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# BOUNDS ON THE FERMIONS AND HIGGS BOSON MASSES IN GRAND UNIFIED THEORIES

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## A B S T R A C T

In the framework of grand unifying theories the requirement that no interaction becomes strong and no vacuum instability develops up to the unification energy, is shown to imply upper bounds to the fermion masses as well as upper and lower bounds to the Higgs boson mass. These bounds are studied in detail for the case of the unifying groups SU(5) or O(10).

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## 1. INTRODUCTION AND OUTLINE OF THE MAIN RESULTS

Grand unified gauge theories [1)–2)] give us a fascinating synthesis of particle interactions. For a theory of this type to be useful, however, an important condition must be met; all interactions, including the Yukawa couplings of the fermions to the Higgs fields, and the self-interactions of the Higgs fields with themselves, should remain in the perturbative domain, and no instability should develop in the whole energy range between the mass scale of the $SU(2)\otimes U(1)$ breaking and the unification mass. The mass scale of the $SU(2)\otimes U(1)$ breaking is defined by the vacuum expectation value of the Higgs field :

$$\langle \varphi \rangle_0 \equiv \eta = 2^{-3/4} G^{-1/2} \simeq 176 \text{ GeV} \tag{1.1}$$

while the unification mass is defined to be the energy scale, $M_U$, where all gauge couplings approach the symmetric value.

The stability requirement we have stated before must be satisfied if we want to compute symmetry breaking effects, like the renormalization [3)] of the Glashow-Weinberg-Salam angle [4)], which is the only way we can put the theory to a test (at least today).

In this paper we shall explore the implications of the stability condition, especially focusing on the Higgs self-coupling, in the particular case of the $SU(5)$ theory of Georgi and Glashow [1)].

We shall work within the simplest possible version of the Georgi and Glashow theory, featuring $N$ fermion generations (i.e., $N$ replicas of the $\bar{5}+10$ structure) with negligible intramultiplet mass splittings, on the scale of $M_U$. For masses smaller than $M_U$ we shall have, furthermore, the $SU(2)\otimes U(1)\otimes SU(3)_{colour}$ gauge fields, and one single Higgs doublet, arising from an $SU(5)$ $5$ multiplet [*)].

Given the unstable character of the renormalization group equations obeyed by the Higgs self-coupling $\lambda$, the requirement that $\lambda$ does not increase too much until $M_U$ is reached gives a significant upper bound to the value of $\lambda$ in the low energy region. This bound is immediately translated into a bound on the Higgs meson mass. In the simplest case, where all fermions are much lighter than the $W$ bosons, we find :

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*) Besides the isospin doublet, the $5$ contains a colour triplet, which, in general, could have a similar mass. The effects of the triplet are in any case much smaller than those of one fermion generation, so that we shall neglect it in what follows.

$$M_H \lesssim 200 \text{ GeV} \tag{1.2}$$

quite independently of the number of generations, N.

The possible presence in the spectrum of fermions heavier than $M_W$ complicates somehow the problem. We can follow precisely what happens in the case of three generations $(N=3)$, where only the t quark may possibly be significantly heavy. Increasing the t mass, the upper bound on $M_H$ varies somehow, but it remains always of the same order, until $M_t$ itself becomes so large that the Yukawa coupling of t to the Higgs field leaves the perturbative domain before $M_U$ is reached. This happens for :

$$M_t \approx 200 \text{ GeV} \tag{1.3}$$

which therefore provides an upper bound to the t mass, in the three-generation case.

Besides the $N=3$ case, we have considered, for illustrative purposes, the extreme case $N=8$, with one single heavy quark. The $N=8$ case is chosen, because this seems to be about the largest value of N compatible with the gauge coupling constants at $M_U$ not being too large, as compared to what is required by the observed proton stability.

The upper bound obtained in the case $N=8$ is very similar to the $N=3$ case. Also a bound similar to (1.3) is obtained, for the heavy quark mass :

$$M_f \leq 250 \text{ GeV} \tag{1.4}$$

Before the bounds (1.3) and (1.4) are reached, a new type of instability in the Higgs self-coupling may appear. The contribution of the Yukawa couplings to the $\beta$ function of $\lambda$ is such as to make $\lambda$ to decrease as the energy scale increases. When the quark mass exceeds the value defined by *) :

$$3 \cdot (M_q)^4_{crit.} = \frac{3}{4} M_W^4 (2 + \sec^4 \theta_W) \tag{1.5}$$

the corresponding Yukawa coupling could drive $\lambda$ to negative values at some energy scale below $M_U$ [5)-7)]. If this happens, the Higgs potential becomes negative at large values of the field, and the minimum at $\varphi = \eta$ is not an absolute minimum anymore. To avoid this pathology, $\lambda$ (i.e., $M_H$) must satisfy a lower bound, as a function of the heavy quark mass.

