to two H-types, whereas in Anglo-Dutch, conditional on getting to the Dutch phase, he faces at most one H-type. Thus in the first-price auction the strong bidder switches to the "always bid H" equilibrium under a broader range of parameters.

The mass of the strong bidder's atom in the Anglo-Dutch auction is always decreasing in μ, and does not exhibit the non-monotonicity seen in the first-price auction. As expected, when the likelihood of facing an H-type increases, the strong bidder plays L with lower probability.

It is straightforward to show that G_H first-order stochastically dominates G_H^*, which means that H-type bidders bid more aggressively in the first-price than in Anglo-Dutch auctions, when an equilibrium in mixed strategies exists in both.[21] If an equilibrium in mixed strategies occurs in the Anglo-Dutch auction, then the H-type weak bidder knows he is only bidding against one strong bidder. In mixed-strategy equilibrium in the first-price auction an H-type knows he is facing a strong bidder, but with probability μ he also faces an H-type weak bidder; H-type expects more competition in the first-price auction, and bids more aggressively.

[21] No similar stochastic dominance ranking is available for the distributions of the strong bidder's bids, G_S and G_S^*; the relative shapes of these two distributions depend on μ and v, and for most $(\mu, v)-$ pairs the two distribution functions intersect at some $b \in (L, b^*)$.

7. Analytical Revenue Comparisons

The integrals in the revenue functions for the first-price and the Anglo-Dutch auctions do not, in general, admit analytical solutions. Proposition 4 summarizes the outcomes for parameter values where analytical comparisons are possible.

Proposition 4. *An analytical revenue ranking among Anglo-Dutch, ascending and first-price auctions can be established in the following three cases:*

Case 1. *When $v = H$, ascending generates most revenue, followed by Anglo-Dutch. The first-price auction gives least revenue.*

Case 2. *When $v \in [v_\beta, v_\gamma)$, such that $\bar{b} \geq H$, first-price gives higher revenue than both ascending and Anglo-Dutch.*

Case 3. *When $v > v_\gamma$, such that $b^* \geq H$, Anglo-Dutch and first-price tie on revenue, and both give higher revenue than ascending.*

Proof. See Online Appendix E, Section 15.4. □

In Case 1 of Proposition 4 the ascending auction gives revenue H when at least one weak bidder is H. The Anglo-Dutch auction only gives revenue H if both weak bidders are H, and the first-price auction never gives such high revenue. This effect dominates over the higher revenues given by ascending and first-price when more of the weak bidders are of L-type.

Cases 2 and 3 of Proposition 4 relate to the switch-over points in first-price and Anglo-Dutch auctions. When $v \in [v_\beta, v_\gamma)$, the strong bidder switches to always bidding H in the first-price auction, while this has not yet occurred in the Anglo-Dutch auction. Hence the first-price revenue is always H, which is more than in the other auctions.[22] At the point where $v > v_\gamma$, the strong bidder always bids H in the Dutch phase of the Anglo-Dutch auction also, giving the same expected as in the first-price. While I have presented Proposition 4 in terms of cut-off values for v, a dual set of propositions could be presented in terms of μ, as these parameters play a dual role in determining $\bar{b}$ and b^*.

7.1. Relating the Results to Maskin and Riley (2000)

The intuition behind the revenue results above can be explained within the framework of Maskin and Riley (2000). They consider thee types of asymmetries, two of which are relevant to my model. When the strong bidder's value distribution is a stretch of the weak bidder's distribution, the authors find that the first-price auction out-performs the ascending auction. However, when the weak bidder's distribution has been obtained from the strong bidder's distribution by shifting some mass from the upper end to the lower end, the ascending auction performs better.

[22] The relative position of ascending and Anglo-Dutch auctions in this case is ex-ante ambiguous, and depends on the model's parameters, as discussed in Section 8.

In my model, the weak bidder has a binary distribution on (L, H), and the strong bidder has a point distribution at v. We can get from one to the other using two transformations of the kind described in Maskin and Riley (2000). First, stretch the upper end of the weak bidder's distribution such that it terminates at v, rather than H. Second, move $(1-\mu)$ of mass from the lower end (L), to the upper end of the distribution, v. This two-step transformation gives a degenerate distribution at v. The magnitude of the stretch-effect depends on the difference between H and v, while the magnitude of the mass-reallocation effect grows as μ becomes smaller, as then more mass is shifted to v.

The Maskin and Riley framework can explain why the first-price auction performs better when v is larger. In the case when $v > H$, the stretch effect and the mass-shift effect both work to favor the first-price auction - and the effects become more pronounced as v increases. When $v < H$, the two effects work in opposite directions: the stretch effect now works to lower the upper end of the distribution, which favors an ascending auction, but the mass-reallocation effect moves $(1-\mu)$ of probability from L to $v > L$, and so still favors the first-price auction. Thus the first-price auction performs particularly badly when v is low. Overall, in my model when v is large the first-price auction is favored by the value structure; when v is low, the value-structure favors the ascending auction. These results are consistent with those in Maskin and Riley (2000).

The results so far suggest that the Anglo-Dutch auction never ranks at either extreme of the revenue ranking, and in the context of Maskin and Riley (2000), it would seem intuitive that a combination of two auctions would give a revenue that is a convex combination of its components. This alone, however, does not take into account the effect of the reserve price that is set after the Anglo phase of the Anglo-Dutch auction, nor the updating in the bidding distributions that occurs after information is revealed at the end of the Anglo phase. Numerical result in the next section show that there exist parameters for which the revenue of the Anglo-Dutch auction is *not* a convex-combination of ascending and first-price revenues.

7.2. Small Entry Costs and Almost-Common Values

If my model is modified to include small entry costs for the weak bidders, the relative performance of the Anglo-Dutch auction improves considerably. Recall that for Case 1 in Proposition 4 when $v = H$, the revenue ranking is $R_{First-price} < R_{Anglo-Dutch} < R_{Ascending}$.

Suppose that the two weak bidders are instead "potential entrants", and they have to pay a small entrance cost $c > 0$ to participate in the auction and observe their value, while the strong bidder pays no such cost. There will be no entry in the ascending auction, since none of the entrants have a positive surplus conditional on entering; the revenue from a no-reserve ascending auction will be minimal. However, for c small enough, both entrants will enter in both the Anglo-Dutch and first-price auctions.

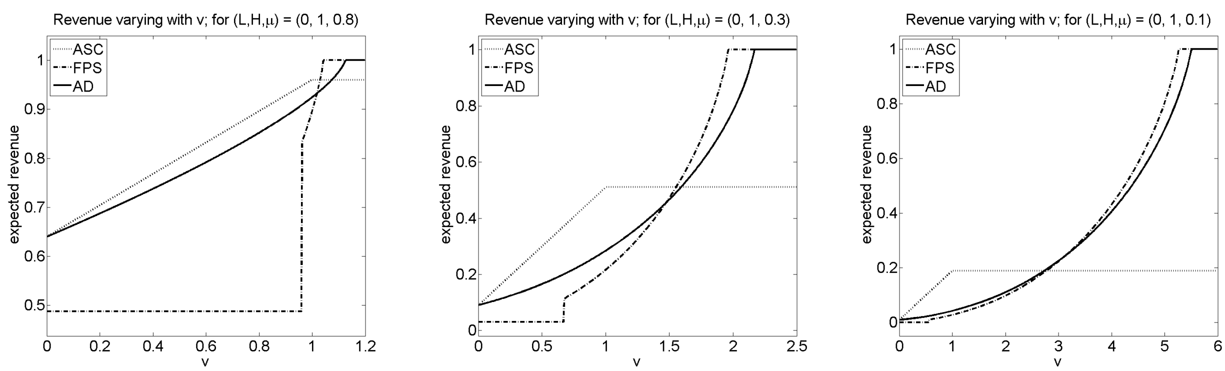
Conditional on both bidders participating, Proposition 4 shows that in this case the Anglo-Dutch auction outperforms the first-price auction on revenue. Since the expected revenue functions from the first-price and Anglo-Dutch auctions are continuous in v, the above argument extends to the case when $v = H + \varepsilon$, which results in an "almost-common value" model with one advantaged incumbent, similar to that in Klemperer (1998). In my model, then, the Anglo-Dutch auction performs particularly well when entry costs and almost-common values are an issue.

8. Numerical revenue comparisons

In this section I present two kinds of graphs which more broadly characterize the behavior of the three auctions at hand. The first kind shows how revenues vary with v, for given values of μ, L and H. The second kind shows how revenue behaves when μ is varied, for a particular fixed set of L,H, and v. While my graphs are drawn with L and H fixed at 0 and 1, this is without loss of generality: different (L, H) pairs would only stretch and shift the graphs. Qualitatively, the graphs would have the same shape, and the same relative relationships would hold.

8.1. Revenue variation with v, with other parameters fixed

Figure 1a shows how revenues from all three auctions behave when μ is high ($\mu = 0.8$). Based on the results from Section 7, we should expect that the Anglo-Dutch auction performs quite poorly for this parameter value. The behaviour of the first-price auction changes depending on whether $v < v_\alpha$: this jump in behavior is due to the strong bidder essentially exiting the market, and not actively trying to win against an H-type.



(a) Anglo-Dutch performs poorly at high μ.

(b) Anglo-Dutch performs better at lower μ.

(c) Anglo-Dutch can be revenue-dominant for small μ

Figure 1: Revenue comparisons with variable v, with other parameters fixed.

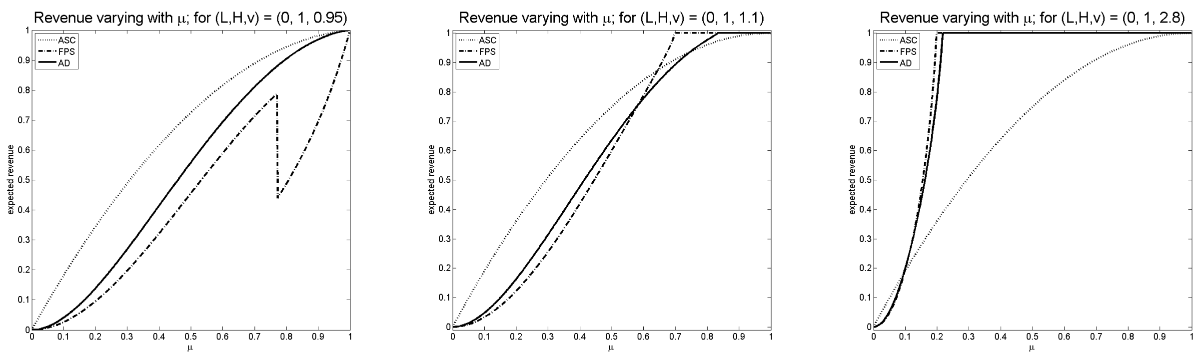
For these parameter values the Anglo-Dutch auction never performs best, and ranks last for v in the range $[1.02, 1.07]$. When v is large enough (e.g. greater than 1.126), the optimal strategy for the strong bidder in both the first-price and Anglo-Dutch auctions is to always bid H; for large v the first-price and Anglo-Dutch auctions both out-perform the ascending auction. Figure 1b shows that when μ is decreased, the range over which the Anglo-Dutch auction performs worst becomes smaller.

When μ becomes even smaller, the Anglo-Dutch auction eventually ranks first for a range of v. The largest value of μ for which this occurs is $\mu = 0.16$. Figures 1c illustrates an example of parameters at which the superiority of the Anglo-Dutch auction is possible.

As suggested by the analytical comparisons, the Anglo-Dutch auction is revenue-dominant at some parameter values. This requires a high value of v, and a low value of μ, for a given pair of H and L. Under such parameters, the ascending auction performs poorly, since the revenue in that auction never depends on v. In the auctions which have a first-price element a higher value of v leads to more aggressive bidding by the strong bidder even if the opponent's signal realization is low.

8.2. Revenue variation with μ, with other parameters fixed.

This section shows how the three auctions perform when the probability of a weak bidder's being an H-type is changed. I start with the case where the strong bidder's valuation is between that of the two weak bidders, as depicted by Figure 5, where $v = 0.95$. The shape of the graph is similar for all $v \in (L, H)$.



