University of Leeds

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Assignment in Mathematics (40cr)

Ramsey Theory

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Abstract

This project will be split into three main sub topics. In the first, we shall look at Ramsey's theorem, which roughly states that if a set, X, has enough elements and if we colour all the elements of $[X]^k$ with r colours, for some k and r, then we have some subset Y of X such that all the elements of $[Y]^k$ are the same colour. We shall see two proofs of this as well as some nice applications. Ramsey numbers are the least possible size the set X has to be for Ramsey's theorem to hold and we shall look at quite a few upper and lower bounds as well as exact values for Ramsey numbers.

The second sub topic we shall look at is Van der Waerden's theorem, which is a fundamental Ramsey-type theorem. Van der Waerden's theorem is concerned with the colourings of the integers and monochromatic arithmetic progressions that arise. Van der Waerden's theorem gives us another important result, Szemerédi's theorem which is the density version of Van der Waerden's theorem and we shall have a brief look at this.

Finally, the last sub topic we shall cover in this project is the generalisation of Van der Waerden's theorem, the Hales-Jewett theorem. The Hales-Jewett theorem is about colourings of the hypercube $C_t^n = \{(x_1, x_2, ..., x_n) : x_i \in \{0, 1, ..., t-1\}\}$ and monochromatic lines that arise. We shall look at the celebrated proof of the Hales-Jewett theorem by Shelah. Like Van der Waerden's theorem the Hales-Jewett theorem has a density version also. We shall see a proof of this which is partly followed from the on line collaboration project called Polymath. In the final chapter, we shall focus on finding upper and lower bounds for the Density Hales-Jewett numbers, $c_{n,3}$ (the size of the largest subset of C_3^n that is line free), and in result of these bounds we shall find the exact values of $c_{n,3}$ for $0 \leq n \leq 6$.

Contents

1 Introduction and Ramsey's theorem **4**
- 1.1 The Pigeonhole Principle 4
- 1.2 Ramsey's theorem 5
  - 1.2.1 Notation 6
  - 1.2.2 Finite Ramsey's theorem 6
  - 1.2.3 Infinite Ramsey's theorem 10

2 Ramsey numbers **15**
- 2.1 Values and bounds for $R(a_1, a_2)$ 15
  - 2.1.1 Upper bounds for $R(a_1, a_2)$ 15
  - 2.1.2 Lower bounds for $R(a_1, a_2)$ 21
- 2.2 Multicolour Ramsey Numbers 24

3 Van der Waerden's theorem **26**
- 3.1 Introduction 26
- 3.2 Proof of Van der Waerden's theorem 28
- 3.3 Szemerédi's Theorem 31

4 Hales-Jewett Theorem **32**
- 4.1 Introduction 32
- 4.2 Shelah's proof 33

5 The density version of The Hales-Jewett theorem **40**
- 5.1 Proof of the Density Hales-Jewett theorem 41
  - 5.1.1 Some new notation, definitions and preliminary tools . 41
  - 5.1.2 Proving the Density Hales-Jewett theorem 45

6 Density Hales-Jewett numbers **53**
- 6.1 Lower bounds for $c_{n,3}$ 53
  - 6.1.1 Lower bounds for $c_{n,3}$ for $4 \leq n \leq 6$ 58
  - 6.1.2 Lower bounds for $c_{n,3}$ for $n > 6$ 61
- 6.2 Upper bounds and exact values for $c_{n,3}$ 67
  - 6.2.1 Upper bound for $c_{4,3}$ 68
  - 6.2.2 Upper bound for $c_{5,3}$ 71
  - 6.2.3 Upper bound for $c_{6,3}$ 71
- 6.3 Summary and further research 72

A Appendix **75**

B Appendix 81

1 Introduction and Ramsey's theorem

Ramsey theory, which is named after the British Mathematician Frank Plumpton Ramsey, is the main topic of this project. There is no one universal definition of Ramsey theory but informally it says that any structure will contain a highly ordered substructure of the same type. [3]

In this chapter we shall look at Ramsey's theorem which is about the colourings of the set $[X]^k$ and monochromatic subsets that arise.

1.1 The Pigeonhole Principle

First we shall look at the pigeonhole principle which is of a similar flavour to Ramsey theory and will be useful in proofs. The simple pigeonhole principle states that if more than n pigeons roost in n holes, then at least one hole must have more than one pigeon in it. The more generalised version of the pigeonhole principle is given below along with a proof as shown by Landman and Robertson [17, pp 3-4].

Lemma 1.1. *(The Pigeonhole principle)*
Suppose $r, m \geq 1$ and $n > mr$. Then if a set of n elements is partitioned into r subsets, then some subset will have at least m+1 elements.

Proof. Let X be a set such that $|X| > mr$ and we partition X into r subsets, X_i for $i = 1, 2, ..., r$. For a contradiction, assume that $|X_i| \leq m$ for all i. Then $|X| = \sum_{i=1}^{r} |X_i| \leq mr$, which is a contradiction and hence there must be at least one i, such that $|X_i| \geq m + 1$. □

The pigeonhole principle will be very useful in some of the proofs to come and so we will look at a few quick examples demonstrating how the pigeonhole principle can be used.

Example 1.2. We will look at the following exercise: 'Prove that if the numbers 1,2,...12 are randomly positioned around a circle, then some set of three consecutively positioned numbers must have a sum of at least 19.' [17, p. 17]

There are 12 different sets of three consecutively positioned numbers around the circle, and each number appears in three of these sets. If A_i is the sum of the three consecutive numbers starting from the i^{th} position on the circle then, $\sum_{i=1}^{12} A_i = 3\sum_{i=1}^{12} i = 234$. Now $234 > 19 \times 12 = 228$, so one of the A_i must be at least 19 by the pigeonhole principle.

Example 1.3. Our second example is: 'An organiser of a party, restricted to those aged between 18 and 30 (inclusive), wanted to ensure that at least three were born in the same year. How many people must be invited to be sure this condition is fulfilled?' [1, p. 296]

There will be 14 years people aged inclusively between 18 and 30 could be born. So here $r = 14$ and as we want to make sure we have at least 3 people born in one of the 14 years we have, $m + 1 = 3$ so $m = 2$. Now, $29 > 2 \times 14 = 28$ so we need 29 people to ensure that three of them are born in the same year.

1.2 Ramsey's theorem

Another example that makes use of the Pigeonhole principle is the party problem. The party problem is the most common starting point of Ramsey's theorem.

Lemma 1.4. *(The party problem)*
If six people are at a dinner party then either three of them are mutual friends or three of them are mutual strangers. [10, p 1]

The proof of this uses the pigeonhole principle discussed before and is rather straightforward.

Proof. If we fix a person out of the six, call them i, and let X be the set of i's friends and Y be the set of the people i has never met. By the pigeonhole principle either $|X| \geq 3$ or $|Y| \geq 3$. Let's assume that $|X| \geq 3$ and if two of the people in X are also friends then we have three mutual friends. If no pair in X know each other then all the people in X are mutual strangers and hence we have three people who are mutual strangers. There is a similar argument if $|Y| \geq 3$. □

Note that here, in order to guarantee that you have three mutual friends or three mutual strangers, you must have at least six people. We can see this by looking at the case with five people and finding an example where there is neither three mutual friends or three mutual strangers. So for example, if we call the five people 1,2,3,4 and 5, then say 1 knows 2, 2 knows 3, 3 knows 4, 4 knows 5 and 5 knows 1 and all the other combinations of people do not know each other. Then there is no three mutual friends or three mutual strangers. [3, p 149]

We can look at other problems similar to the party problem. For example, if we had 17 people at a funeral and two people can either be total strangers, related or friends (if two people are related they are not classed as friends). Then can we guarantee we will have either three people who are strangers, three people who are related or three people who are friends? We shall come back to this problem later. Now, generalising these sort of problems gives us Ramsey's theorem. However, we will first introduce some notation which will be useful before stating Ramsey's theorem.

1.2.1 Notation

Firstly, let X be a set then we shall write,

$$[X]^k = \{Y : Y \subset X, |Y| = k\}$$

For the set $\{1, 2, ...n\}$ we shall write $[n]$, and if $X = [n]$ we shall simply write $[n]^k = \{Y : Y \subset \{1, 2, ...n\}, |Y| = k\}$.

1.2.2 Finite Ramsey's theorem

Theorem 1.5. *(Ramsey's Theorem)*
Let r, k and m be positive integers. Then there exists an N such that: if a set, X, has at least N elements and all elements of $[X]^k$ are coloured with r colours then there exists an m-element subset, Y, such that all elements of $[Y]^k$ are the same colour (i.e. they are monochromatic).
Or more generally, let $r, k, a_1, a_2, ..., a_r$ be positive integers then there exists an N such that: if X is a set of at least N elements and all elements of $[X]^k$ are coloured with r colours, $c_1, ...c_r$. Then there exists an a_i-element subset of X where all of its k-element subsets have colour c_i. [3, p 150]

In order to prove Ramsey's theorem we will need to use the idea of Ramsey numbers which is defined below.

Definition 1.6. $R_k(a_1, a_2, ..., a_r)$ is the least N for which Ramsey's theorem holds. $R_k(a_1, a_2, ..., a_r)$ is called a *Ramsey number* and if $k = 2$ we will just write $R(a_1, a_2, ..., a_r)$.

We shall look at a few examples demonstrating Ramsey numbers in order to ensure the notation is clear.

Example 1.7. The party problem relates to the Ramsey number $R(3, 3)$ and so we know that $R(3, 3) = 6$.

Example 1.8. Earlier we brought up the question: if we had 17 people at a funeral and two people can either be total strangers, related or friends (if two people are related they are not classed as friends). Then can we guarantee we will have either three people who are strangers, three people who are related or three people who are friends? This problem relates to the Ramsey number $R(3, 3, 3)$ and the answer to our question is yes.

If we pick a person, call them A, and then let X be the set of people who A does not know at the funeral, Y the set of A's relatives at the funeral and let Z be the set of A's friends at the funeral. Then by the pigeonhole principle one of X, Y or Z must have at least 6 people in it. If $|Y| \geq 6$, then we have two possibilities:
1) One or more of the people in Y are relatives.
2) Everybody in Y are either friends or strangers.

If 1) is true then we have three people who are relatives as required. If 2) is true then by the party problem we have either three people who are strangers or three people who are friends. We can apply the same to X and Z. So in terms of Ramsey numbers we know $R(3, 3, 3) \leq 17$.

In chapter 2 we will look more into Ramsey numbers, by looking at the known Ramsey numbers and bounds on Ramsey numbers. Now for the proof of Ramsey's theorem we will follow Cameron's [3, p 151] proof, elaborating on the details.

Proof. (Ramsey's theorem)
Firstly, for $k = 1$, we can use the pigeonhole principle to find $R_1(a_1, a_2, ..., a_r)$ to be,

$$R_1(a_1, a_2, ..., a_r) = \sum_{i=1}^{r} a_i - r + 1$$

so now we know the theorem holds for $k = 1$, we shall assume $k > 1$. We shall use induction on the $\sum_{i=1}^{r} a_i$. It is trivial that for any i, if $a_i = k$ then the Ramsey number $R_k(a_1, ..., a_r) = k$.

By the induction hypothesis we know that the following are defined for $i = 1, ..., r$

$$A_i = R_k(a_1, ..., a_{i-1}, a_i - 1, a_{i+1}, ..., a_r)$$

Let $n = 1+R_{k-1}(A_1, ..., A_r)$, then we shall aim to show that $R_k(a_1, a_2, ..., a_r) \leq n$. Let X be a set of n elements and we shall colour $[X]^k$ with the r colours, $c_1, ..., c_r$. Let's pick an element of X, $x \in X$ and let Y be the set $Y = X \backslash \{x\}$. Now, colour $[Y]^{k-1}$ with the colours $c_1^*, ...c_r^*$ according to the following rule: the subset V is colour c_i^* if and only if in the colouring of X, $V \cup \{x\}$ is colour c_i. Rearranging n we get $n - 1 = R_{k-1}(A_1, ..., A_r)$, and so there is a subset of size A_i, let's call it Z_i, of Y which is c^*-monochromatic. Then by the definition of A_i, Z_i has either:
1) A set of size a_j where all the k-element subsets are coloured c_j for $j \neq i$.
2) Or a set, Q , of size $a_i - 1$ where all the k-element subsets are coloured c_i.

In case 1) we have what we desire. In case 2) we have $|Q \cup \{x\}| = a_i$ and all the k-element subsets of $|Q \cup \{x\}|$ are coloured c_i as any k-element subset that does not contain x is coloured c_i by 2) and those k-element subsets that contain x are also coloured c_i by the definition of the c^*-colouring. □

Ramsey's theorem has a lot of beautiful applications which include applications in number theory, logic, set theory, theoretical computer science, Geometry and others [22]. We will focus on a very nice geometrical application which we shall follow from Krishnamurphy [16, p 457], elaborating on his details and providing some examples to demonstrate the proofs.

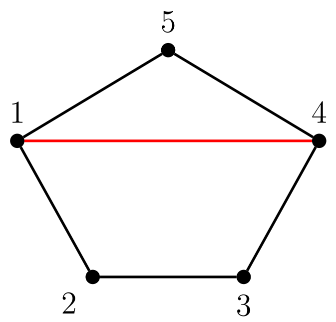
Theorem 1.9. *Let $m \geq 3$ be an integer. Then there exists a positive integer N_m such that:*
If $n \geq N_m$ and we choose n points in the plane such that no three points are collinear (lie on a straight line), then m of these n points are the vertices of a convex (all interior angles are less than $180°$) m-gon.

For the proof of this we need two lemmas.

Lemma 1.10. *If we choose five points in a plane such that no three are collinear, then four of the five points are the vertices of a convex quadrilateral.*

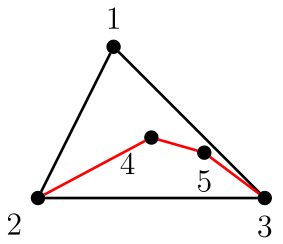
Proof. If we place five points in a plane and join each pair with an edge, so we have ten edges. Then the outer perimeter is a convex polygon. We have three cases here:
1) The outer perimeter is a pentagon.
2) The outer perimeter is a quadrilateral.
3) The outer perimeter is a triangle.

If 1) is true, if we take any four of the vertices we can get a convex quadrilateral. For example in the convex pentagon below by using the red line between 1 and 4 we get a convex quadrilateral.



If 2) is true we have what we want.

If 3) is true then we must have two points inside the triangle. We therefore have a convex quadrilateral with the two points inside the triangle and the two vertices of the triangle that are on the same side as the two points on the inside. For example if we have the following outer perimeter and two inside points, then the lines in red plus the line between 2 and 3 gives us a convex quadrilateral.



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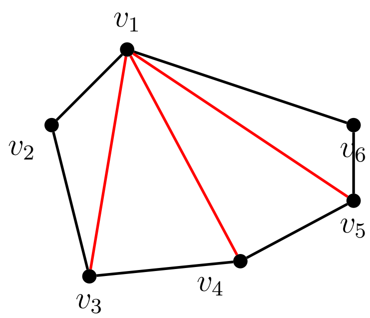
Lemma 1.11. *If we have m points in a plane such that no three points are collinear and if all the quadrilaterals that are formed by the m points are convex then the m points form the vertices of a convex m-gon.*

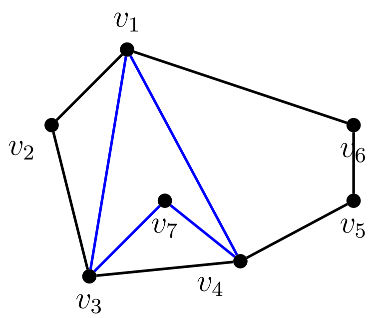
Proof. The m points give $\binom{m}{2}$ edges. The outer perimeter is again a convex polygon. Lets say the outer perimeter is formed by p of the points and $p < m$. Call the points that form the vertices of the outer perimeter $v_1, v_2, v_3, ..., v_p$ in that order. Now any of the points that lie inside the outer perimeter must lie inside one of the triangles $v_1v_2v_3, v_1v_3v_4, ..., v_1v_{p-1}v_p$.