*) $\theta_W$ is the Glashow-Weinberg-Salam angle.

Finally, if the fermion mass is smaller than the value given in Eq. (1.5), a different lower bound on the Higgs may arise, as pointed out by Linde and by Weinberg [8], due to the possible appearance of a new minimum at $\varphi < \eta$.

The plan of the paper is the following. In Section 2 we recall the main results of the grand unification scheme on the behaviour of the gauge couplings. In Section 3 we write down the renormalization group equations satisfied by the Yukawa couplings of the Higgs particles to quarks, and by the quartic self-coupling of the Higgs fields. In Section 4 we derive the upper bounds on the fermion and Higgs boson masses. In Section 5, finally, we derive the lower bounds on the Higgs boson mass. For the sake of brevity, in Section 3 we have considered the case of only one heavy quark. A more complete equation is written in the Appendix.

## 2. GAUGE COUPLINGS

We write in this Section the renormalization group equations for the gauge couplings of $SU(3)_{colour} \otimes SU(2) \otimes U(1)$, with the aim of establishing the notations and of deriving an approximate upper bound for the number of generations, N. To the one loop approximation, and neglecting Higgs contributions, one has [3]:

$$\frac{1}{\alpha} = \frac{8}{3}\left(\frac{1}{\alpha_0} + \frac{L_u}{3\pi} N\right) - \frac{L_u}{3\pi}\left(\frac{11}{2}\right)$$

$$\frac{1}{\alpha_W} = \left(\frac{1}{\alpha_0} + \frac{L_u}{3\pi} N\right) - \frac{L_u}{3\pi}\left(\frac{11}{2}\right) \qquad (2.1)$$

$$\frac{1}{\alpha_S} = \left(\frac{1}{\alpha_0} + \frac{L_u}{3\pi} N\right) - \frac{L_u}{3\pi}\left(\frac{33}{4}\right)$$

where $\alpha$, $\alpha_W$ and $\alpha_S$ are the fine structure constants of QED, of the weak SU(2) and of the colour group, respectively, evaluated at low energy (i.e., at a momentum scale $q = \eta$) and $\alpha_0$ is the SU(5) symmetric value, at the unification mass, $M_U$. Finally :

$$L_u = \ln\left(\frac{M_U^2}{\eta^2}\right) \qquad (2.2)$$

Equations (2.1) give rise to one relation among the low energy couplings, namely [3),9)] :

$$\sin^2\theta_W \equiv \frac{\alpha}{\alpha_W} = \frac{1}{6} + \frac{5\alpha}{9\alpha_s} \approx .2 \tag{2.3}$$

Furthermore, one may obtain :

$$L_u = \frac{2\pi}{11}\left(\frac{1}{\alpha} - \frac{8}{3\alpha_s}\right) \approx 63 \tag{2.4}$$

(In the above equations, and in the following, we use $\alpha_s \approx 0.1$.) Finally, one can deduce from Eqs. (2.1) the value of $\alpha_0$, which depends explicitly upon N :

$$\frac{1}{\alpha_0} = \frac{3}{8\alpha} - \frac{L_u}{3\pi}\left(N - \frac{33}{16}\right) \simeq 6.7\cdot(9.7 - N) \tag{2.5}$$

The omission of higher loop terms, which leads to the successful result Eq. (2.3), can be justified only if $\alpha_0$ is not too large. Looking at Eq. (2.5), we see that this is obeyed only if :

$$N \lesssim 9 \div 10 \tag{2.6}$$

The saturation of the bound for N is rather unlikely, within the SU(5) scheme. A value for $\alpha_0$ much too larger than $\alpha$ may lead to unacceptably fast proton decay. A value as large as $N = 8$ cannot be excluded. In the following we shall consider, as extreme cases, $N = 3$ and $N = 8$.