(a) Anglo-Dutch ranks intermediate when $v < H$.

(b) When $v > H$, auctions with a first-price stage can dominate.

(c) Anglo-Dutch is revenue-dominant for a range of μ.

Figure 2: Revenue comparisons with variable μ, with other parameters fixed.

In Figure 2a, the Anglo-Dutch auction always ranks above the first-price auction, but below the ascending auction. The downward jump in revenue of the first-price auction is due to the strong bidder stopping to compete with the H-types: as μ increases with $v < H$, it is more likely that we end up in the $v < v_\alpha$ case. With high μ and low v, the strong bidder prefers to bid very low, and hope to make a profit if both of his opponents turn out to be L-types. The consequent fall in revenue is substantial.

When $v > H$, there is a range of values of μ for which the auctions with a first-price component dominate the ascending auction. Figure 2b depicts such a case, with $v = 1.1$. When μ is large, both the Anglo-Dutch and first-price auctions dominate the ascending auction, but there is also a range of μ values for which the Anglo-Dutch auction performs worst. Finally, by picking v appropriately, I can also illustrate a range of $\mu-$values for which the Anglo-Dutch auction is revenue-dominant. Figure 2c illustrates one such case, with $v = 2.8$.

8.3. The Overall Picture

Figures 3 and 4 summarize the revenue-dominance and revenue-inferiority results for each pairing of μ and v. The range of parameters at which the Anglo-Dutch auction is revenue-dominant is small, but equally, it ranks worst in an even smaller area.

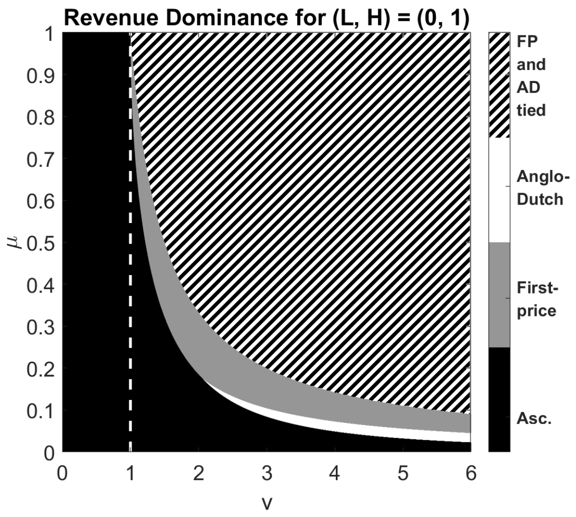


Figure 3: Areas of revenue-dominance, by auction. Anglo-Dutch is dominant for a small subset of parameters only.

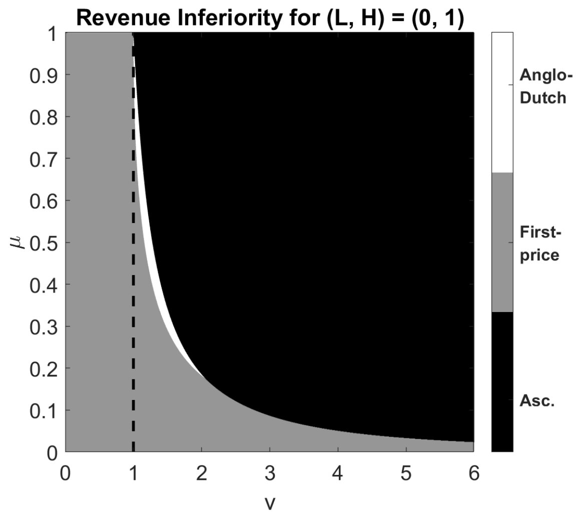


Figure 4: Areas of revenue-inferiority, by auction. Anglo-Dutch performs worst for a similarly small subset of parameters.

As v increases, the area where the Anglo-Dutch auction dominates grows, while it never performs worst when $v > 2$. While the other two auctions are revenue-dominant for larger sets of parameters, as suggested by Figure 9, they also rank last in a larger number of cases (as shown in Figure 10). The ascending auction, in particular, under-performs when v is high - and ranks last in the large area where the first-price and Anglo-Dutch auctions are tied. Depending on the auctioneer's beliefs over the likely values of μ and v, the Anglo-Dutch auction can best "on average". For example, assuming uniform distribution over the parameter space defined by $\mu \in [0, 1]$ and $v \in [0, V]$, the Anglo-Dutch auction is best on average whenever $V > 2.5$.

If small entry costs are present, the Anglo-Dutch auction would also be revenue-dominant in the whole area where the ascending auction was dominant for $v > H$; the ascending auction would rank revenue-last, instead of first-price. In this case, if we average the performance of the three auctions using the same uniform parameter-rectangle as above, the Anglo-Dutch auction ranks first for all V. Overall, then, the Anglo-Dutch auction is rarely best, but even more rarely worst - and may well be best on average.

9. Efficiency and welfare: numerical comparisons

Following the same convention as Bustos and Costinot (2003), I measure efficiency by the expected value of the winning bidder's valuation. In my setup the ascending auction is always efficient, since the highest-value bidder always wins. To obtain a relative measure for the other two auctions, I divide the winner's expected valuations in the Anglo-Dutch and first-price auctions by the valuation of the winning bidder in the ascending auction.

When $v = H$, all auctions are equally efficient, since irrespective the identity of the winner, he will have a valuation of H. Furthermore, all auctions will also be fully efficient when $v > v_\gamma$, since in that case the strong bidder always wins in all three auctions.

Figure 5 shows that when μ is high, the Anglo-Dutch auction may be more efficient than the first-price auction for some values of $v < H$, and is more efficient for moderately high $v > H$. When μ decreases, we see from Figure 6 that the Anglo-Dutch auction is less efficient than the first-price for $v < H$, but is still more efficient for a range of $v > H$. In the range where the Anglo-Dutch revenue-dominates, it is also more efficient than the first-price.

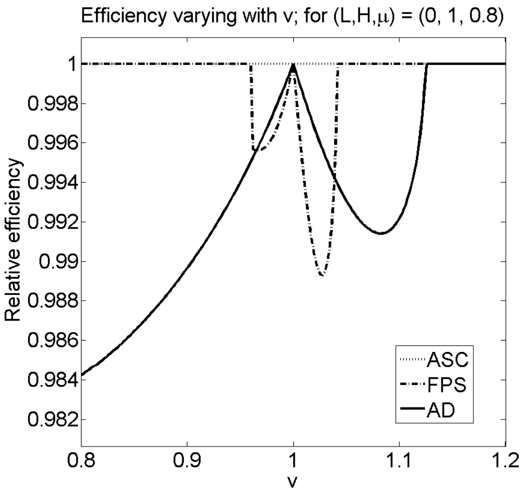


Figure 5: Efficiency comparison with a high μ. No clear dominance pattern emerges.

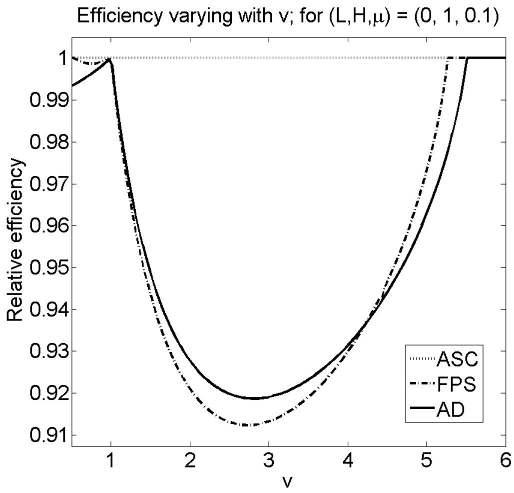


Figure 6: With low μ: Anglo-Dutch outperforms first-price for most $v > H$.

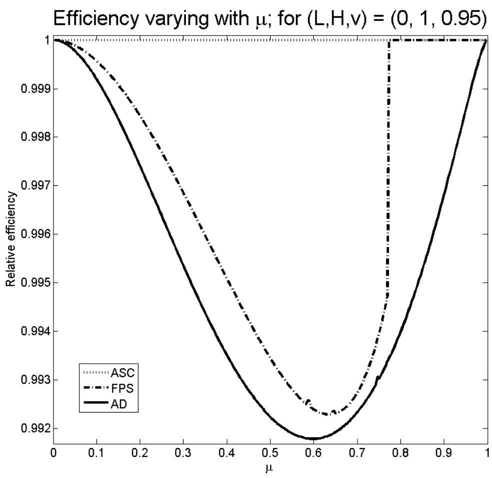


Figure 7: Anglo-Dutch is relatively inefficient in the case when $v < H$.

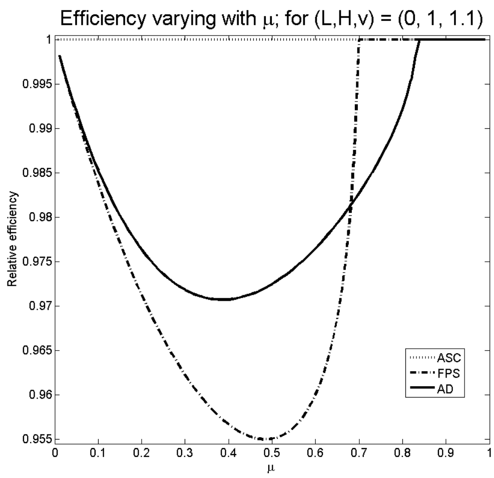


Figure 8: When $v > H$, Anglo-Dutch is relatively efficient for most μ.

From figure 7 we see that for $v < H$, the first-price auction is more efficient than the Anglo-Dutch auction for most μ. Conversely, figure 8 shows that the conclusions are reversed when $v > H$, and for most μ-values the Anglo-Dutch auction is more efficient than the first-price. In the range in which the Anglo-Dutch auction revenue-dominates the first-price auction (for $\mu < 0.16$), the Anglo-Dutch auction is also more efficient.

The relative efficiency of the Anglo-Dutch and first-price auction are sensitive to the assumption of discrete values: when moving around the parameter space, the equilibria in first-price and Anglo-Dutch auctions can switch between hybrid and pure-strategy equilibria. This, in turn, may lead to large shifts in efficiency for small parameter changes. Nonetheless, in the current setting the efficiency differences are small.

Intuitively, we would expect that the Anglo-Dutch is more efficient than the first-price because in the Anglo-Dutch auction it is always one of the two highest-valuing bidders who win. However, when v is very large, the equilibrium in the first-price auction more readily switches to $\min(v, H)$ being played by the strong bidder, and H being played by the H-type - here the first-price becomes more efficient. This effect also dominates for values of v for which v is slightly less than v_β. The argument for the case when $v < H$ is analogous to the argument for revenue: Anglo-Dutch performs better than the first-price for the cases when both weak bidders obtain a value of H, but worse in the case when one weak bidder is H and the other is L, since the H-types bid more aggressively in the first-price auction. Figures 7 and 8 show that either effect can dominate, depending on model parameters.

Figures 9 and 10 summarize the overall efficiency performance of Anglo-Dutch and first price auctions.[23] Each auction ranks worst over a similar size of the parameter space, but at high enough v both also achieve full efficiency. The overall differences in efficiency are small, especially compared to potential differences in revenue. Thus for a policymaker who has a compound objective function which includes both efficiency and revenue, the Anglo-Dutch auction is likely to perform well if revenue is relatively more important.

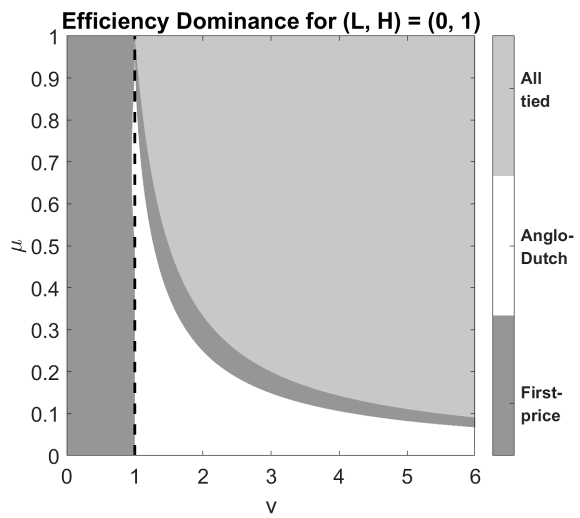


Figure 9: Areas of efficiency-dominance between first-price and Anglo-Dutch auctions. For most parameters, all three auctions tie.