For example if $p = 6$, then the diagram below demonstrates the triangles in which any inner point must lie.



Let s be a point inside the outer perimeter but then the quadrilateral $v_1v_isv_{i+1}$ has an angle greater than 180° (angle v_isv_{i+1}).

For example, carrying on from our previous example if v_7 is an interior point then the we can form a quadrilateral which is concave, which is shown in blue in the following digram.



This contradicts the hypothesis that all quadrilaterals that are formed from the m points are convex and so we must have $p = m$ and so we have a convex m-gon. □

Now using Ramsey's theorem, lemma 1.10 and lemma 1.11 we can give the proof for theorem 1.9.

Proof. (Theorem 1.9.)
It is obvious that $N_3 = 3$ as triangles are convex so any three points will form a convex triangle as long as we have at least three points in the plane.

Let $m \geq 4$ and let $n \geq R_4(5, m)$. If we have n points in the plane and we partition the four-element subsets of the n points into concave and convex quadrilaterals then by Ramsey's theorem we either have:
1) A set of five points with all quadrilaterals concave.
2) Or a set of m points with all quadrilaterals convex.

By lemma 1.10 1) cannot be true. So 2) must be true and by lemma 1.11 we have a convex m-gon. □

1.2.3 Infinite Ramsey's theorem

We will look at the infinite version of the Pigeonhole principle which is stated by Cameron [3, p 312] but we shall give our own proof to the principle. The principle says, if infinitely many pigeons roost in a finite number of holes then some hole must contain infinite pigeons. Below is a more formal definition of the infinite Pigeonhole principle.

Lemma 1.12. *(The infinite Pigeonhole Principle)*
If infinite many items are divided into a finite number of classes, then some class has an infinite number of items.

Proof. Let X be an infinite set of items, then partition X into r classes where r is finite. Then let $X = X_1 \sqcup X_2 \sqcup ... \sqcup X_r$ be the partition. If all X_i are finite then X would also be finite, so there must be an i, where X_i is infinite. □

The Infinite Ramsey's theorem is a generalisation of the infinite Pigeonhole principle. Cameron [3, p 316] states the theorem and proves it for the $k = 2$ case, we shall prove it for general k.

Theorem 1.13. *(Infinite form of Ramsey's theorem)*
Let r and k be positive integers and let X be an infinite set. Colour the elements of $[X]^k$ with r colours. Then there is an infinite subset Y of X such that all the elements of $[Y]^k$ are the same colour.

Proof. We shall prove this theorem by induction on k. The $k = 1$ case is the infinite form of the Pigeonhole principle which we have just proved. So we shall assume the theorem holds for $k - 1$ and we shall show it holds for k.

Suppose X is countable, say $X = \{x_1, x_2, ...\}$. Then let $y_1, y_2, ...$ be a subsequence of distinct elements. Finally let $Y_0, Y_1, ...$ be a sequence of distinct elements such that:
1) $Y_1 \subseteq Y_2 \subseteq ...$
2) $y_i \notin Y_i$ and all $\{y_i\} \cup z$ have the same colour, where z are the $(k - 1)$ element subsets of Y_i.
3) For all $j > i$ we have $y_j \in Y_i$.

We shall construct the Y_i as follows:
Let $Y_0 = X$. Then to construct Y_i, choose a $y_i \in Y_{i-1}$ then there are infinite many $\{y_i\} \cup z$ coloured with r colours, where z are the $(k-1)$ element subsets of $Y_i \backslash \{y_i\}$. By the induction hypothesis there exists an infinite subset Y_i of $Y_{i-1} \backslash \{y_i\}$ where all the $(k - 1)$ element subsets are the same colour and so 2) holds.

So now the colour of a k-element subset only depends on the y_i with the smallest i, call this colour c_i. As we have a finite number of colours, by the Pigeonhole principle there exists an infinite subset M of the natural numbers such that $c_i = c_j$ if i and j are both in M. So the set $\{y_i : i \in M\}$ is a monochromatic subset of X. □

We can deduce another proof for the finite version of Ramsey's theorem from the infinite form using König's infinity lemma. König's infinity lemma is a theorem about infinite graphs so it makes sense to define a graph and some properties of graphs before stating and proving it.

Definition 1.14. A *graph* is a pair $G = (V, E)$, where V is a finite set of points called vertices and E is a set of edges where each edge is associated with two vertices in V (endpoints). A directed graph is a graph G where the edges have a direction associated with them.

We note that in this project we shall only refer to simple graphs, which are graphs with no loops (no edge with both endpoint the same) or no multiple edges (no two edges have the same set of endpoints).

Definition 1.15. Let $G = (V, E)$ be a graph, then the *degree* of vertex $v \in V$, denoted $d(v)$, is the number of incident edges.

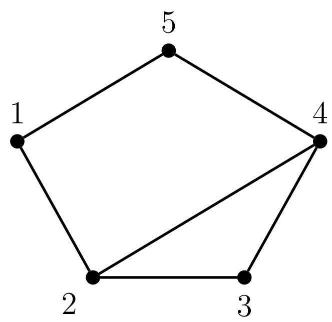
Definition 1.16. A *path* is a sequence of edges that connect a sequence of vertices. All edges and vertices in the sequences (except maybe the first and last vertex) are distinct. A graph is *connected* if for every pair of vertices u and v there is a path from u to v (a path that starts at vertex u and ends at vertex v).

Definition 1.17. A *subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

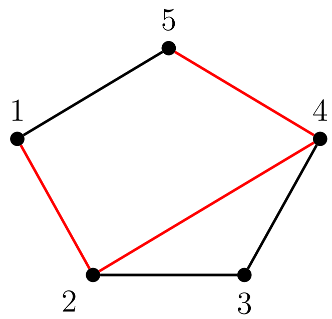
Definition 1.18. A graph G is *complete* if every pair of vertices in G are connected by an edge. A complete graph on n vertices is denoted as K_n.

We shall give an example to demonstrate these graph theory definitions.

Example 1.19. Below is an example of a graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (3, 4), (4, 5), (1, 5), (2, 4)\}$ (here (u, v), where $u, v \in V$, means that the vertices u and v are connected by an edge in the graph).

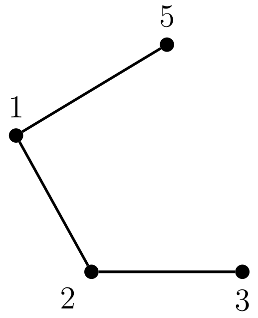


The five vertices in the above graph have the following degrees; $d(1) = 2$, $d(2) = 3$, $d(3) = 2$, $d(4) = 3$, $d(5) = 2$. An example of a path in this graph is shown below in red,



The path begins at vertex 1 then goes to 2, then 4 and then 5. This graph is connected as there is a path between vertices u and v for all $u, v \in V$.

Below is an example of a subgraph of this graph.



Finally this graph is not complete as there is not an edge between every pair of vertices for example there is no edge between 1 and 4.

Below is König's infinity lemma and proof which we follow from Wilson's [27, p 78] proof.

Theorem 1.20. *(König's infinity lemma)*
Let $G = (V, E)$ be a connected infinite graph (infinite number of vertices), where all vertices have finite degree. Then, for any $v \in V$ there exists a one-way infinite path starting at v.

Proof. For every vertex, $z \in V \backslash \{v\}$ there is a path from v to z as the graph is connected. There are infinitely many paths in G that start at v and as v has only finitely many neighbours, there exists a $v_1 \in V$ such that infinitely many of the paths start $v \to v_1 \to ....$ We can keep on repeating this process for v_i infinitely many times and so we have the one way infinite path $v \to v_1 \to v_2 \to ....$ □

We shall discuss a quick example showing König's infinity lemma in action. This example was proposed by König [14] and we shall give a proof.

Example 1.21. Claim: If we assume that humans will never become extinct then we have an infinite line of males.
Proof: Let H_1 be the set of all the males that are alive today and let H_2 be the set of all the sons of the males in H_1, let H_3 be the set of all the male sons of the men in H_2, ..., let H_i be the set of all the male children of the men in the set H_{i-1}. Then as humans can only have a finite number of children and as we have assumed that humans will not become extinct, we know that all H_i are finite and non zero. Now if we create a graph where the vertices are all the males from the sets H_i, and there is an edge from $v \in H_i$ to $u \in H_{i+1}$

if and only if u is the son of v. So in our graph we have an infinite number of vertices (as humans will not become extinct) and each vertex has a finite degree. These are the conditions for us to use König's infinity lemma, so we have an infinite path $m_1, m_2, m_3, ...$ where $m_i \in H_i$ and m_{i+1} is the son of m_i. So this infinite path is our infinite male line.

We can now deduce a second proof for the finite Ramsey's theory from the infinite version along with use from König's infinity lemma. Below is the proof which expands on Cameron's [3, p 316] outline of the proof.

Proof. (Finite Ramsey's theorem)
We shall assume that the theorem does not hold for some r, k and m. So for all positive n there exists an r-colouring of $[X]^k$ where $X = \{1, 2, ...n\}$ such that there is no m-monochromatic set, we shall call these colourings bad colourings. Let V_n be the set of bad colourings of $[X]^k$ where $X = \{1, ..., n\}$. Now form a directed graph where the vertex set is $V = V_0 \cup V_1 \cup ...$ and define the edges as follows: There is an edge going from $v_n \in V_n$ to $v_{n+1} \in V_{n+1}$ if and only if v_n is the exact same colouring as v_{n+1}, with the k-element subsets with $n+1$ in being ignored (v_n is the restriction of the colouring v_{n+1}). There are $\binom{n}{k-1}$ k-element subsets of an $n+1$-element set that contain the element $n+1$ and so there are $r^{\binom{n}{k-1}}$ different ways to colour these subsets with r colours. Therefore, for a vertex $v_n \in V_n$, it is the restriction of at most $r^{\binom{n}{k-1}}$ colourings and so the vertex v_n has finite degree. We can now apply König's infinity lemma. So by the lemma there is a one way infinite path $v_o, v_1, v_2, ...$ that tells us how to colour the k-element subsets of the natural numbers such that there is no monochromatic m-set but then this contradicts the infinite version of Ramsey's theorem. □

2 Ramsey numbers

In chapter 1 we saw that the Ramsey number $R_k(a_1, a_2, ...a_r)$ is the least possible N for which Ramsey's theorem holds. For the $k = 2$ case we write $R(a_1, ...a_r)$. Only a handful of the exact values for Ramsey numbers are known and these are mainly for the $k = r = 2$ case. The best we can do for the other numbers is to try and find upper and lower bounds for them. In this chapter we will look at the exact Ramsey numbers and bounds for the $k = r = 2$ case and then move onto looking briefly at cases where $r > 2$.

2.1 Values and bounds for $R(a_1, a_2)$

We will now think of $R(a_1, a_2)$ to be the least possible N such that when a complete graph with at least N vertices has its edges coloured with two colours it either has a monochromatic complete subgraph on a_1 vertices or a monochromatic complete subgraph on a_2 vertices.

2.1.1 Upper bounds for $R(a_1, a_2)$

We will start by looking at some upper bounds for $R(a_1, a_2)$. First we will look at a few trivial bounds [1, p 306].

Lemma 2.1.
1) $R(a_1, a_2) = R(a_2, a_1)$
2) $R(a_1, 1) = 1 = R(1, a_2)$
3) $R(a_1, 2) = a_1$ *and* $R(2, a_2) = a_2$

Proof. 1) If $R(a_1, a_2) = n$ then for all complete graphs on at least n vertices, for all colourings of the edges with red and blue there is either a red K_{a_1} or a blue K_{a_2}. If we take the complement of any of these colourings, so any edge that is red is now blue and vice versa, then we will have either a blue K_{a_1} or a red K_{a_2} and hence $R(a_1, a_2) = R(a_2, a_1)$.
2) This is trivial as a subgraph on one vertex has no edges, hence it is monochromatic.
3) We shall first prove that $R(a_1, 2) \leq a_1$. If we have a complete graph on a_1 vertices and colour its edges red and blue. If all the edges are red then we have a red K_{a_1} if not we have at least one blue edge and so we have a blue K_2. Hence $R(a_1, 2) \leq a_1$.

Now we will prove that $R(a_1, 2) > a_1 - 1$. If we colour the edges of K_{a_1-1} with all red edges then we have no red K_{a_1} or blue K_2. Hence $R(a_1, 2) = a_1$ and by 1) $R(2, a_2) = a_2$.

□

In chapter 1, the first proof of the finite Ramsey's theorem gives us the following inequality for Ramsey numbers.

$$R_k(a_1, ..., a_r) \leq 1 + R_{k-1}(A_1, ..., A_r) \tag{1}$$

where

$$A_i = R_k(a_1, ...a_{i-1}, a_i - 1, a_{i+1}, ...a_r)$$

and from this we can derive the following result.

Theorem 2.2. *For all $a_1, a_2 \geq 2$ we have,*

$$R(a_1, a_2) \leq R(a_1, a_2 - 1) + R(a_1 - 1, a_2)$$

Proof. From the proof of Theorem 1.5. we know that $R_k(a_1, ..., a_r) \leq 1 + R_{k-1}(A_1, ..., A_r)$ and so for the $k = r = 2$ case we have,

$$R(a_1, a_2) \leq 1 + R_1(A_1, A_2)$$

where

$$A_1 = R(a_1 - 1, a_2)$$

$$A_2 = R(a_1, a_2 - 1)$$

By the pigeonhole principle we have $R_1(A_1, A_2) = A_1 + A_2 - 1$ and so we have $R(a_1, a_2) \leq R(a_1, a_2 - 1) + R(a_1 - 1, a_2)$.

□

We can look at a special case for this theorem, which we expand on Graham's explanations [10, p 89].

Theorem 2.3. *If both $R(a_1, a_2 - 1)$ and $R(a_1 - 1, a_2)$ are even then we have a strict inequality,*

$$R(a_1, a_2) < R(a_1, a_2 - 1) + R(a_1 - 1, a_2)$$

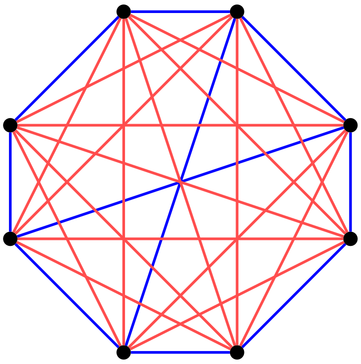
Proof. We shall show that a complete graph, $G = (V, E)$, on $n = R(a_1, a_2 - 1) + R(a_1 - 1, a_2) - 1$ vertices, that has its edges coloured with red and blue either has a red complete subgraph on a_1 vertices or a blue complete subgraph on a_2 vertices.