## 3. YUKAWA AND QUARTIC HIGGS COUPLINGS

We consider first the Yukawa coupling of the Higgs doublet to the fermion fields. For simplicity of notation we consider explicitly the case where we have only one heavy quark. This is certainly a good approximation if $N = 3$, since the Yukawa couplings of the observed quarks and leptons are negligible on the scale of the gauge couplings, and only the t quark could give a sizeable contribution. Defining $h(q^2)$ as the running coupling constant of the heavy quark to the Higgs doublet $\varphi$, and :

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}$$

$$\langle \varphi^0 \rangle_0 = \eta \tag{3.1}$$

$$t = \ln (q^2/\eta^2)$$

one finds the renormalization group equation (see Appendix) :

$$8\pi^2 \frac{d h^2}{dt} = \left( \frac{9}{4} h^2 - 16\pi\alpha_s \right) h^2 \tag{3.2}$$

The quark mass is :

$$M_f = h(\eta^2) \cdot \eta \tag{3.3}$$

In writing Eq. (3.2), terms of order $\alpha_W$ and $\alpha$ have been neglected.

The renormalization group equation for the Higgs self-coupling, $\lambda$, can be easily obtained from the work of Ref. 10). We define the quartic term in the Higgs potential to be :

$$V^{(4)} = \frac{\lambda}{6} (\varphi^+ \varphi)^2 = \frac{\lambda}{4!} \left( \sum_{i=1}^{4} \varphi_i^2 \right)^2 \tag{3.4}$$

One finds :

$$32\pi^2 \frac{d\lambda}{dt} = 4\lambda^2 + 12\lambda h^2 - 3\lambda(3g^2 + g'^2) - 36 h^4 + \frac{9}{4}\left[2g^4 + (g^2 + g'^2)^2\right] \tag{3.5}$$

In Eq. (3.5) we have used the couplings $g$ and $g'$ of the weak SU(2) and U(1), respectively, which are related to $\alpha$ and $\alpha_W$ according to the well-known formulae :

$$\frac{g^2(q^2)}{4\pi} \equiv \alpha_W(q^2) = \frac{\alpha(q^2)}{\sin^2\theta_W(q^2)}$$

$$\frac{g'(q^2)}{g(q^2)} = tg\, \theta_W(q^2) \tag{3.6}$$

## 4. UPPER BOUNDS ON THE FERMION AND ON THE HIGGS BOSON MASSES

According to the general philosophy described in Section 1, we derive now the upper bounds to the Yukawa coupling and to the Higgs self-coupling $\lambda$. These bounds can be directly translated into bounds for the fermion and for the Higgs boson masses.

We consider first the Yukawa coupling. For a sufficiently large initial value, $h^2(\eta^2)$, the term proportional to $h^4$ in the r.h.s. of Eq. (3.2) dominates, and $h(q^2)$ develops a singularity at a finite value of $q^2$. If $\alpha_s$ were constant, this would happen for :

$$h^2(\eta^2) > \frac{64}{9}\pi\alpha_s \tag{4.1}$$

corresponding to $M_f > 250$ GeV. Clearly, in the vicinity of the singular point, Eq. (3.2) does not describe the correct behaviour of $h^2$, in that higher order corrections become significant. The presence of a singularity in the solution of Eq. (3.2), for a certain value of $q^2$, is simply a signal that, at that energy scale, the interaction becomes strong. This should not happen for $q^2$ smaller than $M_U^2$, and this condition leads to an upper bound for the initial value, $h^2(\eta^2)$. We note that the stronger condition that $h^2(q^2)$ remains small for $q^2 < M_U^2$, leads essentially to the same bound. This is due to the fact that $L_U$ is very large, and that the solution to Eq. (3.2) varies very rapidly near the singular point. We shall therefore adopt the weaker bound discussed above.

We have determined the upper bound of $h^2(\eta^2)$ by solving numerically Eq. (3.2). In the case $N=3$, Eq. (4.1) is sufficient to describe the variation of $\alpha_s$ with $q^2$. For $N=8$, we have taken into account the two loop corrections to the evolution of $\alpha_s$, computed in Ref. 11). Using $\alpha_s(\eta^2) \approx 0.1$, we find :

$$\begin{aligned} M_f &< 200 \text{ GeV} \quad (N=3) \\ M_f &< 250 \text{ GeV} \quad (N=8) \end{aligned} \tag{4.2}$$

The case where one has many heavy fermions has been considered in Ref. 12), where it has been shown that the condition previously illustrated leads to a sum rule for the fermion masses *).