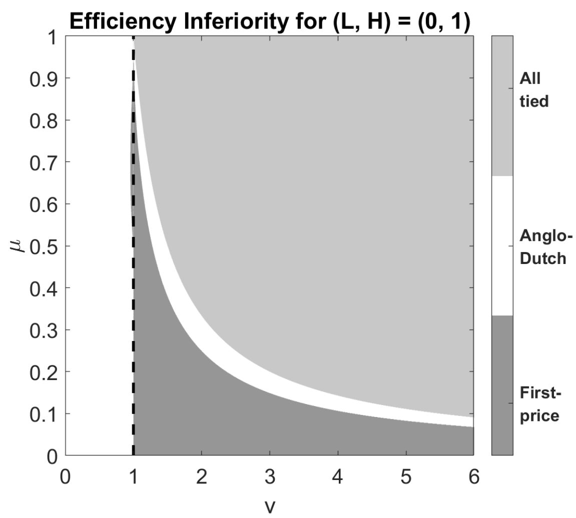


Figure 10: Areas of efficiency-inferiority between first-price and Anglo-Dutch auctions. When $v > H$, Anglo-Dutch rarely ranks last.

[23]The ascending auction is always efficient, and therefore not included in the figures, for clarity.

10. Extensions, limitations, and conclusions

The model in this paper does not include features that would inherently bias the outcome in favor of the Anglo-Dutch auction. Most significantly, there is no endogenous entry or risk aversion - so my model produces results which are conservative with respect to the performance of the Anglo-Dutch auction. In the base model the range of parameters in which the Anglo-Dutch auction performs best is narrow, but if we extend the model to consider small entry costs, we can obtain a broader class of examples in which the Anglo-Dutch auction dominates its rivals: the assumption of almost-common values falls in this category.

Extending the base model to consider budget constraints, and an alternative process of information revelation - as covered in Online Appendix A, Section 11 - does not improve the relative performance of the Anglo-Dutch auction. Overall, the Anglo-Dutch auction is quite robust: across all variants of my model, the range of parameters for which the auction performs worse than both alternatives is usually small, or non-existent. Thus if a policymaker has a relatively inaccurate prior information about the auction context, the Anglo-Dutch auction can perform better on average than both of its components alone.

The assumptions of exogenous entry and discrete values allowed me to derive a complete ranking between the Anglo-Dutch auction, and its component auctions, which is an extension of the existing literature. These assumptions are restrictive, and though my results do not directly generalize to multi-unit hybrid auctions, they indicate that evaluating whether hybrid auctions are "the best of both worlds" is not trivial. Results from more complex settings, including multi-unit and package demand patterns, still remain theoretically intractable - hence obtaining equilibria via numerical optimization is a likely and fruitful next avenue of research in this area.

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References

K. Abbink, B. Irlenbusch, P. Pezanis-Christou, B. Rockenbach, A. Sadrieh, and R. Selten. An experimental test of design alternatives for the British 3G/UMTS auction. *European Economic Review*, 49(2):505–530, 2005.

L. Ausubel and O. Baranov. A practical guide to the combinatorial clock auction. *The Economic Journal*, 127:F334–F350, 10 2017. doi: 10.1111/ecoj.12404.

H. Āzacis and R. Burguet. Incumbency and entry in license auctions: The Anglo–Dutch auction meets another simple alternative. *International Journal of Industrial Organization*, 26(3):730–745, 2008.

C. Brunner, J. K. Goeree, C. A. Holt, and J. O. Ledyard. An experimental test of flexible combinatorial spectrum auction formats. *American Economic Journal: Microeconomics*, 2(1):39–57, 2010.

Christoph Brunner, Audrey Hu, and Jörg Oechssler. Premium auctions and risk preferences: an experimental study. *Games and Economic Behavior*, 87:467–484, 2014.

J. Bulow, M. Huang, and P. Klemperer. Toeholds and takeovers. *Journal of Political Economy*, 107(3):427–454, 1999.

A. Bustos and A. Costinot. The Anglo-Dutch Auction. *Coursework paper, Princeton University*, 2003.

P. Cramton. Spectrum auction design. *Review of Industrial Organization*, 42(2):161–190, 2013.

Jacob K Goeree and Theo Offerman. Notes and comments the amsterdam auction. *Econometrica*, 72(1):281–294, 2004.

Audrey Hu, Theo Offerman, and Liang Zou. Premium auctions and risk preferences. *Journal of Economic Theory*, 146(6):2420–2439, 2011.

J.H. Kagel, Y. Lien, and P. Milgrom. Ascending Prices and Package Bidding: A Theoretical and Experimental Analysis. *American Economic Journal: Microeconomics*, 2(3):160–185, 2010. ISSN 1945-7669.

J.H. Kagel, Y. Lien, and P. Milgrom. Ascending prices and package bidding: Further experimental analysis. *Games and Economic Behavior*, 85:210–231, 2014.

P. Klemperer. Auctions with Almost Common Values: The "Wallet Game" and its Applications. *European Economic Review*, 42(3):757–69, 1998.

D. Levin and L. Ye. Hybrid auctions revisited. *Economics Letters*, 99(3):591–594, 2008.

Shengwu Li. Obviously strategy-proof mechanisms. *American Economic Review*, 107(11): 3257–87, 2017.

A. Lunander and S. Lundberg. Bids and costs in combinatorial and noncombinatorial procurement auctions - evidence from procurement of public cleaning contracts. *Contemporary economic policy*, 31(4):733–745, 2013.

D. Marszalec, A. Teytelboym, and S. Laksá. EPIC fail: how below-bid pricing backfires in multiunit auctions. *CIRJE Working Paper Series,*, CIRJE-F-1096, 2020.

E. Maskin and J. Riley. Asymmetric auctions. *Review of Economic Studies*, 67(3):413–38, July 2000.

National Audit Office. 4g radio spectrum auction: lessons learned. **URL:** https://www.nao.org.uk/wp-content/uploads/2015/03/4G-radio-spectrum-auction-lessons-learned.pdf, 2014.

Marek Pycia and Peter Troyan. A theory of simplicity in games and mechanism design. 2019.

J.G. Riley. Expected Revenue from Open and Sealed Bid Auctions. *The Journal of Economic Perspectives*, 3(3):41–50, 1989.

M.S. Robinson. Collusion and the Choice of Auction. *The RAND Journal of Economics*, 16(1):141–145, 1985.

Christiaan van Bochove, Lars Boerner, and Daniel Quint. Anglo-dutch premium auctions in eighteenth-century amsterdam. Technical report, Diskussionsbeiträge, 2012.

11. Online Appendix A – Modifications of the Base Model

This section discusses two tractable extensions of the base model.

11.1. Different Pattern of Information Revelation - the L^+/L^- Model

My basic model of the Anglo-Dutch auction assumes that both of the weak bidders are identical. This assumption is analogous to saying that the strong bidder "doesn't see" the identity of the bidder that drops out in the first round, if the ascending phase ends at L. In practice this is unlikely to be true, and the identity of the drop-out bidders can be identified. To model a situation where the identity of the drop-out can be seen by the strong bidder, I can modify the valuation structure in the model. Instead of assuming that both weak type bidders are ex-ante identical, I now assume that there are two ex-ante types of weak bidders. One of the weak bidders is of type W^+, and has value of H with probability of μ, and L^+ with probability of $(1-\mu)$. The other bidder is of type W^-, and has valuation of H with probability μ, and L^- with probability $(1-\mu)$. I assume that L^+ is just above L, and L^- is just below L.

Given the new value structure, the strong bidder now gains more information after the Anglo stage is over. In the case in which the Anglo auction terminates at L^-, he still doesn't know whether his rival has a value of L^+ or H; the conditional probability of the rival's being an H-type changes. Is is straightforward to show that the conditional probability of the rival's being H is lower in the $\mathrm{L}^+/\mathrm{L}^-$ variant relative to the base model, so we should expect less aggressive bidding when an equilibrium in mixed strategies exists. Yet if the Anglo auction ends at L^+, the bidder knows that the only situation in which the W^- type would stay in at a price of L^+ is when his valuation is H. Thus in this case the strong bidder behaves "as if" he had observed a price of $\min(v, H)$ at the end of the Anglo phase. He thus bids $\min(v, H)$ for sure, rather than mixing (as was the case in the base model). The equilibrium mixing distributions are then given by:

$$
\begin{aligned}
\hat{G}_H(b) &= \frac{(1-\mu)}{\mu}\left(\frac{b-L}{(v-b)}\right) \\
\hat{G}_S(b) &= \frac{H-\hat{b}}{H-b} \\
\hat{b} &= v-(1-\mu)(v-L) < b^*
\end{aligned}
$$

Thus an equilibrium in mixed strategies exists in the $\mathrm{L}^+/\mathrm{L}^-$ model, whenever it exists in the basic model. Furthermore, mixing now occurs for a broader range of v. Comparing $\hat{G}_H$ and $\hat{G}_S$ with G_H^* and G_S^* it is clear that the G^* distributions first-order stochastically dominate their $\hat{G}$ equivalents. This suggests that bidding is more aggressive in the base model. The reason for this finding is that in the $\mathrm{L}^+/\mathrm{L}^-$ model, it is less likely that actual mixing occurs in the Dutch stage: in the base model mixing occurs with probability $1-\mu^2$, whereas in the new model it occurs only with probability $\mu(1-\mu)$. Conditional on a mixed-strategy equilibrium being played, it is more likely in the basic model that the strong bidder is facing an H-type in the second round (this probability is $\frac{2\mu(1-\mu)}{1-\mu^2}$), compared with the $\mathrm{L}^+/\mathrm{L}^-$ variant (where the probability is $\mu < \frac{2\mu(1-\mu)}{1-\mu^2}$, $\forall \mu \in (0,1)$). There are thus two new effects on revenue, when we compare the $\mathrm{L}^+/\mathrm{L}^-$ model with the base case. Firstly, since the probability of an equilibrium in mixed strategies occurring is now lower, the probability of the revenue being directly $\min(v, H)$ is larger - this effect enhances expected revenue. In the second case when mixing does occur, bidding is less aggressive, and so expected revenue from the mixing scenario falls.

It can be shown analytically that the expected revenue from the $\mathrm{L}^+/\mathrm{L}^-$ model is less than it would be in the base model, so the modified version of the Anglo-Dutch auction performs revenue-wise worse than the base model. I find that in the modified model the range of parameters for which the Anglo-Dutch auction performs worse than both rivals is now increased, but for majority of parameter values the Anglo-Dutch auction still ranks intermediate. Despite being revenue-inferior to the base model, the $\mathrm{L}^+/\mathrm{L}^-$ model of the Anglo-Dutch auction is more efficient than in the base model, for some parameter values.

11.2. Budget Constraints Model with Common Values

The model I have discussed in this paper can be straightforwardly extended to consider different budget constraints, instead of different valuations. I thus assume in this section that the values of L, H and v are in fact budget constraints of the different types of bidders, and the object has a true common value of x to each bidder. All bidders know the value of x, and I assume it is larger than all the budget constraints. I use the same tie-breaking rule as before, though now I make all the decisions contingent on budgets rather than valuations. Deriving equilibrium bidding distributions in this model is exactly analogous to the standard model, and in fact when an equilibrium in mixed strategies exists in the first-price auction, bidding distributions for the first-price and Anglo-Dutch auctions can be obtained by substituting x for v and H in equations (1), (4) and (5). The interesting new comparison in this model is to keep L, H, v and μ fixed, and vary x.

Results from the budget constrained model are unfavorable for the Anglo-Dutch auction. The auction never ranks strictly first in terms of expected revenue, but ranks last for a relatively small range of parameters. As before, for the majority of parameter values the Anglo-Dutch auction ranks as intermediate. Hence while the Anglo-Dutch auction would never be strictly preferred by a policymaker who knows x exactly, it might nonetheless be a desirable option when the policymaker is uncertain of the position of x.