We shall assume that there are no such subgraphs. Let $x \in V$, then x is connected to $n - 1$ other vertices by $R(a_1 - 1, a_2) - 1$ red edges (to avoid having a blue subgraph on a_2 vertices) and by $R(a_1, a_2 - 1) - 1$ blue edges (again to avoid having a red subgraph on a_1 vertices). So our total number of red edges is $\frac{(R(a_1-1,a_2)-1)n}{2}$ and similarly the total number of blue edges is

$\frac{(R(a_1,a_2-1)-1)n}{2}$. As the total number of red and blue edges have to be integers we will need both $R(a_1, a_2-1)$ and $R(a_1-1, a_2)$ to be odd. So in order for us to have a red complete subgraph on a_1 vertices or a blue complete subgraph on a_2 vertices we must have both $R(a_1, a_2-1)$ and $R(a_1-1, a_2)$ even.

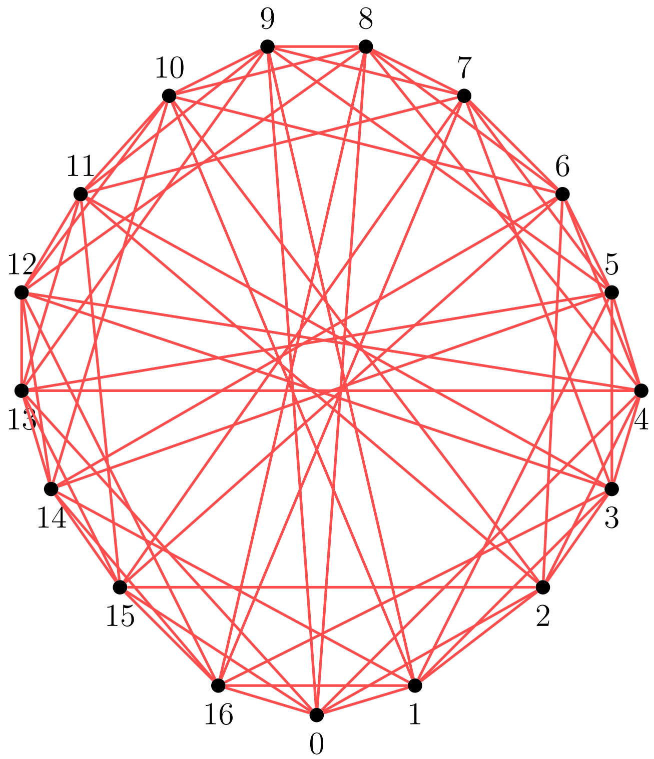
□

For example we have $R(3,4) \leq R(3,3) + R(2,4)$, and we already saw in chapter 1 that $R(3,3) = 6$ and from lemma 2.1 we know $R(2,4) = 4$ so both $R(3,3)$ and $R(2,4)$ are even and therefore $R(3,4) \leq 9$. To show that $R(3,4) = 9$ we must find an example of a red and blue colouring of K_8 that has no red K_3 or blue K_4. If you colour the edges of K_8 as shown in graph H,

$H =$ 

then we can clearly not get a red K_3 or a blue K_4. So this proves that $R(3,4) = 9$.

Now as we know $R(3,4) = 9$ by theorem 2.2 we know $R(4,4) \leq R(3,4) + R(3,4) = 18$ and in fact we know $R(4,4) = 18$. We can see this by defining a two colouring of the complete graph on 17 vertices and showing it has no monochromatic subgraph on four vertices. In the graph K_{17}, if we colour the edge from vertex i to: $i+1 \pmod{17}$, $i+2 \pmod{17}$, $i+4 \pmod{17}$, $i+8 \pmod{17}$ red and the rest blue as shown in graph J (only the red edges are shown in the graph to avoid a very complicated graph) then we can show we have no red K_4. By symmetry of the graph if we can show that vertex 0 is in no red K_4 then we have no red K_4 in the colouring. Vertex 0 is adjacent to 1, 2, 4, 8, 9, 13, 15 and 16. Vertices 0 and 1 cannot be in a red K_4 as 1 is adjacent to 2, 9 and 16 that are also adjacent to 0. But then none of 2, 9 and 16 are adjacent so we have no red K_4 including 0 and 1. We can do similar arguments for 0 and 2, 0 and 4 etc. to show there is no red K_4. We have no blue K_4 by a similar argument. Therefore $R(4,4) = 18$.

$J =$ 

These Ramsey numbers grow extremely quickly as a_1 and a_2 increase and it is therefore very difficult to find their exact values. This point was demonstrated by Paul Erdős' famous quote:

"If aliens offered earthlings the choice of (i) determine $R(5,5)$ within one year or (ii) face intergalactic war, then we should make strenuous efforts to find $R(5,5)$. If the condition (i) were altered to that of finding $R(6,6)$ we should immediately prepare for war!" [1, p 311]

Summing up, we have already seen that $R(3,3) = 6$, $R(3,4) = 9$ and $R(4,4) = 18$. Six other non trivial exact Ramsey numbers have been found by people in the past, table 1 shows these values along with some upper bounds (the upper bounds in the table are in brackets).

Another famous upper bound due to Erdős and Szekeres that we can get from the recursion relation (1) is shown in the following theorem.

Theorem 2.4. *For positive a_1 and a_2,*

$$R(a_1, a_2) \leq \binom{a_1 + a_2 - 2}{a_1 - 1}$$

a_1 \ a_2	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2		2	3	4	5	6	7	8	9	10	11
3			6	9	14	18	23	28	36	(43)	(51)
4				18	25	(41)	(61)	(84)	(115)	(149)	(191)
5					(49)	(87)	(143)	(216)	(316)	(442)	
6						(165)	(298)	(495)	(780)	(1171)	
7							(540)	(1031)	(1713)	(2826)	(4553)
8								(1870)	(3583)	(6090)	(10630)
9									(6588)	(12677)	(22325)
10										(23556)	

Table 1: Exact values and some upper bounds (shown in brackets) for the Ramsey numbers $R(a_1, a_2)$
[23, p 4858]

Proof. To prove this we can either use Theorem 2.2 or equation (1). We will prove it using (1) as done by Cameron [3, p 152]. We will use induction on a_1 and then on a_2. For the base case we will look at the cases where $a_1 = 2$ and $a_2 = 2$. Using lemma 2.1 we get,

$$R(2, a_2) = a_2 = \binom{2 + a_2 - 2}{2 - 1}$$

$$R(a_1, 2) = a_1 = \binom{a_1 + 2 - 2}{2 - 1}$$

and so the theorem is true for $a_1 = a_2 = 2$. Then by the induction hypothesis and equation (1) we have,

$$A_1 = R(a_1 - 1, a_2) \leq \binom{a_1 - 1 + a_2 - 2}{a_1 - 1 - 1} = \binom{a_1 + a_2 - 3}{a_1 - 2}$$

$$A_2 = R(a_1, a_2 - 1) \leq \binom{a_1 + a_2 - 1 - 2}{a_1 - 1} = \binom{a_1 + a_2 - 3}{a_1 - 1}$$

So by (1) we know $R(a_1, a_2) \leq 1 + R_1(A_1, A_2)$ and by the pigeonhole principle $R_1(A_1, A_2) = A_1 + A_2 - 1$ as $A_1 + A_2 - 1 \geq A_1 - 1 + A_2 - 1$. So,

$$R(a_1, a_2) \leq 1 + R_1(A_1, A_2) = A_1 + A_2$$

$$\leq \binom{a_1 + a_2 - 3}{a_1 - 2} + \binom{a_1 + a_2 - 3}{a_1 - 1} = \binom{a_1 + a_2 - 2}{a_1 - 1}$$

The last step here used that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. □

Not much progress was made in improving this bound until the 1980s, where in 1987 Graham and Rödl improved upon the result and then in 1988 Thomason improved upon it further by his result in 'An upper bound for some Ramsey numbers' [25]. To date Conlon's upper bound

$$R(k+1,k+1) \leq k^{-C\frac{logk}{loglogk}}\binom{2k}{k}$$

where C is some constant, is the best upper bound for diagonal Ramsey numbers. This was proved in the article 'A new upper bound for diagonal Ramsey numbers' published in 2009 [4].

In 2012, Samana [23] made the following improvement on theorem 2.4 which is not as strong as the above but the proof of this is rather nice.

Theorem 2.5. *For $a_1 \geq 5$ and $a_2 \geq 5$, we have,*

$$R(a_1,a_2) \leq \binom{a_1+a_2-2}{a_1-1} - \binom{a_1+a_2-4}{a_1-2}$$

Proof. Firstly Samana proved that for $a_2 \geq 7$ we have

$$R(4,a_2) \leq \frac{{a_2}^3+5a_2}{6} = \binom{a_2+2}{3} - \binom{a_2}{2} = \binom{4+a_2-2}{4-1} - \binom{4+a_2-4}{4-2}$$

and for $a_2 \geq 6$ we have

$$R(5,a_2) \leq \frac{{a_2}^4+2{a_2}^3+11{a_2}^2+10a_2}{2}4 = \binom{a_2+3}{4} - \binom{a_2+1}{3}$$

$$= \binom{5+a_2-2}{5-1} - \binom{5+a_2-4}{5-2}$$

These were proved using theorem 2.2 and $R(3,a_2) \leq \binom{a_2}{2} - c$ for some non-negative integer c along with induction on a_2. We will however not go in to detail on this proof as they are rather straightforward.

We will now prove the main result using these two results. We shall use proof by induction on $k = a_1 + a_2$. For the base case let $k = 10$, then if $a_1 = a_2 = 5$ we get $\binom{5+5-2}{5-1} - \binom{5+5-4}{5-2} = 50$. From table 1 we know that $R(5,5) \leq 49 < 50$. So we know it holds for $a_1 = a_2 = 5$. Similarly for $a_1 = 4$ and $a_2 = 6$ we have $\binom{4+6-2}{4-1} - \binom{4+6-4}{4-2} = 41$ and from table 1 we know $R(4,6) \leq 41$.

We will now assume that $k > 10$ and that the theorem holds for $k-1$. By the induction hypothesis, the two above inequalities we quoted and theorem 2.2 we have,

$$R(a_1,a_2) \leq R(a_1-1,a_2) + R(a_1,a_2-1)$$

$$\leq \binom{a_1 + a_2 - 3}{a_1 - 2} - \binom{a_1 + a_2 - 5}{a_1 - 3}$$
$$+ \binom{a_1 + a_2 - 3}{a_1 - 1} - \binom{a_1 + a_2 - 5}{a_1 - 2}$$
$$= \binom{a_1 + a_2 - 2}{a_1 - 1} - \binom{a_1 + a_2 - 4}{a_1 - 2}$$

□

For example table 2 compares the upper bound from theorem 2.4 and the upper bound from theorem 2.5 for $R(6, t)$, for different values of t.

	R(6,6)	R(6,7)	R(6,8)	R(6,9)	R(6,10)	R(6,11)
Theorem 2.4	252	462	792	1287	2002	3003
Theorem 2.5	182	336	582	957	1507	2288

Table 2: Upper bounds for $R(6, t)$ found from theorems 2.4. and 2.5.

You can see that theorem 2.5. improves upon theorem 2.4. quite efficiently.

2.1.2 Lower bounds for $R(a_1, a_2)$

One way to find lower bounds for the Ramsey number $R(a_1, a_2)$ is by searching graphs for monochromatic K_{a_1} and K_{a_2} subgraphs. This is very time consuming for even the smaller Ramsey numbers so we will look at some of the known lower bounds.

We will start by looking at some lower bounds for the diagonal Ramsey number $R(a_1, a_1)$, where the proofs use a probabilistic method (we shall write $R(a_1, a_1)$ as $R(a_1)$ from now on). We will use the theorems and proofs given by Graham [10], Cameron [16] and Allenby [1] expanding on their explanations. We will start with the following theorem which we can then use to find some lower bounds for $R(a_1)$. For this part we expand on Graham's [10, p 92] explanations.

Lemma 2.6. *If,*

$$\binom{n}{a_1} 2^{1-\binom{a_1}{2}} < 1$$

then $R(a_1) > n$

Proof. We want to prove that if $\binom{n}{a_1}2^{1-\binom{a_1}{2}} < 1$ then there exists a 2-colouring of K_n such that there is no monochromatic K_{a_1} subgraph. Now here is where we use the probabilistic method. We consider a random colouring of K_n with the colours red and blue such that an edge is coloured red with probability $\frac{1}{2}$ and an edge is coloured blue with probability $\frac{1}{2}$ (the edges are coloured independently). As there are $\binom{n}{2}$ edges in K_n, there are $2^{\binom{n}{2}}$ different possible colourings and each of these colourings occur with probability $\frac{1}{2^{\binom{n}{2}}}$.

If we colour K_{a_1} with red and blue according to the probabilities as above, the probability all edges are coloured red is $2^{-\binom{a_1}{2}}$ and similarly the probability all edges are coloured blue is $2^{-\binom{a_1}{2}}$. So the overall probability that K_{a_1} is monochromatic is $2^{1-\binom{a_1}{2}}$. There are $\binom{n}{a_1}$ different K_{a_1} subgraphs of K_n, so the probability that there is a monochromatic K_{a_1} subgraph of K_n is

$$\binom{n}{a_1}2^{1-\binom{a_1}{2}}$$

If there was a monochromatic K_{a_1} colouring then we would have $\binom{n}{a_1}2^{1-\binom{a_1}{2}} = 1$ and so if $\binom{n}{a_1}2^{1-\binom{a_1}{2}} < 1$ then $R(a_1) > n$. □

Our aim is now to find an n such that this inequality holds, in the next theorem we find such an n as given by Cameron [16, p 153] but we shall elaborate on the details in the proof Cameron gives.

Theorem 2.7. *For all* $a_1 > 2$,

$$R(a_1) \geq 2^{\frac{a_1-2}{2}}$$

Proof. We have,

$$1 - \binom{a_1}{2} = 1 - \frac{a_1(a_1-1)}{2} = 1 - \frac{{a_1}^2}{2} + \frac{a_1}{2} < -\frac{{a_1}^2}{2} + a_1 = -\frac{a_1(a_1-2)}{2}$$

as $a_1 > 2$.

We also know,

$$\binom{n}{a_1} = \frac{n!}{(n-a_1)!a_1!} = \frac{n(n-1)...(n-a_1+1)}{a_1!} < \frac{n^{a_1}}{a_1!} < n^{a_1}$$

therefore using lemma 2.6,

$$\binom{n}{a_1}2^{1-\binom{a_1}{2}} < n^{a_1}2^{-\frac{a_1(a_1-2)}{2}}$$

if we let n equal the integer part of $2^{\frac{a_1-2}{2}}$ then we have,

$$\binom{n}{a_1}2^{1-\binom{a_1}{2}} < n^{a_1}2^{-\frac{a_1(a_1-2)}{2}} \leq 1$$

So there exists a two colouring of K_n such that there is no monochromatic K_{a_1} subgraph. □

We can improve upon this result for n by the following theorem, which we expand on Allenby's [1, pp 312-313] proof.