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*) The r.h.s. of Eq. (16) of Ref. 12) should be multiplied by a factor of two.

A similar analysis can be applied to Eq. (3.5). The bound on $\lambda(\eta^2)$ for different values of $h^2(\eta^2)$ can be found, by solving numerically Eq. (3.5) and requiring $\lambda(q^2)$ to be finite for $q^2 < M_U^2$. The bound on $\lambda(\eta^2)$ thus found gives a bound on the Higgs boson mass, according to :

$$M_H^2 = \frac{2}{3}\lambda(\eta^2).\eta^2 \tag{4.3}$$

The resulting bound on $M_H$, as a function of the heavy quark mass, is shown in Fig. 1, for the case $N=3$, and in Fig. 2, for the case $N=8$.

The $N$ dependence of the result arises through the influence of $N$ on the evolution of the gauge couplings.

Upper bounds on the Higgs boson and on fermion masses have been previously derived [13] from the requirement that the Yukawa couplings and the Higgs self-coupling be small at the energy scale $\eta$. These bounds ($M_H$, $M_f \lesssim 1$ teV) are considerably weaker than those derived here on the assumption that the two couplings remain small in the whole $q^2$ range up to $M_U^2$. Bounds of the order of 1 teV for the mass splitting between fermion doublets and for the Higgs boson mass have also been derived by Veltman [14].

## 5. LOWER BOUND ON THE HIGGS BOSON MASS

In this Section we discuss the lower bound on $\lambda(\eta^2)$ which can be derived from the requirement that $\lambda(q^2)$ does not take negative values, in the region : $\eta^2 < q^2 < M_U^2$. If $\lambda(q^2) < 0$ for some value of $q$, the self-interaction of the Higgs field becomes attractive, and we may expect a vacuum instability to develop. More precisely, as discussed in Ref. 15), the behaviour of the effective Higgs potential for large values of the field, $\varphi$, is given by Eq. (3.4), with a $\varphi$ dependent coupling :

$$\lambda = \lambda(\varphi^2) \tag{5.1}$$

where $\lambda(\varphi^2)$ obeys the renormalization group equation (3.5), with $q$ replaced by $\varphi$. If the solution to Eq. (3.5) becomes negative at large $q$, the effective Higgs potential becomes negative at large values of the field, and the minimum at $\varphi=\eta$ is not the absolute minimum [5)-7)].

Looking back at Eq. (3.5), we see that $\lambda(q^2)$ can be driven to negative values if the Yukawa coupling dominates the r.h.s. The situation is easy to analyze if we neglect the $q^2$ dependence of $h$, $g$ and $g^2$. If :

$$36 h^4 - \frac{9}{4}\left[2 g^4 + (g^2 + g'^2)^2\right] \propto$$
$$\propto 12 M_f^4 - 3(2 M_W^4 + M_Z^4) > 0 \tag{5.2}$$

the r.h.s. of Eq. (3.5) has one negative and one positive root : $\lambda_- < 0 < \lambda_+$. The negative root $\lambda_-$ is an ultra-violet stable fixed point. Under these conditions, for any initial value in the region :

$$0 \leq \lambda(\eta^2) \leq \lambda_+ \tag{5.3}$$

$\lambda$ becomes negative at some finite value of $q^2$. A lower bound for $\lambda(\eta^2)$ can thus be obtained, by requiring that this does not happen for $q^2$ less than $M_U^2$. In the more complicated case where h, g and g' evolve with $q^2$ according to Eqs. (2.1) and (3.2), the lower bound can be obtained numerically. The corresponding lower bound on $M_H$ is displayed in Figs. 1 and 2, for $N = 3$ and 8 respectively *), as a function of the fermion mass, i.e., $h(\eta^2)$.

We observe that, according to our previous discussion, the lower bound exists only for large enough values of the fermion mass, see Eq. (5.2). If we require the absence of this type of instability even for very small values of $\lambda(\eta^2)$ [e.g., $\lambda(\eta^2) \approx g^4$] inequality (5.2) should not be satisfied, i.e.,

$$12 M_f^4 < 3\left[2 M_W^4 + M_Z^4\right] \tag{5.4}$$

This upper bound on the fermion mass coincides with that given in Refs, 5)-7). The upper bound in Eq. (5.4) is lower than the one derived in the previous Section. As discussed above, the bound (5.4) can be exceeded, provided $\lambda(\eta^2)$, i.e., $M_H$, is large enough. Furthermore, we note that for large values of the fermion mass, the upper and lower bounds on $M_H$ tend to coincide.