12. Online Appendix B: The Degenerate Case, when v$\in$ [0,L]

12.1. First-price auction

Here the strong bidder's value is so low that his maximum bid is of no viable threat to the weak bidders. Hence, the lowest "active" valuation against which weak bidders have to compete is L, which is the lowest possible valuation for a weak bidder. Thus I can construct an equilibrium similar to that of Āzacis and Burguet (2008). The properties of the equilibrium are that:

- Strong bidder bids v, and never wins
- Weak bidder of type L bids L, and can only win of both weak bidders are of L-type.
- H-type weak bidders mix on $[L, \bar{b}_0]$, with cumulative bidding density $G_{H0}(b)$.

By similar reasoning as in the cases when $v > v_\alpha$ for the first-price auction, at most one of the bidders can have an atom at L, so it follows that G_{H0} cannot have such an atom (else, in the case when there are two H types, both players would have an atom, which is a contradiction). Given the above description, the profit function for an H-type is:

$$\Pi_{H0}(b) = ((1-\mu) + \mu G_{H0}(b))(H-b) \tag{7}$$

Using the fact that, $G_{H0}(L) = 0$ and $G_{H0}\left(\bar{b}\right) = 1$, and the fact that all values in the support of the mixing distribution give the same payoff:

$$\Pi_{H0}(L) = \Pi_{H0}\left(\bar{b}\right) = \Pi_{H0}(b)$$

Hence:

$$\begin{aligned} (1-\mu)(H-L) &= \left(H-\bar{b}_0\right) \\ &\implies \bar{b}_0 = H-(1-\mu)(H-L) \end{aligned}$$

And:

$$\begin{aligned} (1-\mu)(H-L) &= ((1-\mu)+\mu G_{H0}(b))(H-b) \\ &\implies G_{H0}(b) = \frac{1-\mu}{\mu}\left(\frac{b-L}{H-b}\right) \end{aligned}$$

Proposition 3. *The equilibrium in the FPS for $v < L$ is thus fully characterized as follows:*
Strong bidder always plays v
L-type weak bidder always plays L
H-type weak bidder mixes according to:

$$G_{H0}(b) = \frac{1-\mu}{\mu}\left(\frac{b-L}{H-b}\right)$$

The expected revenue in Case A is:

$$R^0_{FPS} = \left(1-\mu^2\right)L + \mu^2 H \tag{8}$$

12.2. Ascending Auction

In this case, the strong bidder's value is below the lowest conceivable value for the weak bidders, so the strong bidder never wins in the auction - indeed, he never even determines the final price.

If both weak bidders are of type H, they both bid up to H, and this is where the ascending auction terminates; the probability of this event is μ^2. In all the remaining cases, when at least one bidder is of type L, the auction stops at L; the probability of this happening is $(1-\mu^2)$ .Hence:

Proposition 4. *The revenue from an ASC auction, when $v < L$ is:*

$$R^0_{ASC} = (1-\mu^2)L + \mu^2 H = R^0_{FPS} \tag{9}$$

12.3. Anglo-Dutch auction

In this case the Anglo-Dutch auction will be equivalent to the first-price auction with $v \in (0, L]$: the dropping bidder is always the incumbent, so in the Dutch phase it is the two entrants playing against each other, which is essentially the same as playing in a first-price auction, since the "reserve price" has no bite. That is, in the first-price auction the incumbent also always bids v, which has the same effect as the reserve price in the Anglo-Dutch. In this context, the additional information revealed by the Anglo-Dutch auction has no value.

Proposition 5. *The equilibrium of the Anglo-Dutch auction when $v < L$ is analogous to the equilibrium of the FPS auction for the case when $v < L$. More fully, the complete strategies require the strong bidder to bid up to v in the Anglo-stage, and bid v in the Dutch stage also. The L-type weak bidder bids up to L in the Anglo phase, and submits a bid equal to L in the Dutch phase. Finally, the H-type weak bidder bid up to H in the Anglo-phase, and bid according to G_{H0} in the Dutch phase.*

12.4. Comparison

Comparing the revenues from the first-price and ascending auctions, and noting that the first-price and the Anglo-Dutch auction are revenue-equivalent, we see that indeed all three auctions are revenue equivalent for when $v \in [0, L]$. This result is not surprising, and can be related to Riley (1989) and his derivation of revenue equivalence in discrete valuation models. In my model, as it stands when $v < L$, what we essentially have is a setting where two ex-ante symmetric weak bidders bid against each other, and they both have valuations drawn from the same discrete distribution; we also have risk-neutrality, and other independent private value assumptions. So my model satisfies the assumptions of Riley's model, and hence the conclusion of revenue-equivalence follows.

13. Online Appendix C: Details on the First Price Auction in the Non-Degenerate Case (when $v > L$).

13.1. Deriving the FPS Equilibrium (Proposition 1.2)

Proposition 1.2 covers all values of v, except the case when $v < L$, which generates revenue-equivalence among the three auctions, and is covered in Online Appendix B, Section 12. The behavior of the expected revenue at the two boundary points, v_α and v_β will be very different. At v_β, all mixing types switch to bidding H in a continuous manner - the mixing distributions put more and more mass close to H, until it becomes optimal for the strong bidder to bid this value for sure. However, when v falls below v_α, the change in revenue is discrete: the strong bidder decides 'not to participate', so while the H-type weak bidders still mix in equilibrium, they only expect to be bidding against one viable opponent at most. The lower revenue is thus caused by loss of competition if the value of the strong bidder is too low.

The main characteristics of the equilibrium are thus:

- L-type weak bidder bids L, and expects no surplus.
- The H-type weak bidders and the strong bidder mix on an interval $[L, \bar{b}]$, according to the cumulative distributions G_H and G_S respectively.
- The strong bidder's bidding distribution has an atom at L.

The profit functions are derived step-wise, as follows. If a high type is to win, he must beat the strong bidder, and either beats an L-type opponent for sure (happens with probability $(1-\mu)$), or he must bid higher than another H-type, which happens with probability $\mu G_H(b)$. Thus:

$$\Pi_H(b) = G_S(b)\left((1-\mu) + \mu G_H(b)\right)(H-b) \tag{10}$$

If the strong bidder is to win, he must either beat two L-types (occurs with probability $(1-\mu)^2$), or he must beat one H type, and one L type (occurs with probability $2\mu(1-\mu)G_H(b)$), or he must beat two H-types (with probability $\mu^2 G_H^2(b)$). Hereby:

$$\Pi_S(b) = \left((1-\mu)^2 + 2\mu(1-\mu)G_H(b) + \mu^2 G_H^2(b)\right)(v-b)$$

From the structure of the surplus functions above, it follows that it is the strong bidder who has an atom at L. Indeed, if the valuation realizations were such that we would have two H-types bidding in the auction, we would end up in a situation where two bidders have an atom at L - but I showed in Section 4 that this cannot happen in equilibrium.

Using the fact that, $G_H(L) = 0$ and $G_H(\bar{b}) = G_S(\bar{b}) = 1$, and the fact that all values in the support of the mixing distribution give the same payoff, $\Pi_S(L) = \Pi_S(\bar{b})$ and so:

$$\begin{aligned} (1-\mu)^2(v-L) &= v-\bar{b} \\ &\implies \bar{b} = v-(1-\mu)^2(v-L) \end{aligned}$$

To obtain the bidding distributions, solve the two equations for G_S and G_H:

$$\begin{aligned}\Pi_S(b) &= \Pi_S(L)\\ \Pi_H(b) &= \Pi_H(L)\end{aligned}$$

This yields:[24]

$$G_H(b) = \frac{(1-\mu)}{\mu}\frac{\sqrt{v-L}-\sqrt{v-b}}{\sqrt{v-b}} \tag{11}$$

$$G_S(b) = \frac{1}{1-\mu}\frac{\left(H-\bar{b}\right)}{\sqrt{v-L}}\frac{\sqrt{(v-b)}}{(H-b)} \tag{12}$$

To complete the definition of equilibrium for $v > L$, two issues remain. Firstly, when v is very small, the above bidding distribution for the strong bidder is not well-defined. Secondly, when v is very large, $\bar{b} > H$, and an equilibrium in mixed strategies cannot exist, since it would require bidding distributions where the H-type bidder bids above his valuation with positive probability.

With respect to the first problem I notice that for v close to L, G_S is first increasing, and then decreasing; in this case it cannot be a well-defined (cumulative) equilibrium bidding distribution. To check for the conditions when an admissible cumulative density exists, consider the marginal density:

$$\frac{\partial G_S}{\partial b} = \frac{1}{1-\mu}\frac{\left(H-\bar{b}\right)}{\sqrt{v-L}}\frac{1}{(H-b)(v-b)^{\frac{1}{2}}}\left(\frac{v-b}{H-b}-\frac{1}{2}\right)$$

The expression in brackets is decreasing in b, whence it takes the minimum value when $b=\bar{b}$, and must be non-negative at that point. The condition for a well-defined cumulative density then becomes:

$$\begin{aligned}v-L &\geq \frac{1}{(1-\mu)^2}(H-v_\alpha)\\ v &\geq \left(H-\frac{(1-\mu)^2}{1+(1-\mu)^2}(H-L)\right)\end{aligned}$$

Define the lower bound for v, for which equilibrium in mixed strategies exists, as:

$$v_\alpha = H-\frac{(1-\mu)^2}{1+(1-\mu)^2}(H-L)$$

Subtracting from this $\mu H+(1-\mu)L$ gives us an indication of the magnitude of v_α

[24] Derivations in Online Appendix D, Section 14.

relative to the (ex-ante) expected valuation of a weak bidder. I thus find:

$$v_\alpha - (\mu H + (1-\mu) L) = \left(1 - \frac{(1-\mu)}{1+(1-\mu)^2}\right)(1-\mu)(H-L) > 0$$

We see that v_α is always larger than the expectation of the weak bidder's valuation. This requirement can naturally be interpreted as requirement for the strong bidder to be "strong enough" for an equilibrium in mixed strategies to exist. When $v < v_\alpha$, the strong bidder switches to always bidding L, as described in Proposition 1.2.

With respect to the second issue, observe that for some parameter values $\bar{b} > H$ (for example, when v is very large). In this event, the "top" of the mixing distribution exceeds the valuation of the H-type weak bidder - and no H-type bidder would play according to such a distribution in equilibrium. There is a natural way for the players to behave in this case: the H-type bidder bids H, and the strong bidder also bids H (and obtains the good via the tie-breaking rule). The rationale for this switching of behavior is that when v is high enough, the strong bidder doesn't want to risk losing to the H-type weak bidder, and bids H (and wins) for sure. I show, in Online Appendix D, Section 13.4, that when $\bar{b} > H$, then playing H for sure gives the strong bidder a higher surplus than a mixing strategy would. The values of v at which the strong bidder prefers to always bid H satisfy the following inequality.

$$\begin{aligned}
\bar{b} &\geq H \\
v - (1-\mu)^2 (v_\beta - L) &\geq H \\
v &\geq \left(H + \frac{(1-\mu)^2}{1-(1-\mu)^2}(H-L)\right) > H
\end{aligned}$$

The boundary value of v at which the switch-over in behavior occurs is thus:

$$v_\beta = H + \frac{(1-\mu)^2}{1-(1-\mu)^2}(H-L)$$

Observe again that this critical value depends on L, H and μ - and will change if we change one of those parameters. Combining the above arguments, I obtain a full specification of equilibrium behavior in the first-price auction.