Theorem 2.8. *For all $a_1 \geq 2$,*

$$R(a_1) > \frac{a_1(\sqrt{2})^{a_1-1}}{e}$$

Proof. First we will show that

$$\binom{n}{a_1} < \frac{1}{2}(\frac{n}{a_1})^{a_1}e^{a_1}$$

We saw in the previous proof that

$$\binom{n}{a_1} = \frac{n!}{(n-a_1)!a_1!} = \frac{n(n-1)...(n-a_1+1)}{a_1!} < \frac{n^{a_1}}{a_1!}$$

so we have,

$$\binom{n}{a_1} < \frac{n^{a_1}}{a_1!} = (\frac{n}{a_1})^{a_1}\frac{{a_1}^{a_1}}{a_1!}$$

now,

$$e^{a_1} = 1 + a_1 + \frac{a_1^2}{2} + \frac{a_1^3}{3} + ... > \frac{a_1^{a_1-1}}{(a_1-1)!} + \frac{a_1^{a_1}}{a_1!}$$

$$= \frac{a_1.a_1^{a_1-1}}{a_1.(a_1-1)!} + \frac{a_1^{a_1}}{a_1!} = \frac{2a_1^{a_1}}{a_1!}$$

thus,

$$\frac{a_1^{a_1}}{a_1!} < \frac{1}{2}e^{a_1}$$

and so we have,

$$\binom{n}{a_1} < \frac{1}{2}(\frac{n}{a_1})^{a_1}e^{a_1}$$

We can now use this inequality as follows,

$$\binom{n}{a_1}2^{1-\binom{a_1}{2}} < 2^{1-\binom{a_1}{2}}\frac{1}{2}(\frac{n}{a_1})^{a_1}e^{a_1}$$

and if we substitute $n = \frac{a_1(\sqrt{2})^{a_1-1}}{e}$ into the previous inequality we get,

$$2^{1-\binom{a_1}{2}}\frac{1}{2}(\frac{a_1(\sqrt{2})^{a_1-1}}{e})^{a_1}(\frac{e}{a_1})^{a_1} = 2^{1-\binom{a_1}{2}}2^{-1}2^{\frac{a_1(a_1-1)}{2}} = 1$$

so if $n = \frac{a_1(\sqrt{2})^{a_1-1}}{e}$ then $\binom{n}{a_1}2^{1-\binom{a_1}{2}} < 1$ and hence $R(a_1) > \frac{a_1(\sqrt{2})^{a_1-1}}{e}$. □

So far we have only seen lower bounds for the diagonal Ramsey numbers. We will start by giving a rather weak but satisfying lower bound for the Ramsey number $R(a_1, a_2)$. We shall follow Allenby's [1, p 307] proof, expanding on details.

Theorem 2.9. *For all $a_1, a_2 \geq 2$ we have,*

$$R(a_1, a_2) > (a_1 - 1)(a_2 - 1)$$

Proof. When $a_1 = 2$ we have $R(2, a_2) = a_2 > a_2 - 1 = (2 - 1)(a_2 - 1)$ and similarly for $a_2 = 2$. So we shall assume $a_1, a_2 > 2$. Our aim is to show that there is a red and blue colouring of the complete graph on $(a_1 - 1)(a_2 - 1)$ vertices without a red K_{a_1} or a blue K_{a_2}. We can describe such a colouring as follows, if we arrange the $(a_1 - 1)(a_2 - 1)$ vertices in a grid with $a_1 - 1$ many rows and $a_2 - 1$ many columns. We then colour the edges according to the following rules:
1) Colour an edge blue if the two endpoints of the edge are in the same row.
2) Colour an edge red if the two endpoints of the edge are in the same column.

Then if we pick a_1 vertices at least two must lie in the same row as we only have $a_1 - 1$ columns and therefore there is a blue edge so we do not have a red K_{a_1}. Similarly, if we pick a_2 vertices at least two of these must lie in the same column and hence we have no blue K_{a_2} and so $R(a_1, a_2) > (a_1 - 1)(a_2 - 1)$. □

2.2 Multicolour Ramsey Numbers

There are also bounds out there for Ramsey numbers with more colours and with larger k. We shall however not delve too deep into this and we will just give one basic upper bound for the multicolour case with $k = 2$, that comes from the proof of Ramsey's theorem.

Theorem 2.10. *For $r \geq 2$ we have,*

$$R(a_1, a_2, ..., a_r) \leq 2 - r + R(a_1 - 1, a_2, ..., a_r) + R(a_1, a_2 - 1, a_3, ..., a_r)$$

$$+... + R(a_1, a_2, ..., a_{r-1}, a_r - 1)$$

Proof. Using inequality (1), which came from the proof of Ramsey's theorem, we have

$$R(a_1, a_2, ..., a_r) \leq 1 + R_1(A_1, A_2, ..., A_r)$$

where $A_i = R_1(a_1, ..., a_{i-1}, a_i - 1, a_{i+1}, ...a_r)$.

By the pigeonhole principle,

$$R_1(A_1, A_2, ..., A_r) = \sum_{i=1}^{r}(A_i - 1) + 1$$

hence we get our result,

$$R(a_1, a_2, ..., a_r) \leq 2 - r + R(a_1 - 1, a_2, ..., a_r) + R(a_1, a_2 - 1, a_3, ..., a_r)$$

$$+... + R(a_1, a_2, ..., a_{r-1}, a_r - 1)$$

□

The only known exact Ramsey number with more than two colours is $R(3, 3, 3) = 17$. The most intriguing and studied open case is $R(3, 3, 3, 3)$ (4 colours) we know that $51 \leq R(3, 3, 3, 3) \leq 62$. The lower bound of 51 was found by Fan Rong K. Chung in 1973 with the following lower bound:

$$R(3, ..., 3; r) \geq 3R(3, ..., 3; r - 1) + R(3, ..., 3; r - 3) - 3$$

where $R(3, ..., 3; r)$ means there are r 3's in the brackets (r colours). [21, pp 37-38]

So when $r = 4$ we have

$$R(3, 3, 3, 3) \geq 3R(3, 3, 3) + R(3) - 3 = 3.17 + 3 - 3 = 51$$

3 Van der Waerden's theorem

3.1 Introduction

In this section we will explore Van der Waerden's theorem which is one of the most fundamental results in Ramsey theory, it is named after the Dutch Mathematician Bartel Leendert Van der Waerden. Ramsey's theorem is concerned with the colourings of the elements of the set $[X]^k$ and looking at the monochromatic subsets, where in Van Der Waerden's theorem we will be looking at the colourings of the integers and monochromatic arithmetic progressions which arise.

Definition 3.1. An *arithmetic progression* is a sequence of numbers such that the difference between any two consecutive terms is constant. A k-term arithmetic progression is therefore of the form $a, a+d, a+2d, ...a+(k-1)d$.

Example 3.2. Here are some examples of 4-term arithmetic progressions:

$$2, 4, 6, 8$$
$$5, 8, 11, 14$$
$$\text{etc.}$$

Van der Waerden first published the following result and proof in 1927. He showed that if you partition the set of positive integers into two classes then one class must have an arbitrarily long arithmetic progression. This was then generalised to give what we shall call Van der Waerden's theorem. [10, p 29]

Theorem 3.3. *(Van der Waerden's theorem)*
For all positive integers k and r, there exists a least positive integer $w(k, r)$, such that for all $n \geq w(k, r)$, if the set of integers $\{1, 2, ..., w(k, r)\}$ is partitioned into r classes then at least one of the classes will contain a k-term arithmetic progression.

Following from the notion of Ramsey's theorem we can think of the partitioning of the set of integers into r classes as colouring each element in the set of integers with one of r colours, then we look for a monochromatic k-term arithmetic progression (we shall use the idea of colours from now on). The number $w(k, r)$ is known as the Van der Waerden number and as with Ramsey numbers little of these numbers are known. The only known numbers to date are $w(3, 2) = 9$, $w(4, 2) = 35$, $w(5, 2) = 178$, $w(3, 3) = 27$, $w(3, 4) = 76$ [13] and most recent $w(2, 6) = 1132$ which was found by formulating the problem as a SAT question for a boolean formula in CNF and then using a

SAT solver [15]. We also know the two general results that $w(k, 1) = k$ and $w(2, r) = r + 1$. We shall prove these and look at $w(2, 3) = 9$ before looking at the proof of Van der Waerden's theorem.

Theorem 3.4. *For $r, k \geq 1$ we have,*
1) $w(k, 1) = k$
2) $w(2, r) = r + 1$

Proof. 1) This is trivial.
2) This was left as an exercise (exercise 2.2) by Landman and Robertson [17, p 49] so we will give the proof here. In order to have a two-term arithmetic progression we simply just need at least two integers to be coloured with the same colour. By the pigeonhole principle $w(2, r) = r + 1$ for any $r \geq 1$. □

In order to prove that $w(3, 2) = 9$ we must first show that $w(3, 2) \geq 9$. To show this we show there is a colouring of $\{1, 2, ..., 8\}$ with two colours, say red and blue, such that there is no three-term monochromatic arithmetic progression. Landman and Robertson [17, p22] give an example of a colouring but we will give a different example. If we colour the eight integers as follows:

1 2 3 4 5 6 7 8

Then there is clearly no three-term arithmetic progression. Hence we know that $w(3, 2) \geq 9$.

We now must show $w(3, 2) \leq 9$, to do this we must show that for every two colouring of $\{1, 2, ..., 9\}$ there exists a monochromatic three-term arithmetic progression. Let's assume for a contradiction that there exists a two colouring of $\{1, 2, ..., 9\}$ which has no monochromatic three term arithmetic progression. We shall use the colours red and blue again. Landman and Robertson [17, p22] explain why 3 and 5 can not both be coloured red or both be coloured blue. This is because lets say 3 and 5 are both red then $(1, 3, 5)$ cannot be monochromatic and hence 1 has to be coloured blue. Similarly, $(3, 4, 5)$ and $(3, 5, 7)$ cannot be monochromatic and so 4 and 7 must also be coloured blue. But then we have a blue $(1, 4, 7)$ and similarly if 3 and 5 were both coloured blue we would have a red $(1, 4, 7)$. It also true that 5 and 7 cannot be coloured the same colour and 4 and 6 cannot be coloured with the same colour. Landman and Robertson [17, p22] do not give details on these cases, so we shall fill in the gaps.
1) 5 and 7: If we colour both these red then we cannot have $(5, 6, 7)$, $(5, 7, 9)$ or $(3, 5, 7)$ being red. So we must colour 6, 9 and 3 blue but then that will give us a blue $(3, 6, 9)$. Hence 5 and 7 cannot be both coloured red and similarly they cannot be both coloured blue.

2) 4 and 6: If we colour both these red then we cannot have $(4,5,6)$, $(4,6,8)$ or $(2,4,6)$ being red. So we must colour 5, 8 and 2 blue but then that will give us a blue $(2,5,8)$. Hence 4 and 6 cannot be both coloured red and similarly they cannot be both coloured blue.

Using similar ideas to Landman and Robertson [17, p22], if we set 5 to be coloured red then we know we must have one of the following possibilities of the colouring of $\{3,4,5,6,7\}$:

1): 3 4 5 6 7
2): 3 4 5 6 7

If 1) is true then we must colour 8 red to avoid $(6,7,8)$ being blue. Then 2 must be blue to avoid having a red $(2,5,8)$, 1 must be red so we don't have a blue $(1,2,3)$ and 9 must be red to avoid having a blue $(3,6,9)$. So we colour the integers as follows:

1 2 3 4 5 6 7 8 9

But now $(1,5,9)$ is red, a contradiction. As 2) is the reverse of 1) we can use a symmetric argument to show that if 2) holds then we get a contradiction. Hence, $w(3,2) \leq 9$ and so we know $w(3,2) = 9$.

3.2 Proof of Van der Waerden's theorem

In 1974 Graham and Rothschild [9] published a shorter proof than the original of the Van der Waerden's theorem. We shall give this proof whilst elaborating on the details but before we start we shall give a few essential definitions.

Definition 3.5. An *equivalence relation* is a relation between elements of a set, such that the relation is reflexive, transitive and symmetric. An *equivalence class* is a class of elements of a set such that the elements in each class bear the relation to each other but to no other elements in different classes.

Definition 3.6. The *l-equivalence classes* of $[0,l]^m$ are the set of $(x_1, ..., x_m) \in [0,l]^m$ where l appears in the i^{th} rightmost positions and nowhere else for $0 \leq i \leq m$. We therefore have $m+1$ l-equivalence classes.

Example 3.7. For $m = 2$, $l = 6$ we have the following three 6-equivalence classes:
$i = 0$: All the $(x_1, x_2) \in [0,6]^2$ that does not contain 6.
$i = 1$: $\{(0,6),(1,6),(2,6),(3,6),(4,6),(5,6)\}$
$i = 2$: $\{(6,6)\}$

Definition 3.8. For any $l, m \geq 1$ we define the statement $s(l, m)$ to be:
For any r, there exists an integer $N(l, m, r)$ such that for any function

$$C : [1, N(l, m, r)] \to [1, r]$$

there exists positive integers $a, d_1, d_2, ..., d_m$ such that, $C(a + \sum_{i=1}^{m} x_i d_i)$ is constant on each l-equivalence class on $[0, l]^m$.

Example 3.9. $s(l, 1)$ is equivalent to Van der Waerden's theorem (with $l = k$). This is because if $m = 1$, then there are two l-equivalence classes of $[0, l]^1$ which are: $[0, l - 1]$ and l by itself. So $s(l, 1)$ says that there exists $N(l, 1, r)$ ($w(l, r)$ in Van der Waerden terms) for any r and l, such that for all functions $C : [1, N(l, 1, r)] \to [1, r]$, where we shall let $[1, r]$ be a set of r colours, there exists positive integer a and d such that,

$$C(a + 0) = C(a + d) = C(a + 2d) = ... = C(a + (l - 1)d)$$

and this gives us a monochromatic l-term arithmetic progression. Hence $s(l, 1)$ is equivalent to Van der Waerden's theorem.

We shall now use these definitions to prove that $s(l, m)$ holds for all l and m and then we will have proved Van der Waerden's theorem.

Theorem 3.10. *$s(l, m)$ holds for all $l, m \geq 1$.*

Proof. 1) $s(l, m) \Rightarrow s(l, m + 1)$
We shall use induction on m for this proof. Let $M = N(l, m, r)$ and $M' = N(l, 1, r^M)$, for a fixed r. Let $C : [1, MM'] \to [1, r]$ be given. We now define C' to be the function $C' : [1, M'] \to [1, r^M]$ such that: $C'(k) = C'(k')$ if and only if $C(kM - j) = C(k'M - j)$ for all $0 \leq j < M$.

By the induction hypothesis we know there exists some a' and d' such that $C'(a' + xd')$ is constant for $x \in [0, l - 1]$ (as $m = 1$). This then gives us, by the definition of C', that

$$C((a' + x_i d')M - j) = C((a' + x_j d')M - j)$$

for all $0 \leq j < M$ and $x_i, x_j \in [0, l - 1]$.

Now, define the interval $I = [a'M - (M - 1), a'M]$. Note that, as $a'M + x_i d'M - j$ is defined on C for $0 \leq j < M$ and $x_i \in [0, l-1]$, then $a'M - j$ will also be defined on C and this gives the interval I for the different possible j's (also the size of I is the same as the size of the interval $[1, M]$). So, $s(l, m)$ can clearly apply to I. By choice of M there exists $a, d_2, ..., d_{m+1}$ where all sums $a + \sum_{i=2}^{m+1} x_i d_i$ for $x_i \in [0, l]$ are in I and $C(a + \sum_{i=2}^{m+1} x_i d_i)$ are constant on l-equivalence classes (because $s(l, m)$ holds on I).

If we set $d'_i = d_i$ for $2 \leq i \leq m+1$ and $d'_1 = d'M$. Then, as $a'M - j$ is equivalent to I when $0 \leq j < M$, adding $x_1 d'M$ for $x_1 \in [0, l-1]$ (if $x_1 = l$ then we would be looking at the equivalence class with only one element in, the one with all l's), we know,

$$C(a + \sum_{i=1}^{m+1} x_i d'_i) = C(a + \sum_{i=2}^{m+1} x_i d_i)$$

as $C(a'M - j) = C(a'M + x_i d'M - j)$ for $x_i \in [1, l-1]$. Therefore, $s(l, m+1)$ holds.