This is easy to understand. As $h(\eta^2)$ increases, the r.h.s. of Eq. (3.5) varies more rapidly with $\lambda$, and the equation becomes more and more unstable. As a consequence, the range of initial values $\lambda(\eta^2)$ such

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*) In the $N = 8$ case, two-loop corrections to the evolution of $\alpha_s$ have been taken into account, as in the previous Section.

that $\lambda(q^2)$ neither blows up nor does become negative (for $q^2 < M_U^2$) narrows down. Eventually, when $M_f$ tends to its upper bound, $M_H$ is essentially determined to be :

$$\begin{aligned} M_H &= 220 \text{ GeV} \quad (N=3) \\ M_H &= 280 \text{ GeV} \quad (N=8) \end{aligned} \tag{5.5}$$

The case $N=8$, $M_f$ and $M_H$ given by Eqs. (4.2) and (5.5) corresponds closely to the situation envisaged in Ref. 12). All bounds are nearly saturated and the low energy couplings, $\alpha$'s, h and $\lambda$, are close to an infra-red stable (but ultra-violet unstable) point. The virtues of such a situation have been discussed in Refs. 16),17),12).

The lower bound on $M_H$ discussed in this Section has been obtained by requiring that the minimum at $\varphi = \eta$ is not unstable towards values of $\varphi > \eta$, i.e., in the ultra-violet region. If we require a similar condition with respect to the infra-red region, $\varphi = 0$, we would obtain an independent bound.

For the mass fermions this corresponds to the bound first obtained in Ref. 8). We have reported in Figs. 1 and 2 this bound as a function of the fermion mass, namely :

$$M_H^2 > \frac{3}{32\pi^2\eta^2}\left[(2M_W^4 + M_Z^4) - 4M_f^4\right] \tag{5.6}$$

We note that this bound is effective only when the inequality (5.4) is satisfied. The reason is that if the fermions are too heavy, $\lambda(q^2)$ increases in going towards the infra-red region.

For completeness we have reported in Fig. 1 the prediction for the Higgs boson mass in the massless theory [i.e., $(\partial^2 V/\partial\varphi^2)_{\varphi=0} = 0$] where the spontaneous breaking arises from radiative corrections alone [15]. In this case $M_H^2$ is twice the value of the Linde-Weinberg bound, Eq. (5.6). Again this mechanism can be effective only if fermions are not too heavy, i.e., if they satisfy Eq. (5.4).

## APPENDIX

In this Appendix we write the differential equations obeyed by the Yukawa couplings of the Higgs field to quarks and leptons.

Assuming N generations of fermions and massless neutrinos, these couplings are described by three $N \otimes N$ matrices, D,U,L :

$$\begin{aligned} \mathcal{L}_Y = \varphi^0 [\bar{d}_L D d_R + \bar{u}_R U^+ u_L + \bar{\ell}_L L \ell_R] \\ + \varphi^+ [\bar{u}_L D d_R - \bar{u}_R U^+ d_L + \bar{\nu}_L L \ell_R] + h.c. \end{aligned} \tag{A.1}$$

where d, u, $\ell$ and $\nu$ are N component vectors. The mass term is obtained by substituting in $\mathcal{L}_y$ $\varphi_0$ with its vacuum expectation value, $\eta$. In the unbroken SU(5) limit $D = L$.

The renormalization group equations obeyed by D, U, L can be obtained by applying the results of Ref. 10) to this particular case.

$$\begin{aligned} 32\pi^2 \frac{dD}{dt} &= \frac{3}{2} (DD^+D - UU^+D) + (\Sigma - A_d) D \\ 32\pi^2 \frac{dU}{dt} &= \frac{3}{2} (UU^+U - DD^+U) + (\Sigma - A_u) U \\ 32\pi^2 \frac{dL}{dt} &= \frac{3}{2} LL^+L + (\Sigma - A_L) L \end{aligned} \tag{A.2}$$