13.2. First Price Auction - deriving the mixing distributions for $v \in [v_\alpha, v_\beta]$

For the strong bidder, the profit function is:

$$\Pi_S(b) = \left((\mu)^2 G_H(b)^2 + 2\mu(1-\mu) G_H(b) + (1-\mu)^2\right)(v-b)$$

For the H-type weak bidder:

$$\Pi_H(b) = (\mu G_H(b) + (1-\mu)) G_S(b)(H-b)$$

Using $G_H(L) = 0$:

$$\Pi_S(L) = (1-\mu)^2 (v-L) = \Pi_S(b)$$

The last equality follows due to the fact that all bids which the bidder submits with positive probability, must give the same expected payoff. Writing this out:

$$\left((\mu)^2 G_H(b)^2 + 2\mu(1-\mu) G_H(b) + (1-\mu)^2\right)(v-b) = (1-\mu)^2 (v-L)$$

This is a quadratic in G_H.

$$G_H(b)^2 \underbrace{(\mu)^2 (v-b_I)}_{A} + G_H(b) \underbrace{2\mu(1-\mu)(v-b_I)}_{B} + \underbrace{(1-\mu)^2 [L-b_I]}_{C} = 0$$

Applying the quadratic formula to the above equation, taking the positive root, gives us:

$$\begin{aligned}
G_H(b) &= \frac{-B+\sqrt{B^2-4AC}}{2A} \\
&= \frac{-2\mu(1-\mu)(v-b)}{2(\mu)^2(v-b)} + \\
&\quad + \frac{\sqrt{(2\mu(1-\mu)(v-b))^2 - 4(\mu)^2(v-b)\left[(1-\mu)^2[L-b]\right]}}{2(\mu)^2(v-b)} \\
&= \frac{(1-\mu)}{\mu}\left[\frac{\sqrt{v-L}-\sqrt{(v-b)}}{\sqrt{(v-b)}}\right]
\end{aligned}$$

This gives one of the bidding distributions. For the other, I observe that since $G_H\left(\bar{b}\right) = 1$:

$$\begin{aligned}
\Pi_S\left(\bar{b}\right) &= \left(v-\bar{b}\right) = (1-\mu)^2(v-L) \\
&\implies \bar{b} = v-(1-\mu)^2(v-L)
\end{aligned}$$

Again, since all bids that are played with positive probability must give the same expected payoff, we have:

$$\Pi_H(b) = \Pi_H\left(\bar{b}\right) = \left(H-\bar{b}\right)$$

Writing this out, and solving for G_S :

$$\begin{aligned}
(\mu G_H(b) + (1-\mu)) G_S(b)(H-b) &= \left(H-\bar{b}\right) \\
G_S(b) &= \frac{1}{(1-\mu)} \frac{\left(H-\bar{b}\right)}{\sqrt{v-L}} \frac{\sqrt{(v-b)}}{(H-b)}
\end{aligned}$$

For an equilibrium in mixed strategies to exist, we need:

$$\begin{aligned}
\bar{b} &\leq H \\
&\Longrightarrow v - (1-\mu)^2 (v - L) \leq H \\
&\Longrightarrow v \leq H + \frac{(1-\mu)^2}{1-(1-\mu)^2} (H - L)
\end{aligned}$$

This means that the strong bidder's valuation cannot be "too large" (since then he will just bid H, and win always).

13.3. First Price Auction - deriving the mixing distributions for $v \in< v_\alpha$

In this equilibrium, the strong bidder does not mix. For the H-type weak bidder:

$$\Pi_{H\alpha}(b) = (\mu G_{H\alpha}(b) + (1-\mu))(H - b)$$

Using $G_{H\alpha}(L) = 0$:

$$\Pi_{H\alpha}(L) = (1-\mu)(H - L) = \Pi_{H\alpha}(b)$$

Rearranging this expression to obtain $G_{H\alpha}$ results in:

$$G_{H\alpha}(b) = \frac{1-\mu}{\mu}\frac{b-L}{H-b}.$$

As in the previous case, the upper bound of the mixing distribution, $\overline{b}_\alpha$ is defined by the condition $G_{H\alpha}\left(\overline{b}_\alpha\right) = 1$. Since each strategy played in equilibrium must give the same expected profit:

$$\Pi_{H\alpha}\left(\overline{b}_\alpha\right) = \Pi_{H\alpha}(L)$$

Solving this equation yields $\overline{b}_\alpha = L + \mu(H - L)$.

13.4. Justifying "switch-over" of strong bidder's strategy when $\bar{b} > H$

I now look for an equilibrium in the case when $\bar{b} > H$. Then:

$$v > H + \frac{(1-\mu)^2}{1-(1-\mu)^2}(H - L)$$

The profit the strong bidder would then get by "mixing" would be:

$$(1-\mu)^2 (v - L)$$

Whereas by bidding H he gets the item for sure (via the tie-breaking rule):

$$v - H$$

The difference between these two cases is

$$\begin{aligned}
\Delta &= (v-H) - (1-\mu)^2 (v-L) \\
&= \left(2\mu - \mu^2\right) v + (1-\mu)^2 L - H \\
&> \left(2\mu - \mu^2\right) \left(H + \frac{(1-\mu)^2}{1-(1-\mu)^2} (H-L) \right) + (1-\mu)^2 L - H = 0
\end{aligned}$$

Hence my desired conclusion of $\Delta > 0$. That is:

$$\begin{aligned}
v - H &> (1-\mu)^2 (v-L) \\
when &: \bar{b} > H
\end{aligned}$$

13.5. Revenue in the First Price Auction

13.5.1. When $v \in [0, L]$

I consider different combinations of weak bidders separately, and then aggregate to get total revenue.

Both weak bidders are L. This happens with probability: $(1-\mu)^2$. Revenue is then L.

One weak bidder is L, other is H. This happens with probability $2\mu(1-\mu)$. The cumulative density is of the winning bid is then $G_H(b)$. The density is thus:

$$G'_H = \frac{1-\mu}{\mu} \left(\frac{H-L}{(H-b)^2} \right)$$

The expected revenue can be obtained as:

$$\frac{1-\mu}{\mu} \int_L^{\bar{b}} t \frac{H-L}{(H-t)^2} dt$$

Both weak bidders are H. This happens with probability μ^2. The winning bid has cumulative density $G_H(b)^2$. Hence density of the winning bid is:

$$\begin{aligned}
& 2G_H(b)\, G'_H(b) \\
= \; & 2 \left(\frac{1-\mu}{\mu} \right)^2 (H-L) \frac{b-L}{(H-b)^3}
\end{aligned}$$

So the expected revenue is:

$$2 \left(\frac{1-\mu}{\mu} \right)^2 (H-L) \int_L^{\bar{b}} t \frac{t-L}{(H-t)^3} dt$$

Total expected revenue. Aggregating the above expressions, after proper pre-multiplication by event probabilities, I have:

$$\begin{aligned}R^0_{FPS} &= (1-\mu)^2 L + \\ &+2\mu(1-\mu)\frac{1-\mu}{\mu}\int_L^{\bar{b}} t\frac{H-L}{(H-t)^2}dt \\ &+\mu^2 2\left(\frac{1-\mu}{\mu}\right)^2 (H-L)\int_L^{\bar{b}} t\frac{t-L}{(H-t)^3}dt\end{aligned}$$

$$\begin{aligned}R^0_{FPS} &= (1-\mu)^2 L + \\ &+2(1-\mu)^2(H-L)\int_L^{\bar{b}} t\left(\frac{1}{(H-t)^2}+\frac{t-L}{(H-t)^3}\right)dt\end{aligned}$$

Using partial fractions:

$$\begin{aligned}R^0_{FPS} &= (1-\mu)^2 L + \\ &+2(1-\mu)^2(H-L)\int_L^{\bar{b}} \left(H\frac{H-L}{(H-t)^3}-\frac{H-L}{(H-t)^2}\right)dt\end{aligned}$$

$$\begin{aligned}R^0_{FPS} &= (1-\mu)^2 L + \\ &+2(1-\mu)^2(H-L)\frac{1}{2}\left[\frac{(L-H)}{(t-H)^2}(H-2t)\right]_L^{\bar{b}}\end{aligned}$$

$$\begin{aligned}R^0_{FPS} &= (1-\mu)^2 L + \\ &+2(1-\mu)^2(H-L)\frac{1}{2}\left[\begin{array}{c}\frac{(L-H)}{((1-\mu)(H-L))^2}(H-2(H-(1-\mu)(H-L))) \\ -\frac{(L-H)}{(L-H)^2}(H-2L)\end{array}\right]\end{aligned}$$

Simplifying the above I get:

$$R^0_{FPS} = (1-\mu)^2 L + \mu^2 H + 2\mu(1-\mu)L = R^A_{ASC}$$

13.5.2. When $v \in [L, v_\alpha]$

I consider different combinations of weak bidders separately, and then aggregate to get total revenue.

Both weak bidders are L. This happens with probability: $(1-\mu)^2$. Revenue is then L.

One weak bidder is L, other is H. This happens with probability $2\mu(1-\mu)$. The cumulative density is of the winning bid is then $G_{H\alpha}(b)$. The density is thus:

$$G'_{H\alpha}(b) = \frac{1-\mu}{\mu}\frac{H-L}{(H-b)^2},$$

and the corresponding expected revenue is:

$$\frac{1-\mu}{\mu}\int_{L}^{L+\mu(H-L)}\frac{H-L}{(H-t)^{2}}tdt$$

Both weak bidders are H. This happens with probability μ^2. The cumulative density is of the winning bid is then $[G_{H\alpha}(b)]^2$. The density is thus:

$$2G_{H\alpha}(b)\,G'_{H\alpha}(b)=2\left(\frac{1-\mu}{\mu}\right)^{2}\frac{(H-L)(b-L)}{(H-b)^{3}},$$

and the corresponding expected revenue is:

$$2\left(\frac{1-\mu}{\mu}\right)^{2}\int_{L}^{L+\mu(H-L)}\frac{(H-L)(t-L)}{(H-t)^{3}}tdt$$

Overall revenue . Aggregating the three above cases with appropriate probability-weights yields:

$$\begin{aligned}
& R^{\alpha}_{FPS}(Total)\\
= \ & (1-\mu)^{2}L\\
& +2\mu(1-\mu)\frac{1-\mu}{\mu}\int_{L}^{L+\mu(H-L)}\frac{H-L}{(H-t)^{2}}tdt\\
& +2\mu^{2}\left(\frac{1-\mu}{\mu}\right)^{2}\int_{L}^{L+\mu(H-L)}\frac{(H-L)(t-L)}{(H-t)^{3}}tdt\\
= \ & (1-\mu)^{2}\left(L+2\int_{L}^{L+\mu(H-L)}\frac{(H-L)^{2}}{(H-t)^{3}}tdt\right)
\end{aligned}$$

13.5.3. When $v > v_\alpha$

The following derivation only applies under parameter values under which the strong bidder's bidding distribution is well behaved. The equilibrium bidding behavior is defined in Proposition 1.2. I proceed by calculating revenue separately from all the possible combinations of realized strong and weak bidder valuations, and then I aggregate

Expected revenue calculations - the sub-cases. The following list of cases is jointly exhaustive (and mutually exclusive):

- Two weak bidders are L, strong bidder plays Atom
- Two weak bidders are L, strong bidder mixes
- Exactly one weak bidder is L, one is H, strong bidder plays Atom
- Exactly one weak bidder is L, one is H, strong bidder mixes

- Both weak bidders are H, strong bidder plays Atom
- Both weak bidders are H, strong bidder mixes

Two weak bidders are L, strong bidder plays Atom. The probability of this event is:

$$(1-\mu)^2 \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}$$

Revenue in this case is

$$R_{FPS}(2L, atom) = L$$

Two weak bidders are L, strong bidder mixes. The probability of this event is:

$$(1-\mu)^2 \left(1 - \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)$$

The cumulative density of the winning bid is then:

$$\frac{G_S(b) - G_S(L)}{1 - G_S(L)}$$

The corresponding density of winning bid is:

$$\frac{1}{1-G_S(L)} G'_S(b)$$

The relevant derivative is given by:

$$\begin{aligned} G'_S(b) &= \left(\frac{1}{1-\mu}\right) \frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}} \left(\frac{-\frac{1}{2}(v-b)^{-\frac{1}{2}}(H-b) + (v-b)^{\frac{1}{2}}}{(H-b)^2}\right) \\ &= \left(\frac{1}{1-\mu}\right) \frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}} \left(\frac{-\frac{1}{2}(v-b)^{-\frac{1}{2}}}{(H-b)} + \frac{(v-b)^{\frac{1}{2}}}{(H-b)^2}\right) \end{aligned}$$