2) $s(l, m) \Rightarrow s(l+1, 1)$
Let $m = r$ and let $C : [1, N(1, r, r)] \to [1, r]$ be given for a fixed r. Then as we are assuming $s(l, m)$ holds, there exists $a, d_1, d_2, ..., d_r$ such that:
1) $a + \sum_{i=1}^{r} x_i d_i$ is in $[1, N(1, r, r)]$ for $x_i \in [0, l]$.
2) $C(a + \sum_{i=1}^{r} x_i d_i)$ is constant on l-equivalence classes.

We have $r+1$ equivalence classes and so by the Pigeonhole principle we must have two l-equivalence classes where $C(a + \sum_{i=1}^{r} x_i d_i)$ is constant for both classes. If we let these two l-equivalence classes be the ones where l appears only in the u and v ($1 \leq u < v \leq r+1$) rightmost positions, then as $\{0, 0, ..., l, l, ...\}$ is in these equivalence classes (with corresponding number of l's for each case), we have

$$C(a + \sum_{i=u}^{r} l d_i) = C(a + \sum_{i=v}^{r} l d_i)$$

and therefore

$$C((a + \sum_{i=v}^{r} l d_i) + x(\sum_{i=u}^{v-1} d_i))$$

is constant for $x \in [0, l]$. This proves that $s(l+1, 1)$ holds if $s(l, m)$ holds. This is because in the $m = 1$ case we have two $l+1$-equivalence classes, which are $\{\{0\}, \{1\}, ..., \{l\}\}$ and $\{l+1\}$.

Finally, as $s(1, 1)$ clearly holds (we will have two 1-equivalence classes with only a one or a zero in), we know $s(l, m)$ holds for all $l, m \geq 1$ □

Proof. (Van der Waerden's theorem)
By theorem 3.10., $s(l, m)$ holds for all $l, m \geq 1$ and so $s(l, 1)$ holds for all l. As discussed in example 3.9. $s(l, 1)$ is equivalent to Van der Waerden's theorem.

□

3.3 Szemerédi's Theorem

When Van der Waerden gave the original proof of his theorem, the bounds it gave for the numbers $w(k, r)$ were very weak. In 1936, Paul Erdös and Paul Turán gave two conjectures which would imply Van der Waerden's theorem. The stronger of the two conjectures was disproved in 1942. The other conjecture was what we now know as Szemerédi's theorem, which is a very important result in Mathematics. Szemerédi's theorem was essential in the proofs of many other theorems and results. For example it was one of the three major theorems behind Tao and Greens [12] proof that there are arbitrarily long arithmetic progressions of primes. We shall now state Szemerédi's theorem. [8, 26, 11]

Theorem 3.11. *(Szemerédi's theorem)*
For any $\delta > 0$ and any positive integer k, there exists a positive integer n such that: For any $N \geq n$, any subset of $\{1, 2, ..., N\}$ with at least δN elements contains a k-term arithmetic progression.

Szemerédi's Theorem says that in Van der Waerden's theorem we can always find a monochromatic arithmetic progression in any colour that is used often enough (i.e. $\delta = \frac{1}{r}$).
It wasn't until 1953 until some progress was made in proving the theorem, where K. F. Roth proved it for the $k = 3$ case, using methods from Fourier analysis. Unfortunately, Roth's proof did not extend to general k as he had hoped. In 1969, E. Szemerédi proved the theorem for the $k = 4$ case by a intricate combinatorial argument. Finally, in 1974 E. Szemerédi proved the theorem for general k by 'a masterpiece of combinatorial reasoning' [10, p46]. Szemerédi's proof used graph theoretic methods and introduced some new important tools like the Szemerédi Regularity Lemma for graphs. Not long after, in 1977 Furstenberg gave a different proof for the theorem which used methods from ergodic theory [7]. Since then numerous new proofs (roughly 16) have been published for the theorem for example in 2001 Timothy Gowers used Fourier analysis and combinatorics in 'A new proof of Szemerédi's Theorem' [8] and in 2011 Henry Towsner gave a model theoretic proof in the paper 'A model theoretic proof of Szemerédi's theorem' [26].

[8, 26, 11]

4 Hales-Jewett Theorem

4.1 Introduction

The Hales-Jewett theorem, named after Mathematicians Alfred W. Hales and Robert Jewett, is a generalisation of Van der Waerden's theorem. In the words of Graham and Rothschild the theorem is the heart of Ramsey theory and without the result 'Ramsey theory would more properly be called Ramseyian theorems' [10, p35]. It is concerned with the colourings of n-cubes over t elements and the monochromatic lines that arise. We shall first introduce a few new definitions before stating the theorem.

Definition 4.1. The *hypercube* C_t^n is defined as,

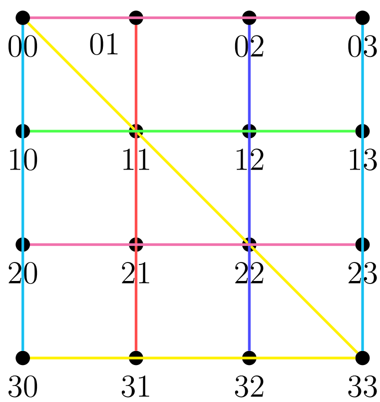
$$C_t^n = \{(x_1, x_2, ..., x_n) : x_i \in \{0, 1, ..., t-1\}\}$$

Definition 4.2. A *line* in C_t^n is a set of t points in C_t^n, call these $X_0, X_1, ..., X_{t-1}$ where $X_i = (x_{i1}, x_{i2}, ..., x_{in})$, and we have,
1) In at least one coordinate, j, we have $x_{aj} = a$ for $0 \leq a < t$.
2) In the other coordinates we have $x_{0j} = x_{1j} = ... = x_{t-1,j}$

Example 4.3. For example if $n = 2$ and $t = 4$ then we have

$$C_4^2 = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3), (2,0), (2,1), (2,2), (2,3), (3,0), (3,1), (3,2), (3,3)\}$$

and in the diagram below the coloured lines demonstrates the lines in C_4^2.



Note: only one of the diagonals is classed as a line in C_2^4.

We are now ready to state one of the fundamental results of Ramsey theory, the Hales-Jewett theorem.

Theorem 4.4. *(Hales-Jewett theorem)*
For all positive integers r and t there exists some $N = HJ(r,t)$ *such that if* $n \geq N$ *then the following holds:*
If the vertices of C_t^n *are coloured with r colours then there exists a monochromatic line.*

We can see how Van der Waerden's theorem can be obtained from Hales-Jewett theorem by naming each n-tuple in C_t^n as follows:
If $(a_1, a_2, ..., a_n)$ is the n-tuple with $a_i \in \{0, 1, ..., t-1\}$ then name this $a = \sum_{i=1}^{n} a_i t^{i-1}$.

This will then give us the set $\{0, 1, ..., t^n - 1\}$, if we colour this with r colours then this gives us a colouring of C_t^n. If n is large enough we have a monochromatic line and this then relates to a monochromatic arithmetic progression of length t.

Example 4.5. Using C_4^2 as in the previous example, we would label the elements as follows:

$$\begin{gathered}(0,0) = 0,\\ (1,0) = 1,\\ (2,0) = 2,\\ (3,0) = 3,\\ (0,1) = 4,\\ (1,1) = 5,\\ .\\ .\\ .\\ (3,3) = 15\end{gathered}$$

Then any line in C_4^2 will give us an arithmetic progression in the labels. For example the line $(0,0), (1,1), (2,2), (3,3)$ gives us the 4-term arithmetic progression $\{0, 5, 10, 15\}$. So if we have a monochromatic line in C_4^2 we will have a monochromatic arithmetic progression of length four of $\{0, 1, ..., 15\}$.

In section 3 we introduced the Van der Waerden numbers $w(k,r)$ similarly we call $HJ(r,t)$ (the minimum N such that the theorem holds) the Hales-Jewett numbers but again we will not look too much at these.

4.2 Shelah's proof

In 1987 Sharon Shelah found a new proof for the Hales-Jewett theorem and therefore for Van der Waerden's theorem too. This proof was very much celebrated, it improved massively upon the upper bounds for $w(k,r)$ and

$HJ(r,t)$ that had been previously found in previous proofs. In this section we will give the proof of the Hales-Jewett theorem using Shelah's proof which we follow from Graham's et al. account [10], however we will expand on their explanations and provide our own examples to aid understanding of the definitions used.

Throughout this section we will make the following small change to C_t^n:

$$C_t^n = \{(x_1, x_2, ..., x_n) : x_i \in \{1, ..., t\}\}$$

We start by defining a Shelah line.

Definition 4.6. We call $L \subset C_t^n$ a *Shelah line* if there is some ordering of L, $l_1, l_2, ..., l_t$, where $l_k = (x_{k1}, ..., x_{kn})$ and there exists i and j with $0 \leq i < j \leq n$ so that:

$$x_{ks} = \begin{cases} t-1 & s \leq i \\ k & i < s \leq j \\ t & j < s \end{cases}$$

It will be very useful to see an example of a Shelah line.

Example 4.7. If we have $t = 5$, $n = 4$ and $i = 1$, $j = 2$ then the Shelah line would be:

$$\begin{array}{c} (4,1,5,5) \\ (4,2,5,5) \\ (4,3,5,5) \\ (4,4,5,5) \\ (4,5,5,5) \end{array}$$

Example 4.8. If we have $t = 26$, $n = 10$ and $i = 4$, $j = 7$ then the Shelah line would be:

$$\begin{array}{c} (25,25,25,25,1,1,1,26,26,26) \\ (25,25,25,25,2,2,2,26,26,26) \\ . \\ . \\ . \\ (25,25,25,25,25,25,25,26,26,26) \\ (25,25,25,25,26,26,26,26,26,26) \end{array}$$

In general, in a Shelah line the first i coordinates are $t-1$, then the next $j-i$ are the moving coordinate and the last $n-j$ are t.

Definition 4.9. The point $l = (x_1, ..., x_n) \in C_t^n$ is a *Shelah point* if it belongs to a Shelah line.

So for example in C_5^4, $(4,1,5,5)$ is a Shelah point as it is in the Shelah line in Example 4.7.

It will be useful to bound the number of Shelah points in C_t^n for future use. As $0 \leq i < j \leq n$ there are $\binom{n+1}{2}$ ways to pick i and j and as there are t elements of C_t^n in a Shelah line, there are at most $\binom{n+1}{2}t$ Shelah points.

We next define a Shelah s-space.

Definition 4.10. Suppose $n_1, ..., n_s$ are given and $n = n_1 + ... + n_s$. We associate C_t^n with $C_t^{n_1} \times C_t^{n_2} \times ... \times C_t^{n_s}$ (the Cartesian product). For $1 \leq j \leq s$, let L_j be a Shelah line of $C_t^{n_j}$ then $L_1 \times ... \times L_s$ is a Shelah *s-space* of C_t^n.

Example 4.11. If we have C_{26}^{10} with $n_1 = 4$, $n_2 = 1$ and $n_3 = 5$ then a Shelah 3-space is:

$$\{(25, \alpha, \alpha, 26, \beta, 25, 25, \gamma, \gamma, 26) : \alpha, \beta, \gamma \in \{1, 2, ..., 26\}\}$$

Definition 4.12. Let $\phi : L_1 \times ... \times L_s \to C_t^s$, where $\phi(\xi) = \alpha_1\alpha_2...\alpha_s$ (α_j is the value of the changing coordinate in L_j) be the *canonical isomorphism.*

Example 4.13. Going back to example 4.11 we have

$$\phi((25, \alpha, \alpha, 26, \beta, 25, 25, \gamma, \gamma, 26)) = \alpha\beta\gamma$$

Definition 4.14. A colouring of C_t^n is called *fliptop* if:
For any two points, P and Q, in C_t^n that have the exact same coordinates except in one position where one has t and the other has $t-1$, then P and Q have the same colour.

The idea of colourings being fliptop will be used often later so we shall look at a few examples of this idea.

Example 4.15. If we have $t = 20$ and $n = 5$, if a colouring of C_{20}^5 is fliptop then we have the following examples of elements in C_{20}^5 that must have the same colour.
1) $(1, 20, 5, 9, 11)$ and $(1, 19, 5, 9, 11)$ have the same colour.
2) $(20, 20, 20, 20, 20)$, $(19, 20, 20, 20, 20)$, $(19, 20, 19, 20, 20)$, $(19, 19, 19, 20, 20)$, $(19, 19, 19, 19, 20)$ and $(19, 19, 19, 19, 19)$ must all have the same colour.
3) There is no condition of the colour of any two points that do not have the exact same coordinates except for in one place where it is 20 or 19. For example there is no condition on colour for any point that does not have a 20 or 19 in.

Definition 4.16. Let χ be a colouring that maps a set of elements to a set of colours. Let $L_1 \times ... \times L_s$ be a Shelah s-space with canonical isomorphism $\phi : L_1 \times ... \times L_s \to C_t^s$. A colouring χ of $L_1 \times ... \times L_s$ is *fliptop* if the colouring χ' of C_t^s given by $\chi'(P) = \chi[\phi^{-1}(P)]$ is fliptop.

Example 4.17. If we have $n = 10$, $t = 20$, $n_1 = 4$ and $n_2 = 6$ then a Shelah plane would be:

$$\{(19, \alpha, \alpha, 20, 19, 19, \beta, \beta, \beta, 20) : \alpha, \beta \in \{1, 2, ..., 20\}\}$$

and so for a colouring of this Shelah plane to be fliptop we would need the following points to be the same colour:

$$\begin{array}{c}(19, 20, 20, 20, 19, 19, 20, 20, 20, 20)\\(19, 19, 19, 20, 19, 19, 20, 20, 20, 20)\\(19, 20, 20, 20, 19, 19, 19, 19, 19, 20)\\(19, 19, 19, 20, 19, 19, 19, 19, 19, 20)\end{array}$$

This is because a colouring of C_{20}^2 is fliptop if $(20, 20)$ and $(19, 20)$ and $(20, 19)$ and $(19, 19)$ are the same colour.

Note, for a Shelah line to be fliptop all we need is for the last two points to be the same colour. For example, for a colouring of the Shelah line in example 4.7 to be fliptop we would need $(4, 5, 5, 5)$ and $(4, 4, 5, 5)$ to be the same colour. All that is left now until we can prove Hales-Jewett theorem is to state and prove two lemmas and a theorem.

Lemma 4.18. *Assume $n \geq c$. If we colour C_t^n with c colours then there exists a fliptop Shelah line (i.e. a Shelah line where the two points which have a t and $t-1$ in the block of changing coordinates, are the same colour).*

Proof. For $0 \leq i \leq n$ define the $n + 1$, $P_i = (x_{i1}, ..., x_{in})$ by the following:

$$x_{ij} = \begin{cases} t-1 & j \leq i \\ t & j > i \end{cases}$$

For example, in C_{26}^4 our 5 P_i's are,

$$P_0 = (26, 26, 26, 26)$$
$$P_1 = (25, 26, 26, 26)$$
$$P_2 = (25, 25, 26, 26)$$
$$P_3 = (25, 25, 25, 26)$$
$$P_4 = (25, 25, 25, 25)$$

By the Pigeonhole principle there must be two of the P_i's that have the same colour, as $n+1 > c$. Let these two points be P_i and P_j and they are the last two points in the Shelah line $l_1, ...l_t$, where $l_k = (x_{k1}, ..., x_{kn})$ with,

$$x_{ks} = \begin{cases} t-1 & s \leq i \\ k & i < s \leq j \\ t & j < s \end{cases}$$

hence we have a fliptop Shelah line. To demonstrate the last idea in this proof let us look back at our example for C_{26}^4. If for example P_1 and P_3 are the two that are coloured the same then

$$\{(25, \alpha, \alpha, 26) : \alpha \in \{1, ..., 26\}\}$$

is our fliptop Shelah line. □

Using this lemma we can prove the following theorem and after that all that is left to do before we can finally prove the Hales-Jewett theorem is to prove one more lemma.