where

$$\Sigma = \mathrm{Tr}\left[3(D^+D + U^+U) + L^+L\right] \tag{A.3}$$

and

$$A_d = 32\pi\alpha_s + \frac{5}{3}\pi\alpha' + 9\pi\alpha_W$$

$$A_u = 32\pi\alpha_s + \frac{17}{3}\pi\alpha' + 9\pi\alpha_W \qquad (A.4)$$

$$A_L = 15\pi\alpha' + 9\pi\alpha_W$$

The gauge coupling constant $\alpha'$ is defined by

$$\frac{1}{\alpha} = \frac{1}{\alpha_W} + \frac{1}{\alpha'}, \qquad (A.5)$$

Equation (3.2) of Section 3 can be obtained from (A.2) by assuming only one heavy (up) quark. This corresponds to set $L = D = 0$, and to retain in $U$ only its largest eigenvalue, $h$. To obtain Eq. (3.2), we have also neglected the weak couplings $\alpha_W$ and $\alpha'$ in $A_U$.

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FIGURE CAPTIONS

Figure 1 : Bounds on the mass of the Higgs boson $(m_H)$ as a function of the top quark mass $(m_f)$ in the case of three generations. We have taken $\sin^2\theta_W = 0.2$. The dashed line and the full line represent the upper and the lower bound, respectively. The dotted line is the prediction of the massless theory. The curves end in correspondence to the upper bound on $m_f$, Eq. (4.2).

Figure 2 : Same as in Fig. 1, for the case of eight generations, with only one very heavy fermion, i.e., a quark of mass $m_f$.