Revenue under the appropriate density is then:

$$R_{FPS}(2L, mix) = \frac{1}{\left(1 - \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)} \left(\frac{1}{1-\mu}\right) \frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}} \cdot \\ \cdot \int_L^{\bar{b}} t \left(\frac{-\frac{1}{2}(v-t)^{-\frac{1}{2}}}{(H-t)} + \frac{(v-t)^{\frac{1}{2}}}{(H-t)^2}\right) dt$$

Exactly one weak bidder is L, one is H, strong bidder plays Atom. The probability of this event is:

$$2\mu(1-\mu) \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} \tag{13}$$

The cumulative density of a winning bid is then $G_H(b)$. Hence the density of the winning bid is:

$$\begin{aligned} G'_H(b) &= \left(\frac{1-\mu}{\mu}\right)\left(\frac{\frac{1}{2}(v-b)^{-\frac{1}{2}}(v-b)^{\frac{1}{2}}+\frac{1}{2}(v-b)^{-\frac{1}{2}}\left((v-L)^{\frac{1}{2}}-(v-b)^{\frac{1}{2}}\right)}{(v-b)}\right) \\ &= \left(\frac{1-\mu}{2\mu}\right)\left(\frac{1}{(v-b)}+\frac{(v-L)^{\frac{1}{2}}-(v-b)^{\frac{1}{2}}}{(v-b)^{\frac{3}{2}}}\right) \end{aligned}$$

The expected revenue is then:

$$R_{FPS}(1L, Atom) = \left(\frac{1-\mu}{2\mu}\right)\int_L^{\bar{b}} t\left(\frac{1}{(v-t)}+\frac{(v-L)^{\frac{1}{2}}-(v-t)^{\frac{1}{2}}}{(v-t)^{\frac{3}{2}}}\right)dt$$

Exactly one weak bidder is L, one is H, strong bidder mixes. This occurs with probability

$$2\mu(1-\mu)\left(1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)$$

The cumulative density of the winning bid is:

$$\begin{aligned} F &= \frac{G_S(b)-G_S(L)}{1-G_S(L)}G_H(b) \\ &= \left[\frac{1}{1-G_S(L)}\frac{1}{\mu}\frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}}\right]\left[\frac{(v-b)^{\frac{1}{2}}}{(H-b)}-\frac{(v-L)^{\frac{1}{2}}}{(H-L)}\right]\frac{(v-L)^{\frac{1}{2}}-(v-b)^{\frac{1}{2}}}{(v-b)^{\frac{1}{2}}} \end{aligned}$$

Differentiating:

$$\begin{aligned} F' &= \left[\frac{1}{1-G_S(L)}\frac{1}{\mu}\frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}}\right]\cdot \\ &\quad \cdot\left[\frac{(v-L)^{\frac{1}{2}}-(v-b)^{\frac{1}{2}}}{(H-b)^2}-\frac{\frac{1}{2}(v-L)}{(H-L)(v-b)^{\frac{3}{2}}}+\frac{\frac{1}{2}(v-b)^{\frac{1}{2}}}{(H-b)(v-b)}\right] \\ &= \left[\frac{1}{1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}}\frac{1}{\mu}\frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}}\right]\cdot \\ &\quad \cdot\left[\frac{(v-L)^{\frac{1}{2}}-(v-b)^{\frac{1}{2}}}{(H-b)^2}-\frac{\frac{1}{2}(v-L)}{(H-L)(v-b)^{\frac{3}{2}}}+\frac{\frac{1}{2}(v-b)^{\frac{1}{2}}}{(H-b)(v-b)}\right] \end{aligned}$$

The expected revenue is then:

$$R_{FPS}(1L, mix) = \left[\frac{1}{1 - \frac{(H-\bar{b})}{(1-\mu)(H-L)}} \frac{1}{\mu} \frac{(H-\bar{b})}{(v-L)^{\frac{1}{2}}} \right] * \\ * \int_L^{\bar{b}} t \left[\begin{array}{c} \frac{(v-L)^{\frac{1}{2}}-(v-t)^{\frac{1}{2}}}{(H-t)^2} - \\ -\frac{\frac{1}{2}(v-L)}{(H-L)(v-t)^{\frac{3}{2}}} + \frac{\frac{1}{2}(v-t)^{\frac{1}{2}}}{(H-t)(v-t)} \end{array} \right] dt$$

Both weak bidders are H, strong bidder plays atom. The event has probability:

$$\mu^2 \left(\frac{(H-\bar{b})}{(1-\mu)(H-L)} \right)$$

In this case the winning bid has the cumulative density $G_H(b)^2$. The relevant density is then:

$$2G_H(b)\, G'_H(b)$$

Substituting in for G_H and G'_H:

$$2\left(\frac{1-\mu}{\mu}\right) \frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{1}{2}}} \left(\frac{1-\mu}{2\mu}\right) \left(\frac{1}{(v-b)} + \frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{3}{2}}} \right) \\ = \left(\frac{1-\mu}{\mu}\right)^2 \left(\frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{3}{2}}} + \frac{\left((v-L) + (v-b) - 2(v-L)^{\frac{1}{2}}(v-b)^{\frac{1}{2}}\right)}{(v-b)^2} \right)$$

The revenue is then:

$$R_{FPS}(2H, Atom) = \left(\frac{1-\mu}{\mu}\right)^2 \int_L^{\bar{b}} t \left(\begin{array}{c} \frac{(v-L)^{\frac{1}{2}}-(v-t)^{\frac{1}{2}}}{(v-t)^{\frac{3}{2}}} \\ + \frac{\left((v-L)+(v-t)-2(v-L)^{\frac{1}{2}}(v-t)^{\frac{1}{2}}\right)}{(v-t)^2} \end{array} \right) dt$$

Both weak bidders are H, strong bidder mixes. The probability of this event is:

$$\mu^2 \left(1 - \frac{(H-\bar{b})}{(1-\mu)(H-L)} \right)$$

The cumulative density of the winning bid is:

$$
\begin{aligned}
& \frac{G_S(b) - G_S(L)}{1 - G_S(L)} \left[G_H(b)\right]^2 \\
= \; & \frac{1}{1 - G_S(L)} \left[\left(\frac{1}{1-\mu} \right) \frac{\left(H - \bar{b}\right)}{(v-L)^{\frac{1}{2}}} \frac{(v-b)^{\frac{1}{2}}}{(H-b)} - \frac{\left(H - \bar{b}\right)}{(1-\mu)(H-L)} \right] \cdot \\
& \cdot \left[\left(\frac{1-\mu}{\mu} \right) \frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{1}{2}}} \right]^2 \\
= \; & \frac{\left(H - \bar{b}\right)}{1 - G_S(L)} \left(\frac{1-\mu}{\mu^2} \right) \left[\frac{1}{(v-L)^{\frac{1}{2}}} \frac{(v-b)^{\frac{1}{2}}}{(H-b)} - \frac{1}{(H-L)} \right] \cdot \\
& \cdot \left[\frac{(v-L) + (v-b) - 2(v-b)^{\frac{1}{2}}(v-L)^{\frac{1}{2}}}{(v-b)} \right]
\end{aligned}
$$

The corresponding density is:

$$
\begin{aligned}
& \frac{\left(H - \bar{b}\right)}{1 - G_S(L)} \left(\frac{1-\mu}{\mu^2} \right) * \\
& \frac{1}{(v-b)} \left(\begin{array}{c} \left[\frac{\left[(v-L)(v-b)^{\frac{1}{2}} + (v-b)^{\frac{3}{2}} - 2(v-b)(v-L)^{\frac{1}{2}}\right]}{(v-L)^{\frac{1}{2}}(H-b)^2} \right] \\ + \left[\frac{\frac{1}{2}(v-L) - \frac{1}{2}(v-b)}{(v-L)^{\frac{1}{2}}(H-b)(v-b)^{\frac{1}{2}}} - \frac{(v-L) - (v-b)^{\frac{1}{2}}(v-L)^{\frac{1}{2}}}{(v-b)(H-L)} \right] \end{array} \right)
\end{aligned}
$$

Whereby the expected revenue is:

$$
\begin{aligned}
& R_{FPS}(2H, mix) = \frac{\left(H - \bar{b}\right)}{1 - \frac{\left(H - \bar{b}\right)}{(1-\mu)(H-L)}} \left(\frac{1-\mu}{\mu^2} \right) * \\
& \int_L^{\bar{b}} t \left(\begin{array}{c} \left[\frac{\left[(v-L)(v-t)^{\frac{1}{2}} + (v-t)^{\frac{3}{2}} - 2(v-t)(v-L)^{\frac{1}{2}}\right]}{(v-L)^{\frac{1}{2}}(H-t)^2(v-t)} \right] \\ + \left[\frac{\frac{1}{2}(v-L) - \frac{1}{2}(v-t)}{(v-L)^{\frac{1}{2}}(H-t)(v-t)^{\frac{3}{2}}} - \frac{(v-L) - (v-t)^{\frac{1}{2}}(v-L)^{\frac{1}{2}}}{(v-t)(H-L)(v-t)} \right] \end{array} \right) dt
\end{aligned}
$$

Overall Expected Revenue.

$$
\begin{aligned}
& R_{FPS}(Total) \\
= \; & (1-\mu)^2 \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} L \\
& + (1-\mu)^2 \left(1 - \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right) \frac{1}{\left(1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)} \left(\frac{1}{1-\mu}\right) \frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}} * \\
& \quad * \int_L^{\bar{b}} t \left(\frac{-\frac{1}{2}(v-t)^{-\frac{1}{2}}}{(H-t)} + \frac{(v-t)^{\frac{1}{2}}}{(H-t)^2}\right) dt \\
& + 2\mu(1-\mu) \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} \left(\frac{1-\mu}{2\mu}\right) \int_L^{\bar{b}} t \left(\frac{1}{(v-t)} + \frac{(v-L)^{\frac{1}{2}} - (v-t)^{\frac{1}{2}}}{(v-t)^{\frac{3}{2}}}\right) dt \\
& + 2\mu(1-\mu) \left(1 - \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right) \left[\frac{1}{1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}} \frac{1}{\mu} \frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}}\right] * \\
& \quad * \int_L^{\bar{b}} t \left[\frac{(v-L)^{\frac{1}{2}} - (v-t)^{\frac{1}{2}}}{(H-t)^2} - \frac{\frac{1}{2}(v-L)}{(H-L)(v-t)^{\frac{3}{2}}} + \frac{\frac{1}{2}(v-t)^{\frac{1}{2}}}{(H-t)(v-t)}\right] dt \\
& + \mu^2 \left(\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right) \left(\frac{1-\mu}{\mu}\right)^2 * \\
& \quad * \int_L^{\bar{b}} t \left(\frac{(v-L)^{\frac{1}{2}} - (v-t)^{\frac{1}{2}}}{(v-t)^{\frac{3}{2}}} + \frac{\left((v-L) + (v-t) - 2(v-L)^{\frac{1}{2}}(v-t)^{\frac{1}{2}}\right)}{(v-t)^2}\right) dt \\
& + \mu^2 \left(1 - \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right) \frac{\left(H-\bar{b}\right)}{1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}} \left(\frac{1-\mu}{\mu^2}\right) * \\
& \quad * \int_L^{\bar{b}} t \left(\begin{array}{c} \left[\frac{\left[(v-L)(v-t)^{\frac{1}{2}} + (v-t)^{\frac{3}{2}} - 2(v-t)(v-L)^{\frac{1}{2}}\right]}{(v-L)^{\frac{1}{2}}(H-t)^2(v-t)}\right] \\ + \left[\frac{\frac{1}{2}(v-L) - \frac{1}{2}(v-t)}{(v-L)^{\frac{1}{2}}(H-t)(v-t)^{\frac{3}{2}}} - \frac{(v-L)-(v-t)^{\frac{1}{2}}(v-L)^{\frac{1}{2}}}{(v-t)(H-L)(v-t)}\right] \end{array}\right) dt
\end{aligned}
$$