Theorem 4.19. *Let r,s and t be fixed positive integers and define $n_1, ..., n_s$ by:*

$$n_1 = r^{t^{s-1}}$$

$$n_2 = r^{\binom{n_1+1}{2} t^{s-1}}$$

and then

$$n_{i+1} = r^{A_i}$$

where,

$$A_i = \left[\prod_{j \le i} \binom{n_j + 1}{2}\right] t^{s-1}$$

We shall set $n = n_1 + ... + n_s$ and then given an r-colouring, χ, of C_t^n we have a fliptop Shelah s-space.

Proof. Associate C_t^n with $C_t^{n_1} \times ... \times C_t^{n_s}$ and write $y \in C_t^n$ as $y = y_1, ..., y_s$ where $y_j \in C_t^{n_j}$. Now, we shall define the equivalence relation $\equiv$ on $C_t^{n_s}$ as follows:

$$y_s \equiv y_s' \iff \chi(y_1, ..., y_{s-1}, y_s) = \chi(y_1, ..., y_{s-1}, y_s')$$

for all Shelah points $y_1, ...y_{s-1}$.

We saw before that C_t^n contains at most $\binom{n+1}{2}t$ Shelah points. So $C_t^{n_i}$ contains at most $\binom{n_i+1}{2}t$ Shelah points and hence there are at most A_{s-1} choices for the Shelah points $y_1, ..., y_{s-1}$. This then tells us that we have at most $n_s = r^{A_{s-1}}$ equivalence classes. Therefore, we can think of $\equiv$ as being an n_s-colouring, $\hat{\chi}$, of $C_t^{n_s}$. We can therefore apply lemma 4.18. (as $n_s \geq n_s$), so there exists a Shelah line $L_s \subset C_t^{n_s}$, which is fliptop under $\hat{\chi}$.

We shall use reverse induction, so we shall assume that $L_s, L_{s-1}, ..., L_{i+1}$ have been found and we shall look for L_i. We shall now define the equivalence relation $\equiv$ on $C_t^{n_i}$ by setting:

$$y_i \equiv y_i' \iff \chi(y_1, ..., y_{i-1}, y_i, z_{i+1}, ..., z_s) = \chi(y_1, ..., y_{i-1}, y_i', z_{i+1}, ..., z_s)$$

for all Shelah points $y_1, ..., y_{i-1}$ and all choices of $z_{i+1} \in L_{i+1}, z_{i+2} \in L_{i+2}, ..., Z_s \in L_s$. For each y_j there are at most $\binom{n_j+1}{2}t$ choices for $1 \leq j \leq i-1$. For each z_j, $i+1 \leq j \leq s$, there are t choices as the L_j are already determined. Hence, there are at most A_{i-1} choices for $y_1, ..., y_{i-1}, z_{i+1}, ..., z_s$ and so there are at most $n_i = r^{A_{i-1}}$ equivalence classes. As before we can think of $\equiv$ as being an n_i-colouring, $\hat{\chi}$, of $C_t^{n_i}$. We can then again apply lemma 4.18., which gives us a Shelah line $L_i \subset C_t^{n_i}$ that is fliptop under $\hat{\chi}$.

We are now ready to prove that the Shelah s-space $L_1, ...L_s$ is fliptop under χ. Fix i, for $1 \leq i \leq s$ and let y_i, y_i' be the last two points of L_i and so they have the same colour under $\hat{\chi}$ and hence they are in the same equivalence class. We therefore have,

$$\chi(y_1, ..., y_{i-1}, y_i, z_{i+1}, ..., z_s) = \chi(y_1, ..., y_{i-1}, y_i', z_{i+1}, ..., z_s)$$

for all Shelah points $y_1, ..., Y_{i-1}$ and all $z_{i+1} \in L_{i+1}, z_{i+2} \in L_{i+2}, ..., Z_s \in L_s$. But for $1 \leq j < i$ all $z_j \in L_j$ are obviously Shelah points so,

$$\chi(z_1, ..., z_{i-1}, y_i, z_{i+1}, ..., z_s) = \chi(z_1, ..., z_{i-1}, y_i', z_{i+1}, ..., z_s)$$

for all $z_j \in L_j$ where $1 \leq j \leq s$ and $j \neq i$. So $L_1, ..., L_s$ is a fliptop Shelah s-space as χ' is fliptop where $\chi'(P) = \chi[\phi^{-1}(P)]$ for $P \in C_t^s$.

□

Lemma 4.20. *Let $s = HJ(r, t-1)$, so that in any given r-colouring of C_{t-1}^s there exists a monochromatic line. Then we have under any fliptop colouring with r colours of C_t^s there exists a monochromatic line.*

Proof. We know there is a monochromatic line $l_1, ..., l_{t-1}$ for any r-colouring of C_{t-1}^s. Let l_t be the point in C_t^s which is found by setting all the moving coordinates to a t and copying the none moving coordinates from l_{t-1}. To demonstrate this idea let us look at an example. If we had the line $(1,3),(2,3),(3,3),(4,3)$ in C_4^2 then l_5 would be $(5,3)$ where $t = 5$.

Then $l_1, ..., l_{t-1}, l_t$ is a line in C_t^s. If the colouring is fliptop, l_t will be the same colour as l_{t-1} as the only places the coordinates vary in l_t and l_{t-1} are where in l_t they are t and in l_{t-1} they are $t-1$. Then as $l_1, ..., l_{t-1}$ are all the same colour, the line $l_1, ..., l_{t-1}, l_t$ is a monochromatic line in C_t^s. In our example if the line $(1,3),(2,3),(3,3),(4,3)$ is monochromatic then $(1,3),(2,3),(3,3),(4,3),(5,3)$ is a monochromatic line as the colouring is fliptop so $(4,3)$ and $(5,3)$ must be the same colour. □

We are finally ready to prove the Hales-Jewett theorem using the definitions, lemmas and theorems we have discussed so far in this section.

Proof. (Hales-Jewett theorem)
Let's fix r and we shall use induction on t. For the base case, when $t = 1$ it is trivial that $HJ(r, 1) = 1$ as we will only have one element in C_1^n. So we shall assume that $s = HJ(r, t-1)$ exists and we want to show that there exists an $n = HJ(r, t)$. Let n be given by theorem 4.19. and so by the theorem given an r-colouring, χ, of C_t^n there is a fliptop Shelah s-space $(L_1 \times ... \times L_s)$. Let χ' be the r-colouring of C_t^s given by $\chi'(y) = \chi(\phi^{-1}(y))$ where ϕ is the canonical isomorphism $\phi : L_1 \times ... \times L_s \to C_t^s$. Then as χ is fliptop χ' must be fliptop and by lemma 4.20. there exists a monochromatic line $L \subset C_t^s$. We then have $\phi^{-1}(L)$ which is a monochromatic line in C_t^n by definition of χ' and ϕ. $\square$

5 The density version of The Hales-Jewett theorem

We have already seen the density version of Van der Waerden's theorem, it is also true that the Hales-Jewett theorem has a density version also. The density Hales-Jewett theorem was first stated and proved by H. Furstenburg and Y. Katznelson and it is given below.

Theorem 5.1. *For every positive integer t and all non zero real numbers δ there exists an integer N such that the following holds:*
If $n \geq N$ and A is any subset of $C_t^n = \{(x_1, x_2, ..., x_n) : x_i \in \{1, ..., t\}\}$ with $|A| \geq \delta t^n$, then A contains a line (as in definition 4.2.).
The least integer N, such that the above holds will be denoted by DHJ(t,δ).

H. Furstenburg and Y. Katznelson proved this for the $t = 3$ case in 1989 and then they extended their proof for general t in 1991 in [6]. They used ergodic techniques that extended the techniques used by Furstenburg in his first proof of Szemerédi's theorem. In 2009 Timothy Gowers set up the Polymath project, a massive on line collaboration of Mathematicians carrying out research together, which focused on the Density Hales-Jewett theorem. There are two parts to this Polymath project, the first focused on proving the theorem using a non-ergodic method and the other part focused on computing Density Hales-Jewett numbers and other related values. The project was initially started as an experiment but it was a large success and many other Polymath projects followed. The Density Hales-Jewett theorem has been proved by a couple of other Mathematicians since the Polymath project started including Tim Austin in 2009 ('Deducing the Density Hales-Jewett Theorem from an infinitary removal lemma' [2]) and P. Dodos et al. in 2012 in 'A Simple Proof of the Density Hales-Jewett Theorem' [5]. In 'A Simple Proof of the Density Hales-Jewett Theorem' they give a purely combinatorial proof that follows Polymath's proof except that they simplify the argument some what.

One of the reasons this theorem has attracted so much attention from Mathematicians in the recent years, is that proving the Density Hales-Jewett theorem then implies the Szemerédi's Theorem. These new proofs for the Density Hales-Jewett theorem by Polymath and P.Dodos et al. give 'arguably the simplest proof yet known of Szemerédi's Theorem' [19, p 4].

5.1 Proof of the Density Hales-Jewett theorem

In this section we shall look in some detail at the proof of the Density Hales-Jewett theorem given by P. Dodos et al. [5]. Their proof is modeled after Polymath's proof, though it does differ in certain parts. Both the Polymath and P. Dodos et al. proofs are based on the density increment method.

In general the density increment method is as follows:
If A has density δ in S and A does not contain a subset of the desired kind, then there is a substructure S' of S such that the density of A inside S' is at least $\delta + a$, where a is a positive constant that only depends on δ. We can then iterate this argument and if S is large enough then we can keep iterating until the density is greater than 1, which gives us a contradiction. For the density Hales Jewett case, S is C_t^n and the subset of the desired kind is a line.

In order to achieve this we must first show that there exists some 'structured' set B such that the density of A inside B is at least $\delta + a$. We then need to show that B can be partitioned into subspaces such that A has density at least $\delta + a$ in one of these subspaces of B. Both the Polymath and Dodos et al. proofs prove the second part of this in the same way (we shall come to this later) but Dodos et al. give a simpler way to solve the first step.

For the first step we need to have a probabilistic version of the Density Hales-Jewett theorem, to do this Polymath used the idea of the equal-slices measure (a probability measure on C_t^n). However, P. Dodos et al. give a different way to give a probabilistic version of the theorem which makes the argument clearer, which we shall see soon.

5.1.1 Some new notation, definitions and preliminary tools

Before giving details of the proof of the Density Hales-Jewett theorem, we shall introduce some new definitions and notations and give a few tools that will be useful throughout the proof. The following definitions and notations are given in [5, pp 3342-3345] but we shall provide our own examples. As in section 4.2, we have $C_t^n = \{(x_1, x_2, ..., x_n) : x_i \in \{1, ..., t\}\}$.

Definition 5.2. For a fixed set of m distinct letters, say $v_1, ..., v_m$, an m-*variable word* of C_t^n is a sequence of n elements of $\{1, ..., t\} \cup \{v_1, ..., v_m\}$, where every v_i for $i \in \{1, ..., m\}$ appears at least once.

Example 5.3. An example of a 2-variable word of C_4^4 is $(1, 3, a, b)$, where here $v_1 = a$ and $v_2 = b$.

Definition 5.4. Let z be an m-variable word then $z(a_1, ..., a_m)$, where $a_1, ..., a_m \in \{1, ..., t\}$, is obtained by replacing v_i in z with a_i for all $i \in \{1, ..., m\}$. Then an *m-dimensional subspace* of C_t^n is a set of the form

$$\{z(a_1, ..., a_m) : a_1, ..., a_m \in \{1, ..., t\}\}$$

If V is an m-dimensional subspace of C_t^n, then the set of lines of V will be denoted by $lines(V)$.

Example 5.5. 1) A 1-dimensional subspace of C_t^n is just a line.
2) An example of a 2-dimensional subspace of C_4^4, where z is the 2-variable word from the previous example, is:

$$\{(1,3,1,1),(1,3,1,2),(1,3,1,3),(1,3,1,4),(1,3,2,1),(1,3,2,2),(1,3,2,3),(1,3,2,4),$$

$$(1,3,3,1),(1,3,3,2),(1,3,3,3),(1,3,3,4),(1,3,4,1),(1,3,4,2),(1,3,4,3),(1,3,4,4)\}$$

Definition 5.6. The *density* of a subset A of C_t^n in V is given by $\frac{|A \cap V|}{|V|}$, we denote it as $dens_V(A)$. The density of A in C_t^n will simply be denoted by $dens(A)$.

Example 5.7. Let $A = \{(1,2),(1,1)\}$. The density of A in $C_2^2 = \{(1,1),(1,2),(2,1),(2,2)\}$ is $dens(A) = \frac{2}{4} = \frac{1}{2}$.

Definition 5.8. Let V be an m-dimensional subspace. For $k \in \mathbb{N}_0$ where $2 \leq k \leq t$, we have,

$$V|k = \{z(a_1, ..., a_m) : a_1, ..., a_m \in \{1, ..., k\}\}$$

Example 5.9. Let V be the 2-dimensional subspace of C_4^4 given in example 5.5. and let $k = 2$, then we have

$$V|2 = \{(1,3,1,1),(1,3,1,2),(1,3,2,1),(1,3,2,2)\}$$

Definition 5.10. For $x \in C_t^n$ and $y \in C_t^l$ we denote the *concatenation* of x and y as $x \frown y$. For subsets, $A \subseteq C_t^n$ and $B \subseteq C_t^l$, the *concatenation* of A and B is $A \frown B = \{x \frown y : x \in A, y \in B\}$.

Example 5.11. If $x = (1,5,3,1)$ and $y = (1,4,2)$ then $x \frown y = (1,5,3,1,1,4,2)$.

Finally, we introduce the numbers $GR(t,m)$ which will be used later on.

Proposition 5.12. *For all integers $t \geq 2$ and $m \geq 1$ there exists a least integer $GR(t,m)$ such that the following holds:*
For every integer $n \geq GR(k,m)$ and all sets L of lines of C_t^n, there exists an m-dimensional subspace, V, of C_t^n such that either:
1) $Lines(V) \subseteq L$, or
2) $Lines(V) \cap L = \emptyset$

We shall not give the proof of this proposition but it can be done by repeatedly applying the Hales-Jewett theorem.

From now on in this section we shall write DHJ_t to denote that for every $0 < \delta \leq 1$, $DHJ(t, \delta)$ is finite. We need just a few more tools before we can move on to proving the density Hales-Jewett theorem.

The first of the results we need is that the density Hales-Jewett theorem implies the multidimensional version of it. We shall start by stating the multidimensional version of the Hales-Jewett theorem and then go on to prove that DHJ_t implies it.

Theorem 5.13. *For all positive δ, and integers t and $m \geq 1$, there exists a positive integer $MDHJ(t, m, \delta)$ such that the following is true:*
For every $n \geq MDHJ(t, m, \delta)$, every subset of C_t^n with density at least δ contains an m-dimensional subspace of C_t^n.