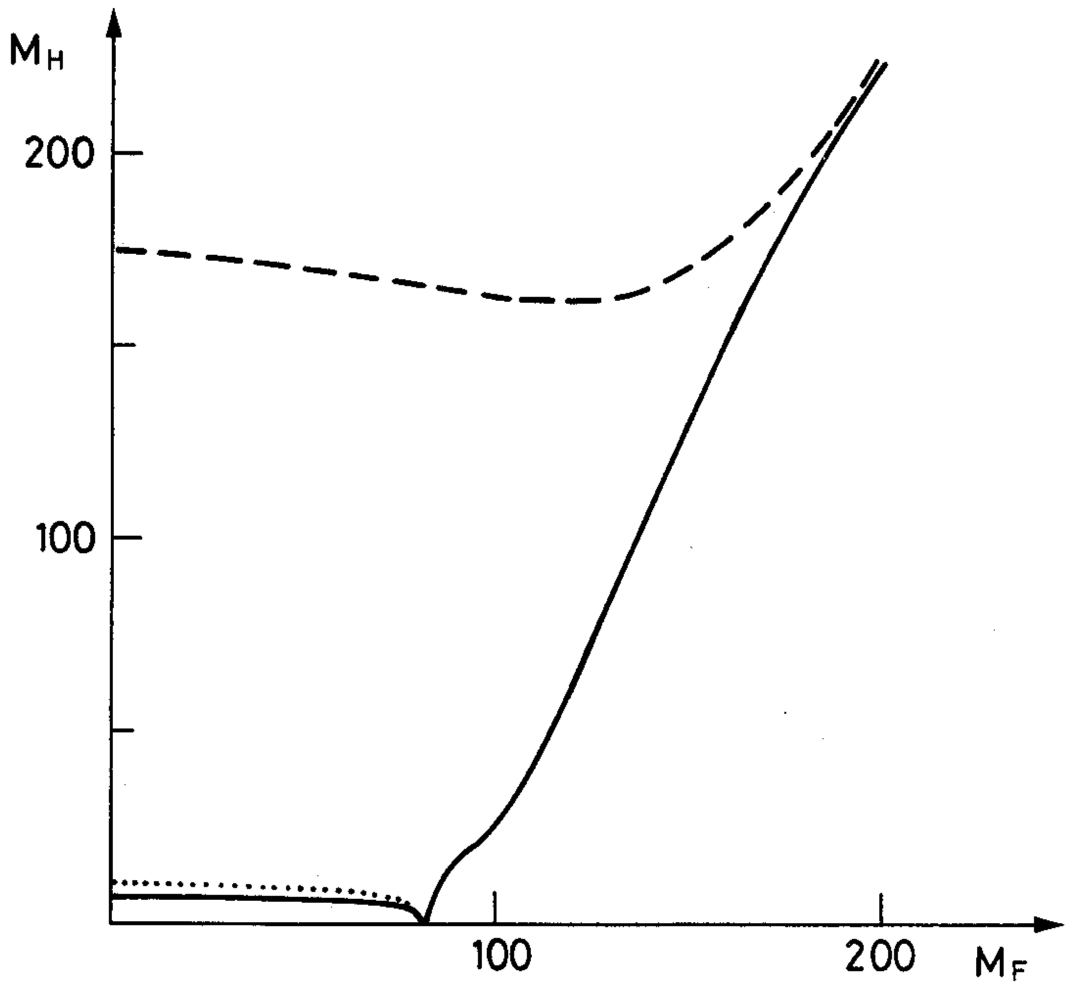


FIG. 1

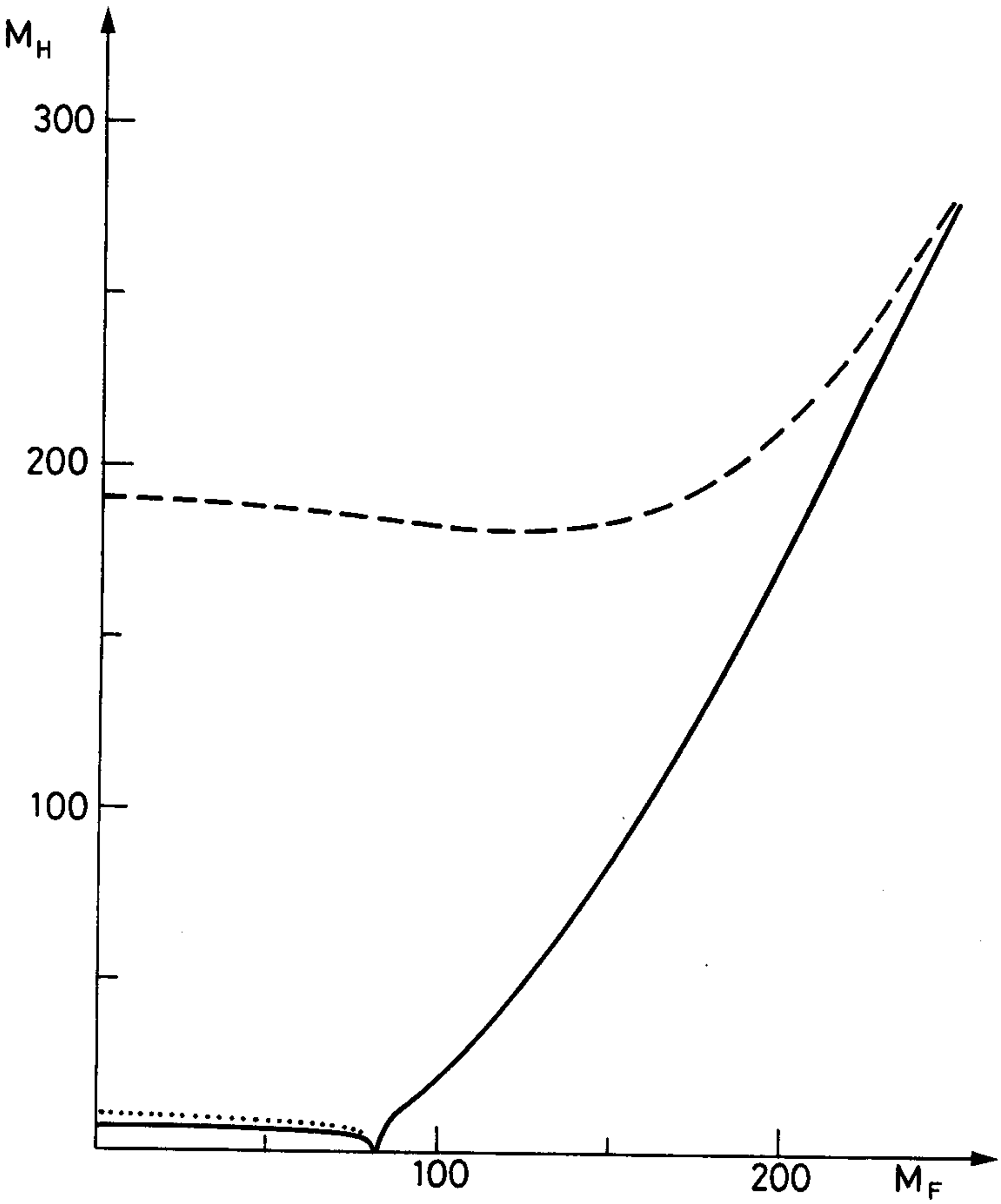


FIG.2