After some algebra, the above simplifies to:

$$
\begin{aligned}
R_{FPS} = \; & (1-\mu)^2 \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} L + \\
& + (1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \int_L^{\bar{b}} t \left(\frac{(v-t)^{\frac{1}{2}}}{(H-t)^2 (v-t)} + \frac{\frac{1}{2}(v-t)^{\frac{1}{2}}}{(H-t)(v-t)^2}\right) dt
\end{aligned}
$$

Integrating by parts:

$$\begin{aligned} R_{FPS} &= (1-\mu)^2 \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} L + \\ &+ (1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \left(\left[\frac{t}{(H-t)(v-t)^{\frac{1}{2}}} \right]_L^{\bar{b}} - \int_L^{\bar{b}} \frac{1}{(H-t)(v-t)^{\frac{1}{2}}} dt \right) \end{aligned}$$

Substituting for boundaries:

$$\begin{aligned} R_{FPS} &= (1-\mu) \frac{\left(H-\bar{b}\right)}{(H-L)} L + \\ &+ (1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \left(\left(\frac{\bar{b}}{\left(H-\bar{b}\right)(v-\bar{t})^{\frac{1}{2}}} \right) - \left(\frac{L}{(H-L)(v-L)^{\frac{1}{2}}} \right) \right) \\ &- (1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \int_L^{\bar{b}} \frac{1}{(H-t)(v-t)^{\frac{1}{2}}} dt \end{aligned}$$

Simplifying and substituting for $\bar{\text{b}}$:

$$\begin{aligned} R_{FPS} &= (1-\mu)(v-L)^{\frac{1}{2}} \left(\frac{v-(1-\mu)^2(v-L)}{(1-\mu)(v-L)^{\frac{1}{2}}} \right) \\ &- (1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \int_L^{\bar{b}} \frac{1}{(H-t)(v-t)^{\frac{1}{2}}} dt \end{aligned}$$

Canceling the appropriate terms, I finally obtain:

$$R_{FPS} = v - (1-\mu)^2(v-L) - (1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \int_L^{\bar{b}} \frac{1}{(H-t)(v-t)^{\frac{1}{2}}} dt$$

14. Online Appendix D: Details of the Anglo-Dutch Auction in the Non-Degenerate Case (when $v > L$).

14.1. Deriving the Anglo-Dutch Equilibrium (Proposition 3)

The profit functions in this case are as constructed as follows.

If the Anglo phase terminated at L, and an H-type bidder is present in the first-price stage, he knows that he is the only H-type left in the auction, and he is bidding against the single strong bidder. Given a bid of b, the H-type beats the strong bidder with probability $G_S^*(b)$. The surplus from bidding b is thus:

$$\Pi_H^*(b) = G_S^*(b)(H-b)$$

If the strong bidder is present in the Dutch stage, he doesn't know his opponent's identity for sure - the only thing he knows at this point is that not both of his opponents were H-types. Thus with the (posterior) probability of $\frac{(1-\mu)^2}{1-\mu^2}$ the strong bidder is in fact facing an L-type bidder, whom he would beat with certainty by bidding any $b \geq L$. However, with the probability $\frac{2\mu(1-\mu)}{1-\mu^2}$ the strong bidder's opponent is in fact an H-type, and the probability of beating the H-type by bidding b is $G_H^*(b)$. The expected surplus of the strong bidder from bidding b is then:

$$\Pi_S^*(b) = \left(\frac{2\mu(1-\mu)}{1-\mu^2} G_H^*(b) + \frac{(1-\mu)^2}{1-\mu^2}\right)(v-b)$$

Solving these two equations proceeds exactly analogously to the first-price case, and the derivations are provided in Online Appendix B. The upper bound of the mixing distribution is given by:

$$b^* = v - \frac{(1-\mu)}{(1+\mu)}(v-L) = v - \frac{(1-\mu)^2}{1-\mu^2}(v-L) \leq \bar{b}$$

14.2. Deriving the mixing distributions for the Anglo-Dutch Auction

The only case in which I need to derive an equilibrium bidding distribution for the Anglo-Dutch auction is in the case when the Anglo phase terminates at L. So suppose the Anglo phase stops at L. Then with probability $\frac{2\mu(1-\mu)}{1-\mu^2}$ the strong bidder is facing an H type weak bidder, and with probability $\frac{(1-\mu)^2}{1-\mu^2}$ he is facing a type L. A low type weak bidder will bid L, and an H-type weak bidder will mix over $[L, b^*]$, for some $b^* \leq H$. The profit for the strong bidder will be:

$$\begin{aligned}\Pi_S^*(b) &= \left(\frac{2\mu(1-\mu)}{1-\mu^2} G_H^*(b) + \frac{(1-\mu)^2}{1-\mu^2}\right)(v-b) \\ &= \left(\frac{2\mu}{(1+\mu)} G_H^*(b) + \frac{(1-\mu)}{(1+\mu)}\right)(v-b)\end{aligned}$$

The profit for H-type weak bidder will be:

$$\Pi_H^*(b) = G_H^*(b)(H-b)$$

Using $G_H^*(L) = 0$

$$\Pi_S^*(L) = \frac{(1-\mu)}{(1+\mu)}(v-L) = \Pi_S^*(b)$$

The last equality follows from the fact that each bid played with positive probability must give the same expected payoff in equilibrium. Hence:

$$\begin{aligned}\frac{(1-\mu)}{(1+\mu)}(v-L) &= \left(\frac{2\mu}{(1+\mu)} G_H^*(b) + \frac{(1-\mu)}{(1+\mu)}\right)(v-b) \\ G_H^*(b) &= \frac{(1-\mu)}{2\mu}\left(\frac{b-L}{v-b}\right)\end{aligned}$$

Using $G_H^*(b^*) = 1$:

$$\begin{aligned}
\frac{(1-\mu)}{2\mu}\left(\frac{(v-L)}{(v-b^*)}-1\right) &= 1 \\
b^* &= v - \frac{1-\mu}{1+\mu}(v-L) = v - \frac{(1-\mu)^2}{1-\mu^2}(v-L) < \bar{b}
\end{aligned}$$

Similarly, since $G_S(b^*) = 1$:

$$\begin{aligned}
\Pi_H^*(b^*) &= (H-b^*) = \Pi_H^*(b) \\
(H-b^*) &= G_S^*(b)(H-b) \\
G_S^*(b) &= \frac{(H-b^*)}{(H-b)} = \frac{\left(H - v + \frac{1-\mu}{1+\mu}(v-L)\right)}{(H-b)}
\end{aligned}$$

14.3. Justifying "switch-over" of strong bidder's strategy when $b^* > H$

I now also look for the equilibrium when $b^* > H$. We must have:

$$v > H + \frac{(1-\mu)}{2\mu}(H-L)$$

The profits obtainable from just bidding H are:

$$(v-H)$$

Whereas the mixing strategy delivered:

$$\frac{1-\mu}{1+\mu}(v-L)$$

The difference is thus:

$$\begin{aligned}
\Delta &= (v-H) - \frac{1-\mu}{1+\mu}(v-L) \\
&= \frac{2\mu}{1+\mu}v - H + \frac{1-\mu}{1+\mu}L \\
&> \left(\frac{2\mu}{1+\mu}\right)\left(H + \frac{(1-\mu)}{2\mu}(H-L)\right) - H + \frac{1-\mu}{1+\mu}L = 0
\end{aligned}$$

In this case also we have the conclusion of $\Delta > 0$. That is:

$$\begin{aligned}
(v-H) &> (1-\mu)^2(v-L) \\
when &: b^* > H
\end{aligned}$$

14.4. Revenue in the Anglo-Dutch Auction

14.4.1. When $v \in [0, L]$.

In this case the Anglo-Dutch auction is equivalent to the first-price.

14.4.2. When $v > L$

As in the first-price auction, here I also proceed by considering the expected revenues from sub-cases first, and then aggregate them.

Expected revenue calculations - the sub-cases. The following list of cases is jointly exhaustive (and individually mutually exclusive):

- Two weak bidders are L, strong bidder plays Atom
- Two weak bidders are L, strong bidder mixes
- Exactly one weak bidder is L, one is H, strong bidder plays Atom
- Exactly one weak bidder is L, one is H, strong bidder mixes
- Both weak bidders are H, and strong bidder plays $\min(v, H)$.

Two weak bidders are L, strong bidder plays Atom. Probability of this event is:

$$(1-\mu)^2 \frac{(H-b^*)}{(H-L)}$$

Revenue is :

$$R_{AD}(2L, atom) = L$$

Two weak bidders are L, strong bidder mixes. Probability of this case is:

$$(1-\mu)^2 \left(1 - \frac{(H-b^*)}{(H-L)}\right)$$

The cumulative density of the winning bid is then:

$$\frac{1}{1-G_S^*(L)}\left(G_S^*(b) - G_S^*(L)\right)$$

The relevant density is given by:

$$\frac{1}{1-G_S^*(L)} G_S^{*\prime}(b)$$

Since:

$$G_S^{*\prime}(b) = \frac{(H-b^*)}{(H-b)^2}$$

Substituting in for G_S^* and $G_S^{*\prime}$:

$$\frac{1}{1-\frac{(H-b^*)}{(H-L)}} \frac{(H-b^*)}{(H-b)^2}$$

The expected revenue is:

$$R_{AD}\left(2L,mix\right)=\frac{1}{1-\frac{(H-b^{*})}{(H-L)}}\int_{L}^{b^{*}}t\frac{(H-b^{*})}{(H-t)^{2}}dt$$

Exactly one weak bidder is L, one is H, strong bidder plays Atom. The probability of this event is:

$$2\mu\left(1-\mu\right)\frac{(H-b^{*})}{(H-L)}$$

The cumulative density of the winning bid is then:

$$\left(\frac{1-\mu}{2\mu}\right)\left(\frac{b-L}{v-b}\right)$$

giving a density of:

$$\left(\frac{1-\mu}{2\mu}\right)\left(\frac{(v-b)+(b-L)}{(v-b)^{2}}\right)=\left(\frac{1-\mu}{2\mu}\right)\frac{v-L}{(v-b)^{2}}$$

So the expected revenue is:

$$R_{AD}\left(1L,atom\right)=\left(\frac{1-\mu}{2\mu}\right)(v-L)\int_{L}^{b^{*}}t\frac{1}{(v-t)^{2}}dt$$

Exactly one weak bidder is L, one is H, strong bidder mixes. The probability of this event is:

$$2\mu\left(1-\mu\right)\left(1-\frac{(H-b^{*})}{(H-L)}\right)$$

The cumulative density of the winning bids will be:

$$\frac{G_{S}^{*}\left(b\right)-G_{S}^{*}\left(L\right)}{1-G_{S}^{*}\left(L\right)}G_{H}^{*}\left(b\right)$$

The relevant density is then:

$$\frac{1}{1-G_{S}^{*}\left(L\right)}\left[G_{S}^{*\prime}\left(b\right)G_{H}^{*}\left(b\right)+G_{H}^{*\prime}\left(b\right)\left(G_{S}^{*}\left(b\right)-G_{S}^{*}\left(L\right)\right)\right]$$

$$\begin{aligned}G_{H}^{*\prime}\left(b\right)&=\left(\frac{1-\mu}{2\mu}\right)\frac{v-L}{(v-b)^{2}}\\G_{S}^{*\prime}\left(b\right)&=\frac{(H-b^{*})}{(H-b)^{2}}\end{aligned}$$

Substituting in the above results gives:

$$
\begin{aligned}
& \frac{1}{1-\frac{(H-b^*)}{(H-L)}}\left(\frac{1-\mu}{2\mu}\right)\left[\frac{(H-b^*)}{(H-b)^2}\left(\frac{b-L}{v-b}\right)+\left(\frac{(H-b^*)}{(H-b)}-\frac{(H-b^*)}{(H-L)}\right)\frac{v-L}{(v-b)^2}\right] \\
= & \frac{1}{1-\frac{(H-b^*)}{(H-L)}}\left(\frac{1-\mu}{2\mu}\right)(H-b^*)\left[\frac{b-L}{(H-b)^2(v-b)}+\left(\frac{1}{(H-b)}-\frac{1}{(H-L)}\right)\frac{v-L}{(v-b)^2}\right]
\end{aligned}
$$

The expected revenue is then:

$$
\begin{aligned}
R_{AD}(1L,mix) = & \frac{1}{1-\frac{(H-b^*)}{(H-L)}}\left(\frac{1-\mu}{2\mu}\right)(H-b^*) * \\
& \int_L^{b^*} t\left[\frac{t-L}{(H-t)^2(v-t)}+\left(\frac{1}{(H-t)}-\frac{1}{(H-L)}\right)\frac{v-L}{(v-t)^2}\right]dt
\end{aligned}
$$

Both weak bidders are H, and strong bidder plays min(v,H). The probability of this event is μ^2, and the revenue in this case is $R_{AD}(2H) = H$.