As with the density Hales-Jewett theorem, we shall write $MDHJ_t$ to denote that for all positive δ and integers m, $MDHJ(t, m, \delta)$ is finite.

Proposition 5.14. *For every t, DHJ_t implies $MDHJ_t$*

We shall follow the proof from Polymath [24, p 676], however we will discuss it in more detail.

Proof. We shall use induction on m. As we saw earlier a 1-dimensional subspace of C_t^n is just a line, hence as we are assuming DHJ_t then the $m = 1$ case holds. So we can assume the result for $m - 1$ and we shall show the result holds for m.

Let A be a subset of C_t^n with density at least δ and let $M = MDHJ(t, m-1, \frac{\delta}{2})$. For all $y \in C_t^{n-M}$ let $A_y = \{x \in C_t^M : x \frown y \in A\}$. Now, let G be the set of all $y \in C_t^{n-M}$ such that A_y has density at least $\frac{\delta}{2}$. Then G must have density at least $\frac{\delta}{2}$ in C_t^{n-M}. We can see this by assuming for a contradiction that $\frac{|G|}{t^{n-M}} \leq \frac{\delta}{2}$, so $|G| \leq \frac{\delta t^{n-M}}{2}$. We therefore have at most $\frac{\delta t^{n-M}}{2}$ y's where $y \in C_t^{n-M}$ such that $x \frown y \in A$ for at least $\frac{\delta t^M}{2}$ x's. Therefore,

$$|A| \leq \frac{\delta t^{n-M} t^M}{2} = \frac{\delta t^n}{2} \leq \delta t^n$$

This is a contradiction as we know that A has density at least δ in C_t^n, therefore G must have density at least $\frac{\delta}{2}$ in C_t^{n-M}.

By induction, for any $y \in G$, A_y must contain an $m-1$-dimensional subspace. There are at most $D = (t + m - 1)^M$ such subspaces. This is because there is an injection from the set of all $m-1$-dimensional subspaces of C_t^M to the set of all $m - 1$ dimensional subspaces of C_{t+m-1}^M .

There must exist a subspace $\sigma \subseteq C_t^M$, such that $G_\sigma = \{y \in C_t^{n-M} : x \frown y \in A, \forall x \in \sigma\}$, where G_σ has density at least $\frac{\delta}{2D}$ in C_t^{n-M}. We can see this by the Pigeon hole principle. We know that, for all $y \in G$, A_y has at least one $m-1$-dimensional subspaces and there are at most D possible subspaces available, also

$$\frac{|G|}{t^{n-M}} \geq \frac{\delta}{2} \Longrightarrow |G| \geq \frac{\delta t^{n-M}}{2} = \frac{\delta t^{n-M} D}{2D}$$

so if we partition the elements of G into D subsets then by the Pigeon hole principle at least one of these must have at least $\frac{\delta t^{n-M}}{2D}$ elements and hence there exists a subspace σ such that G_σ has density of at least $\frac{\delta}{2D}$.

Therefore if $n - M \geq DHJ(t, \frac{\delta}{2D})$, then G_σ must contain a line, call this line λ. Then $\sigma \times \lambda$ is an m-dimensional subspace of C_t^n which is contained in A. □

For the next two results before proving the density Hales-Jewett theorem, we shall only state them, not prove them. In the following lemma we need the following new notation:

$$A_x = \{y \in C_t^{n-l} : x \frown y \in A\}$$

where $x \in C_t^l$ and A is a subset of C_t^n.

Lemma 5.15. *Let $t \geq 2$ and $m \geq 1$ be integers and let $0 < \epsilon < 1$. If $n \geq \epsilon^{-1} t^m m$, then the following holds:*
For every subset A of C_t^n with density greater than ϵ there exists some $l < n$ and an m-dimensional subspace of C_t^l such that for all $x \in V$, $dens(A_x) \geq dens(A) - \epsilon$.

This lemma says that when a dense subset of C_t^n is restricted to a suitable subspace of C_t^n, the dense subset becomes very uniformly distributed. The proof of this theorem is fairly short and straightforward and just uses a logical argument to show the result holds.

The next result combines both Proposition 5.14. and Lemma 5.15. to get the following corollary which will be useful later on.

Corollary 5.16. *Let $t \geq 2$ be an integer and assume DHJ_t. Then for all integers $m \geq 1$ and all $0 < \delta \leq 1$, there exists an integer $MDHJ^*(t, m, \delta)$ such that the following holds:*
If $n \geq MDHJ^*(t, m, \delta)$, then for all subsets of C_{t+1}^n with density at least δ, there exists an m-dimensional subspace V of C_{t+1}^n such that $V|t$ is contained in A.

To prove this we would set $MDHJ^*(t, m, \delta) = (\frac{\delta}{2})^{-1}(t+1)^M M$, where $M = MDHJ(t, m, \frac{\delta}{2})$. We would then show that if $n \geq MDHJ^*(t, m, \delta)$ and if A is a subset of C_{t+1}^n with density at least δ, then there exists an m-dimensional subspace V of C_{t+1}^n such that $V|t$ is contained in A. To show this we make use of lemma 5.15.

5.1.2 Proving the Density Hales-Jewett theorem

We shall prove the Density Hales-Jewett theorem by induction on t. The $t = 2$ case follows from Sperner's theorem, which is given below.

Theorem 5.17. *Let A be a collection of subsets of $[n] = \{1, ..., n\}$ such that A has more than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements. Then there exists distinct sets $B, C \in A$ such that B is a proper subset of C.*

We can see that Sperner's theorem gives DHJ_2, there are 2^n different subsets of $[n]$ and each one can be mapped to one of the 2^n points of C_2^n. This is a one-one correspondence. A set X of $[n]$ is mapped to a point of C_2^n in the following way, the elements of X tell us in what positions there is a 1 and the elements in $[n]$ that are not in X tell us where there is a 2. For example if $n = 5$ and $X = \{1, 2, 4\}$ then our point in C_2^5 is $(1, 1, 2, 1, 2)$. Two subsets B, C of $[n]$ corresponds to a line if one is a proper subset of the other, for example if $n = 5$ and $B = \{1, 5, 4\}$ and $C = \{1, 5\}$ then C is a proper subset of B. The set B corresponds to the point $(1, 2, 2, 1, 1)$ and C corresponds to the point $(1, 2, 2, 2, 1)$ which is a line. Hence, proving Sperner's theorem proves DHJ_2. P. Dodos et al. [5] do not give the proof of this but we shall include it. We shall follow Polymath's [24, p 673-674] proof expanding on their details.

Proof. (Theorem 5.17)
Let us assume that no element of A is a proper subset of any other. To start the proof we shall look at a way of choosing a random subset of $[n]$. Lets choose a random permutation π of $[n]$ and a random number $m \in \{0, ..., n\}$. Let $B = \{\pi(1), ..., \pi(m)\}$ and since no element of A is a proper subset of another, for each possible π at most one m can be used so the set belongs to A. Therefore the probability of choosing a set in A is at most $\frac{1}{n+1}$ as there are $n + 1$ m to choose from.

Now the probability of choosing a particular set B that is of size m, is equal to $\frac{1}{(n+1)\binom{n}{m}}$ as $\frac{1}{n+1}$ is the probability of choosing the size m and there are $\binom{n}{m}$ different sets of size m. So to make A as large as possible but to make sure the probability of choosing a set in A is at most $(n+1)^{-1}$, we must choose A to consist of sets of size m such that $\frac{1}{\binom{n}{m}}$ is minimum. Equivalently we want to make $\binom{n}{m}$ maximum. To find out when this is maximum let us look at the following:

$$\binom{n}{m-1} \leq \binom{n}{m}$$

we have,

$$\binom{n}{m-1} = \frac{n!}{(m-1)!(n-m+1)!} = \frac{n!}{m!(n-m)!}\frac{m}{n-m+1} = \binom{n}{m}\frac{m}{n-m+1}$$

so we get,

$$\binom{n}{m}\frac{m}{n-m+1} \leq \binom{n}{m} \Longrightarrow m \leq n-m+1 \Longrightarrow m \leq \frac{n+1}{2} \Longrightarrow m \leq \lceil \frac{n}{2} \rceil$$

and as $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$, if no element of A is a proper subset of another then $|A| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

□

Before we start to carry out the main part of the proof, we shall give a table (table 3) of all the different constants that will be used throughout the proof for easy access.

$m_0 = DHJ(t, \frac{\delta}{4})$	$\theta = \frac{\frac{\delta}{4}}{(t+1)^{m_0} - t^{m_0}}$
$\eta = \frac{\delta\theta}{48}$	$\gamma = \frac{\delta\eta^2}{t}$
$n(m, \epsilon) = \epsilon(t+1)^m m$	$\lambda = \frac{t+1}{t}$
$M_0 = max\{m_0, \frac{log\eta^{-1}}{log\lambda}\}$	$M_1 = MDHJ^*(t, m, \beta)$
$F(m, \beta) = \lceil \beta^{-1}(t+1+m)^{M_1}(t+1)^{M_1-m}M_1 \rceil$	$F^{(1)}(m, \beta) = F(m, \beta)$
$F^{(r+1)}(m, \beta) = F^{(r)}(F(m, \beta), \beta)$	

Table 3: Table of constants that will be used throughout the proof of the density Hales-Jewett theorem [5].

From here onwards all the lemmas, propositions, corollaries and definitions come from [5], we shall however give our own explanations and expand on theirs and also give our own examples. We shall start by giving the proposition that completes the induction step, that is if we assume DHJ_t, we have DHJ_{t+1}.

Proposition 5.18. *Let $t \geq 2$ be an integer and assume DHJ_t. For all $0 < \delta \leq 1$ and all integers $d \geq 1$ there exists an integer $N(t, d, \delta)$ such that*

the following holds:
If $n \geq N(t, d, \delta)$ and A is a subset of C_{t+1}^n with density at least δ in C_{t+1}^n, then either,
1) A contains a line of C_{t+1}^n, or
2) There exists a d-dimensional subspace V of C_{t+1}^n such that $dens_V(A) \geq \delta + \frac{\gamma}{2}$.

This is enough to prove the Density Hales-Jewett theorem. Assuming Proposition 5.18. holds, if A has no line then there exists a V as above such that $dens_V(A) \geq \delta + \frac{\gamma}{2}$. We can now repeat the argument on $A \cap V$, if $A \cap V$ does not contain a line then we can go to another subspace which has relative density at least $\delta + \frac{\gamma}{2} + \frac{\gamma}{2}$. As the density cannot exceed 1, there can be at most $\frac{2}{\gamma}$ iterations before the density exceeds 1 and so if n is large enough we have a contradiction and hence we have a line. [19, p 31]

As discussed earlier in order to prove Proposition 5.18., we must first start by showing that if A has no line then A must correlate with a structured set of C_{t+1}^n more than we would expect. To get this result we must first state a few lemmas and definitions. We shall start with the following lemma.

Lemma 5.19. *Let $m \geq m_0$ be an integer and let $0 < \delta \leq 1$. If $n \geq n(GR(t, m), \frac{\eta^2}{2})$ then we have:*
For every subset A of C_{t+1}^n with density at least γ there exists some $l < n$ and an m-dimensional subspace U of C_{t+1}^n such that the following two statements hold:
1) For all $u \in U$, $dens(A_u) \geq \delta - \frac{\eta^2}{2}$
2) For all lines $l \in Lines(U|t)$, $(\bigcap_{u \in l} A_u) \geq \theta$.
Where $A_u = \{y \in C_{t+1}^{n-l} : u \frown y \in A\}$ for $u \in U$.

To prove this we would make use of lemma 5.15. and proposition 5.12 and we can now use this lemma to help us prove the following result.

Lemma 5.20. *Let $m \geq m_0$ be an integer and let $0 < \delta \leq 1$. Let $n \geq n(GR(t, m), \frac{\eta^2}{2})$ and let A be a subset of C_{t+1}^n with density at least δ. Then either:*
1) There exists an m-dimensional subspace X of C_{t+1}^n with $dens_X(A) \geq \delta + \frac{\eta^2}{2}$, or
2) There exists an m-dimensional subspace W of C_{t+1}^n with $dens_W(A) \geq \delta - 2\eta$ and $|\{l \in Lines(W|t) : l \subseteq A\}| \geq (\frac{\theta}{2})|Lines(W|t)|$.

This lemma tells us that if we have a lack of density increment (1) doesn't hold) then we can find a subspace W of C_{t+1}^n such that the density of A inside W is very close to the density of A inside C_{t+1}^n and there are plenty of lines in $A \cap (W|t)$.

From here onwards the proof will be of the same ideas as Polymath's. We shall introduce the important idea of insensitive sets in the following definitions and we shall provide some examples of these. We shall use the notation that for $x, y \in C_{t+1}^n$, we shall write $x = (x_r)_{r=1}^n$ and $y = (y_r)_{r=1}^n$.

Definition 5.21. The elements $x, y \in C_{t+1}^n$ are (i,j)-*equivalent*, for $i, j \in \{1, ..., t+1\}$ with $i \neq j$, if for every $s \in \{1, ..., t+1\} \setminus \{i, j\}$

$$\{r \in \{1, ..., n\} : x_r = s\} = \{r \in \{1, ..., n\} : y_r = s\}$$

Example 5.22. If $n = 5$ and $t + 1 = 4$ then $x = (3, 1, 1, 4, 3)$ and $y = (3, 2, 1, 4, 3)$ are $(1, 2)$-equivalent.

Definition 5.23. A subset A of C_{t+1}^n is (i,j)-*insensitive*, where $i, j \in \{1, ..., t+1\}$ and $i \neq j$, if for all $x \in A$ and all $y \in C_{t+1}^n$, where x and y are (i,j)-equivalent, then $y \in A$.

If A is a subset of V, where V is an m-dimensional subspace of C_{t+1}^n, for each element $z(a_1, ..., a_m)$ in V (where z is the m-variable word that generates V and $a_i \in \{1, ..., t+1\}$ for all $i \in [m]$) we identify it with the corresponding $(a_1, ..., a_m) \in C_{t+1}^m$. If A becomes an (i,j)-*insensitive subset* of C_{t+1}^m, then A is (i,j)-insensitive in V.

Example 5.24. Let $t + 1 = 4$ and $n = 3$, then $A = \{(1, 2, 4), (1, 3, 4)\}$ is $(2, 3)$-insensitive.

Now we know the notion of an insensitive set we can move on to the next lemma and then we will have all we need to be able to prove the first part of proposition 5.18..

Lemma 5.25. *Let $m \geq M_0$ be an integer and let $0 < \delta \leq 1$. Now let $n \geq n(GR(t,m)\frac{\eta^2}{2})$ and let A be a subset of C_{t+1}^n with density at least δ. Assuming that A contains no line of C_{t+1}^n and for every m-dimensional subspace X of C_{t+1}^n, $dens_X(A) = \delta + \frac{\eta^2}{2}$. Then there exists an m-dimensional subspace W of C_{t+1}^n and a subset C of W such that the following holds:*
1) $dens_W(C) \geq \frac{\theta}{4}$ and $C = \bigcap_{i=1}^t C_i$ where C_i is (i,t+1)-insensitive in W for all $i \in [t]$.
2) $dens_W(A \cap (W \setminus C)) \geq (\delta + 6\eta) dens_W(W \setminus C)$ and $dens_W(A \cap (W \setminus C)) \geq \delta - 3\eta$.