Overall expected revenue. Aggregating across the above cases, after appropriate pre-multiplication with probabilities, I obtain:

$$
\begin{aligned}
R_{AD} = & (1-\mu)^2\frac{(H-b^*)}{(H-L)}L \\
& +(1-\mu)^2\left(1-\frac{(H-b^*)}{(H-L)}\right)\frac{1}{1-\frac{(H-b^*)}{(H-L)}}\int_L^{b^*} t\frac{(H-b^*)}{(H-t)^2}dt \\
& +2\mu(1-\mu)\frac{(H-b^*)}{(H-L)}\left(\frac{1-\mu}{2\mu}\right)(v-L)\int_L^{b^*} t\frac{1}{(v-t)^2}dt \\
& +2\mu(1-\mu)\left(1-\frac{(H-b^*)}{(H-L)}\right)\frac{1}{1-\frac{(H-b^*)}{(H-L)}}\left(\frac{1-\mu}{2\mu}\right)(H-b^*) * \\
& * \int_L^{b^*} t\left[\frac{t-L}{(H-t)^2(v-t)}+\left(\frac{1}{(H-t)}-\frac{1}{(H-L)}\right)\frac{v-L}{(v-t)^2}\right]dt \\
& +\mu^2 H
\end{aligned}
$$

This simplifies to:

$$
\begin{aligned}
R_{AD} = & (1-\mu)^2(H-b^*)\frac{L}{(H-L)}+\mu^2 H \\
& +(1-\mu)^2(H-b^*)(v-L)\int_L^{b^*} t\left[\frac{1}{(H-t)^2(v-t)}+\frac{1}{(H-t)(v-t)^2}\right]dt
\end{aligned}
$$

Integrating by parts:

$$\begin{aligned}
R_{AD} &= (1-\mu)^2 (H-b^*) \frac{L}{(H-L)} + \mu^2 H \\
&\quad + (1-\mu)^2 (H-b^*)(v-L) \left(\left[\frac{t}{(H-t)(v-t)} \right]_L^{b^*} - \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt \right)
\end{aligned}$$

Substituting for boundaries:

$$\begin{aligned}
R_{AD} &= (1-\mu)^2 (H-b^*) \frac{L}{(H-L)} + \mu^2 H \\
&\quad + (1-\mu)^2 (H-b^*)(v-L) \left(\left[\frac{b^*}{(H-b^*)(v-b^*)} \right] - \left[\frac{L}{(H-L)(v-L)} \right] \right) \\
&\quad - (1-\mu)^2 (H-b^*)(v-L) \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt
\end{aligned}$$

Simplifying and substituting for b^* :

$$\begin{aligned}
R_{AD} &= \mu^2 H \\
&\quad + (1-\mu)^2 (v-L) \left(\left[\frac{v - \frac{(1-\mu)^2}{1-\mu^2}(v-L)}{\left(\frac{(1-\mu)^2}{1-\mu^2}(v-L) \right)} \right] \right) \\
&\quad - (1-\mu)^2 (H-b^*)(v-L) \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt
\end{aligned}$$

Simplifying again, I finally get:

$$\begin{aligned}
R_{AD} = \mu^2 H \quad &+ \left(1-\mu^2\right) v - (1-\mu)^2 (v-L) \\
&- (1-\mu)^2 (H-b^*)(v-L) \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt
\end{aligned}$$

15. Online Appendix E: Derivation of Efficiency, and the Revenue Comparison

15.1. Efficiency in the Ascending Auction

When $v \in [L, H]$, if both weak bidders are low, the winning bidder is the strong bidder, with valuation v. If at least one weak bidder is of H type, then an H-type will win the auction. The efficiency in this case is thus:

$$\text{Eff}_{ASC} = (1-\mu)^2 v + \left(1 - (1-\mu)^2\right) H$$

However, when $v > H$, the strong bidder always wins the ascending auction, and so the winning valuation will be v.

15.2. Efficiency in the First-price auction

In this section I use expressions derived in Online Appendix C, Section 13.5. Efficiency, measured by the expected valuation of the winning bidder, can be calculated as follows.

If $\bar{b} > H$ (i.e., $v > v_\beta$) then efficiency is v. When $v < v_\alpha$, the efficiency is $(1-\mu)^2 v + \left(1-(1-\mu)^2\right) h$ - the same as in the ascending auction. If $v \in [v_\alpha, v_\beta]$, the efficiency is obtained by the following formula.

$$
\begin{aligned}
\text{Eff}_{FPS} &= (1-\mu)^2 v \\
&\quad +2\mu(1-\mu) G_S(L) H \\
&\quad +2\mu(1-\mu)(1-G_S(L)) \left(\begin{array}{c} \left(v * \int_L^{\bar{b}} \frac{G'_S(t)}{1-G_S(L)} G_H(t)\, dt\right) \\ +H * \left(1 - \int_L^{\bar{b}} \frac{G'_S(t)}{1-G_S(L)} G_H(t)\, dt\right) \end{array} \right) \\
&\quad +\mu^2 \left(G_S(L) H + (1-G_S(L)) \left(\begin{array}{c} \left(v * \int_L^{\bar{b}} \frac{G'_S(t)}{1-G_S(L)} (G_H(t))^2\, dt\right) \\ +H * \left(1 - \int_L^{\bar{b}} \frac{G'_S(t)}{1-G_S(L)} (G_H(t))^2\, dt\right) \end{array} \right) \right)
\end{aligned}
$$

The individual terms in the above expression contribute to the efficiency measure by in the following way. In the case when both weak bidders are L (occurs with probability $(1-\mu)^2$) the strong bidder always wins, and his valuation is v. With probability $2\mu(1-\mu)$, exactly one weak bidder is H. If in this case the strong bidder plays the atom (occurs with probability $G_S(L)$), the H-type weak bidder wins for sure, with value H. However, in the case when the strong bidder mixes (occurs with probability $(1-G_S(L))$), the strong bidder wins with probability $\int_L^{\bar{b}} \frac{G'_S(t)}{1-G_S(L)} G_H(t)\, dt$, with valuation v, and the H-type weak bidder wins with the residual probability, $1 - \int_L^{\bar{b}} \frac{G'_S(t)}{1-G_S(L)} G_H(t)\, dt$, with value H. The reasoning for the case when there are two H-type weak bidders is precisely analogous to the case with one H-type weak bidder, with the probabilities adjusted accordingly.

15.3. Efficiency in the Anglo-Dutch auction

The reasoning used to derive the expressions for efficiency for the Anglo-Dutch auction is analogous to the first-price case. Thus I just provide a summary of the results. When $v > v_\gamma$, the strong bidder always wins, and the efficiency will be v. For the case when $v \in (L, v_\gamma)$ the efficiency is given by:

$$
\begin{aligned}
\text{Eff}_{AD} &= (1-\mu)^2 v \\
&\quad +2\mu(1-\mu) G^*_S(L) H + \\
&\quad +2\mu(1-\mu)(1-G^*_S(L)) \left(\begin{array}{c} \left(v * \int_L^{\bar{b}} \frac{(G^*_S(t))'}{1-G^*_S(L)} G^*_H(t)\, dt\right) \\ +H * \left(1 - \int_L^{\bar{b}} \frac{(G^*_S(t))'}{1-G^*_S(L)} G^*_H(t)\, dt\right) \end{array} \right) \\
&\quad +\mu^2 \max(v, H)
\end{aligned}
$$

15.4. Showing the Analytical Revenue Ranking (Proposition 4)

Showing the first part of Proposition 4 proceeds as follows. In the case when $v = H$, we have:

$$\bar{b} = H - (1-\mu)^2 (H - L) < H$$

$$b^* = H - \frac{(1-\mu)^2}{1-\mu^2}(H-L) < H$$

$$R_{ASC} = (1-\mu)^2 L + \left(1-(1-\mu)^2\right) H = H - (1-\mu)^2 (H-L)$$

$$\begin{aligned} R_{FPS} &= H - (1-\mu)^2 (H-L) - \\ &\quad - (1-\mu)(1-\mu)^2 (H-L)(H-L)^{\frac{1}{2}} \int_L^{\bar{b}} \frac{1}{(H-t)(H-t)^{\frac{1}{2}}} dt \\ &= H - (1-\mu)^2 (H-L) - 2(1-\mu)^2 \mu (H-L) < R_{ASC} \end{aligned}$$

$$\begin{aligned} R_{AD} &= \mu^2 H + \left(1-\mu^2\right) H - (1-\mu)^2 (H-L) - \\ &\quad - (1-\mu)^2 (H-b^*)(H-L) \int_L^{b^*} \frac{1}{(H-t)(H-t)} dt \\ &= H - (1-\mu)^2 (H-L) - 2\mu \frac{(1-\mu)^3}{1-\mu^2} (H-L) < R_{ASC} \end{aligned}$$

And also:

$$R_{AD} - R_{FPS} = (1-\mu)^3 (H-L) \left[\frac{2\mu^2}{1-\mu^2}\right] > 0$$

The revenue ranking in this case is:

$$R_{FPS} < R_{AD} < R_{ASC}$$

This establishes the first statement in Proposition 4. For the second statement, when $H = \bar{b}$, w have:

$$(v-H) = (1-\mu)^2 (v-L)$$

Then the expression for $R_{AD} - R_{FPS}$ becomes:

$$\begin{aligned} R_{AD} - R_{FPS} = -\mu^2 (1-\mu)^2 (v-L) - (1-\mu)^2 (H-b^*)(v-L) \cdot \\ \cdot \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt < 0 \end{aligned}$$

which establishes that AD does worse than FPS. I can also show that:

$$R_{FPS} = v - (1-\mu)^2 (v-L)$$

$$\begin{aligned} R_{ASC} &= (1-\mu)^2 L + \left(1-(1-\mu)^2\right)\left(v-(1-\mu)^2 (v-L)\right) \\ &= v - (1-\mu)^2 (v-L) - \mu(2-\mu)(v-L) < R_{FPS}, \end{aligned}$$

whereby we know that in this case:

$$\begin{aligned} R_{AD} &< R_{FPS} \\ R_{ASC} &< R_{FPS} \end{aligned}$$

When $\bar{b} > H$, this means that the first-price auction does not sustain an equilibrium in mixed strategies, and always returns revenue of H; both other auctions give lower revenues than this. The ranking between R_{AD} and R_{ASC} is ambiguous. This establishes the second statement in Proposition 4. For the third part, when $b^* = H$, we have:

$$b^* = v - \frac{(1-\mu)^2}{1-\mu^2}(v-L) = H$$

$$\implies \bar{b} = v - (1-\mu)^2 \frac{1-\mu^2}{(1-\mu)^2} v - H = \mu^2 v + \left(1-\mu^2\right) H > H,$$

whereby an equilibrium in mixed strategies does not exist in a first-price auction. Hence:

$$\begin{aligned} R_{FPS} &= H \\ R_{ASC} &= (1-\mu)^2 L + \left(1-(1-\mu)^2\right) H < H \\ R_{AD} &= \mu^2 H + \left(1-\mu^2\right) v - (1-\mu)^2 (v-L) = H \end{aligned}$$

Therefore the revenue ranking is:

$$R_{ASC} < R_{AD} = R_{FPS}$$

When $b^* > H$, neither the first-price or the Anglo-Dutch auction can sustain an equilibrium in mixed strategies, and so both auctions return revenues of H; the ascending auction generates less revenue, so long as $\mu < 1$. This concludes the proof of Proposition 4.