To prove this we use an argument that uses lemma 5.20. Now we have this result we can prove that if A contains no line then it must have density, in a 'simple' subset of C_{t+1}^n, of at least $\delta+$ a constant that only depends on δ. This will complete the first part of the proof, we shall give a proof of the following Corollary which we follow from [5, p 3349] but as always we shall provide more detailed explanations.

Corollary 5.26. *Let $m \geq M_0$ be an integer and let $0 < \delta \leq 1$. Now let $n \geq n(GR(t,m)\frac{\eta^2}{2})$ and let A be a subset of C_{t+1}^n with density at least δ. Assuming that A contains no line of C_{t+1}^n, there exists an m-dimensional subspace W of C_{t+1}^n and a set $\{D_1, ..., D_t\}$ of subsets of W such that:*
1) D_i is (i,t+1)-insensitive in W for all $i \in [t]$,and
2) If we set $D = D_1 \cap ... \cap D_t$ then we have $dens_W(D) \geq \gamma$ and $dens_W(A \cap D) \geq (\delta + \gamma)dens_W(D)$.

Proof. We shall look at two cases for this proof.
1) Firstly, let us assume there exists an m-dimensional subspace X of C_{t+1}^n with $dens_X(A) \geq \delta + \frac{\eta^2}{2}$. We shall set $W = X$ and $D_i = X$ for all $i \in [t]$. As $D_i = X$ for all i, clearly each D_i is $(i, t+1)$-insensitive in $W = X$. If we set $D = D_1 \cap ... \cap D_t$ then $D = X$, so we have $dens_W(D) = dens_X(X) = 1 \geq \gamma$ and $dens_W(A \cap D) = dens_X(A \cap X) = dens_X(A) \geq \delta + \frac{\eta^2}{2} \geq \delta + \gamma$ as $\frac{\eta^2}{2} \geq \gamma$.

2) Now let us assume that for all m-dimensional subspaces X of C_{t+1}^n, we have $dens_X(A) < \delta + \frac{\eta^2}{2}$. So by lemma 5.25., there exists an m-dimensional subspace W of C_{t+1}^n and a subset $C = C_i \cap ... \cap C_t$ of W, where each C_i is $(i, t+1)$-insensitive in W for all $i \in [t]$. We also know that $dens_W(A \cap (W \setminus C)) \geq (\delta + 6\eta)dens_W(W \setminus C)$ and $dens_W(A \cap (W \setminus C)) \geq \delta - 3\eta$.

Let

$$P_1 = W \setminus C_1 \text{ and } P_i = (W \setminus C_i) \cap C_1 \cap ... \cap C_{i-1}$$

for $i \in \{2, ..., t\}$. Also let, for each $i \in [t]$,

$$\lambda_i = \frac{dens_W(P_i)}{dens_W(W \setminus C)} \text{ and } \delta_i = \frac{dens_W(A \cap P_i)}{dens_W(P_i)}$$

(if P_i is empty then let $\delta_i = 0$). As the set $\{P_1, ..., P_t\}$ is a partition of $W \setminus C$, we get

$$\sum_{i=1}^{t} \lambda_i \delta_i = \sum_{i=1}^{t} \frac{|P_i \cap W||W|}{|W||(W \setminus C) \cap W|} \frac{|A \cap P_i \cap W||W|}{|W||P_i \cap W|}$$

$$= \sum_{i=1}^{t} \frac{|A \cap P_i \cap W||W|}{|(W \setminus C) \cap W||W|} = \frac{dens_W(A \cap (W \setminus C))}{dens_W(W \setminus C)} \geq \delta + 6\eta$$

Hence, there exists an $i_o \in [t]$ such that both $\lambda_{i_0} \geq \frac{3\eta}{t}$ and $\delta_{i_0} \geq \delta + 3\eta$ holds. Otherwise, if for all $i \in [t]$, $\lambda_{i_0} < \frac{3\eta}{t}$ and $\delta_{i_0} < \delta + 3\eta$ then $\sum_{i=1}^{t} \lambda_i \delta_i < 3\eta(\delta + 3\eta) = 3\eta\delta + 9\eta^2 < \delta + 6\eta$, as clearly $\eta < \frac{1}{3}$, this gives a contradiction.

We set the D_i's as follows : $D_i = C_i$ if $i < i_0$, $D_{i_0} = W \setminus C_{i_0}$ and $D_i = W$ if $i > i_0$. The set D_i is $(i, t+1)$-insensitive in W for all $i \in [t]$, as all C_i are. Clearly W is also and as C_{i_0} is (i,t+1)-insensitive in W for all $i \in [t]$ $W \setminus C_{i_0}$ is too. Now, $P_{i_0} = (W \setminus C_{i_0}) \cap C_1 \cap ... \cap C_{i_0-1} = (W \setminus C_{i_0}) \cap C_1 \cap ... \cap C_{i_0-1} \cap W = D_1 \cap ... \cap D_k$. So,

$$dens_W(P_{i_0}) = \frac{dens_W(P_{i_0})}{dens_W(W \setminus C)} dens_W(W \setminus C) = \lambda_{i_0} dens_W(W \setminus C)$$

$$\geq (\frac{3\eta}{t}) dens_W(A \cap (W \setminus C) \geq \frac{3\eta}{t}(\delta - 3\eta) \geq \gamma$$

and,

$$dens_W(A \cap P_{i_0}) = \frac{dens_W(A \cap P_{i_0})}{dens_W(P_{i_0})} dens_W(P_{i_0}) = \delta_{i_0} dens_W(P_{i_0})$$

$$\geq (\delta + 3\eta) dens_W(P_{i_0}) \geq (\delta + \gamma) dens_W(P_{i_0})$$

as required.

□

We are now ready to prove the second part of Proposition 5.18.. We want to show that any subset of C_{t+1}^n that is of the form $D = D_1 \cap ... \cap D_t$, where D_i is $(i, t+1)$-insensitive in C_{t+1}^n can be partitioned into subspaces with sufficient dimension. We shall start by showing the following result.

Lemma 5.27. *Let $m \geq 1$ be an integer, let $0 < \beta \leq 1$ and let $i \in [t]$. If $n \geq F(m, \beta)$, then for every subset D of C_{t+1}^n that is (i,t+1)-insensitive and has $dens(D) \geq 2\beta$, there exists a family $\mathcal{V}$ of pairwise disjoint m-dimensional subspaces of C_{t+1}^n which are contained in D and have $dens(D \setminus \cup\mathcal{V}) < 2\beta$.*

This has a slightly lengthy proof which uses an algorithm which gives a new set of subspaces at each iteration with the density increasing. The algorithm will eventually terminate and the result follows.

The following corollary uses this to complete the second part of the proof of the Density Hales-Jewett theorem. We shall give a proof for this which we follow from [5, p 3351] expanding on the details.

Corollary 5.28. *Let $m \geq 1$ be an integer, let $0 < \beta \leq 1$ and let $r \in [t]$. Also, let $n \geq F^r(m, \beta)$ and for all $i \in [r]$ let D_i be an (i,t+1)-insensitive subset of C_{t+1}^n. Set $D = D_1 \cap ... \cap D_r$ if $dens(D) \geq 2r\beta$, then there exists a family $\mathcal{V}$ of pairwise disjoint m-dimensional subspaces of C_{t+1}^n which are contained in D and have $dens(D \setminus \cup\mathcal{V}) < 2r\beta$.*

Proof. We shall prove this by induction on r, the $r = 1$ case is given by lemma 5.27.. So we shall assume that the result holds for up to r. We shall prove it holds for $r + 1$. Let $n \geq F^{(r+1)}(m, \beta)$ and let $D_1, ..., D_{r+1}$ be subsets of C_{t+1}^n

as described in the corollary. As $F^{(r+1)}(m,\beta) = F^{(r)}(F(m,\beta),\beta)$, by the induction hypothesis there exists a family, $\mathcal{V}_1$, of pairwise disjoint $F(m,\beta)$-dimensional subspaces of C^n_{t+1} which are all contained in $D' = D_1 \cap ... \cap D_r$ and we have $dens(D' \setminus \cup\mathcal{V}_1) < 2r\beta$. Let $\mathcal{V}_2 = \{V \in \mathcal{V}_1 : dens_V(D_{r+1}) \geq 2\beta\}$. For all $V \in \mathcal{V}_2$ let $\mathcal{B}_V$ be the set of m-dimensional subspaces of V which are given by applying lemma 5.27. to $V \cap D_{r+1}$. If we set $\mathcal{V} = \{W : V \in \mathcal{V}_2, W \in \mathcal{B}_V\}$, then $\mathcal{V}$ is a family of pairwise disjoint m-dimensional subspaces of C^n_{t+1} which are clearly all contained in $D = D_1 \cap ... \cap D_{r+1}$ as each $W \in \mathcal{B}_V$ is contained in $V \cap D_{r+1}$ and as $V \in \mathcal{V}_1$, V is contained in D'. We also have $dens(D \setminus \cup\mathcal{V}) < 2(r+1)\beta$ as required. □

We are finally ready to put everything together and complete the proof of proposition 5.18. and therefore complete the proof of the density Hales-Jewett theorem.

Proof. (Proposition 5.18.)
For every $0 < \delta \leq 1$ and integer $d \geq 1$, let $\beta = \frac{\gamma^2}{4t}$ and $m(d) = max\{M_0, F^{(t)}(d,\beta)\}$. Fix $n \geq N(t, d\delta) = n(GR(t, m(d)), \frac{\eta^2}{2})$ and fix a subset A of C^n_{t+1} with density at least δ. Assume that A contains no line of C^n_{t+1}. By corollary 5.26., with $m = m(d)$ here, there exists an $m(d)$-dimensional subspace W of C^n_{t+1} and a set $\{D_1, ..., D_t\}$ of subsets of W such that D_i is $(i, t+1)$-insensitive in W for all $i \in [t]$. Let $D = D_1 \cap ... \cap D_t$ and we have $dens_W(D) \geq \gamma$ and $dens_W(A \cap D) \geq (\delta + \gamma) dens_W(D)$.

We know that, $n \geq N(t,d,\delta) = n(GR(t,m(d)), \frac{\eta^2}{2}) = \frac{2}{\eta^2}(t+1)^{GR(t,m(d))} GR(t,m(d)) \geq \frac{2}{\eta^2}(t+1)^{GR(t,m(d))} m(d) \geq m(d) \geq F^{(t)}(d,\beta)$, and then by corollary 5.28., there exists a family $\mathcal{V}$ of pairwise disjoint d-dimensional subspaces that are contained in D and $dens_W(D \setminus \cup\mathcal{V}) < 2t\beta = \frac{\gamma^2}{2}$.

We know,

$$dens_W(A \cap D \setminus \cup\mathcal{V}) \leq dens_W(D \setminus \cup\mathcal{V}) < \frac{\gamma^2}{2} < \gamma \leq dens_W(D)$$

and using this and the other previous inequalities we get,

$$dens_W(A \cap \cup\mathcal{V}) = dens_W(A \cap D) - dens_W(A \cap (D \setminus \cup\mathcal{V})) \geq (\delta+\gamma) dens_W(D) - \frac{\gamma^2}{2}$$

$$\geq (\delta+\gamma) dens_W(D) - \frac{\gamma}{2} dens_W(D) = (\delta + \frac{\gamma}{2}) dens_W(D) \geq (\delta + \frac{\gamma}{2}) dens_W(\cup\mathcal{V})$$

Hence, there exists $V \in \mathcal{V}$ with,

$$dens_W(A \cap V) \geq (\delta + \frac{\gamma}{2}) dens_W(V) \implies \frac{dens_W(A \cap V)}{dens_W(V)} \geq \delta + \frac{\gamma}{2}$$

$$\Longrightarrow dens_V(A) \geq \delta + \frac{\gamma}{2}$$

as required. □

We have now essentially proved three important theorems with one clever proof. The Multidimensional Hales-Jewett theorem follows from the density version as we have seen and probably most importantly the Szemerédi's Theorem follows from this. In the next chapter we will take a look at the sister polymath project of the one we have just looked at.

6 Density Hales-Jewett numbers

The Polymath project split into two projects. The first project focused on proving the Density Hales-Jewett theorem whilst the other looked at Density Hales-Jewett and Moser numbers. We shall focus on finding exact numbers and bounds of Density Hales-Jewett numbers but we shall not look at Moser numbers unfortunately. We shall mainly follow the article 'Density Hales-Jewett and Moser Numbers' by Polymath [20], however we shall make use of Polymath's on line resources and our own input. We shall start by defining the Density Hales-Jewett number and giving some basic results.

Definition 6.1. The *density Hales-Jewett number*, denoted $c_{n,t}$, is the size of the largest subset of $C_t^n = \{(x_1, x_2, ..., x_n) : x_i \in \{1, ..., t\}\}$ which is line (as in definition 4.2.) free.

We can clearly see that we have the bound $c_{n,t} \leq t^n$, as C_t^n has t^n elements. We also know the trivial exact values, $c_{n,1} = 1$ for all n, as there is only one element in C_1^n. Furthermore, in section 5.1.2 we proved Sperner's theorem, this tells us that $c_{n,2} = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

We shall mainly focus on establishing upper and lower bounds for the non-trivial $t = 3$ case, and whilst doing so we shall find a few exact values for $c_{n,3}$.

6.1 Lower bounds for $c_{n,3}$

Polymath [20] use the idea of Fujimura sets in order to establish lower bounds for $c_{n,3}$. We shall start by introducing Fujimura sets and other notations which we take from [20] but we provide our own examples and explanations.

Definition 6.2. Let $\triangle_{n,t}$ be the set of all tuples $(a_1, ..., a_t)$ that sum to n, where a_i is non-negative for all i.

Example 6.3. For $n = 4$ and $t = 3$, we have:

$$\triangle_{4,3} = \{(4,0,0),(3,0,1),(3,1,0),(2,0,2),(2,2,0),(2,1,1),(1,0,3),(1,3,0)$$
$$(1,2,1),(1,1,2),(0,0,4),(0,4,0),(0,3,1),(0,1,3),(0,2,2)\}$$

Next, we introduce simplices.

Definition 6.4. A *simplex* is a set of t points from $\triangle_{n,t}$ of the form:

$$(a_1 + r, a_2, ..., a_t), (a_1, a_2 + r, ..., a_t), ..., (a_1, a_2, ..., a_t + r)$$

where $0 < r \leq n$ and $a_1 + a_2 + ... + a_t = n - r$.

Example 6.5. $\{(4,0,0),(2,2,0),(2,0,2)\}$ is an example of a simplex in $\triangle_{4,3}$, with $r=2$, $a_1=2$ and $a_2=a_3=0$.

We now have everything we need to define a Fujimura set.

Definition 6.6. A *Fujimura set* is a subset of $\triangle_{n,t}$ which contains no simplices.

Example 6.7. Following on from our previous examples, an example of a Fujimura set of $\triangle_{4,3}$ is,

$$B=\{(4,0,0),(3,0,1),(2,2,0),(2,1,1),(1,3,0),(1,1,2),(0,4,0),(0,1,3)\}$$

There are other sets that are also Fujimura sets of $\triangle_{4,3}$.

From a Fujimura set, B, of $\triangle_{n,t}$ we can create a line free subset of C_t^n. The set,

$$A_B=\cup_{\vec{a}\in B}\Gamma_{\vec{a}}$$

where the cell $\Gamma_{a_1,\ldots,a_t}$ is the set of elements in C_t^n which have exactly a_i i's, is line free as B contains no simplices. This is because for any line, $(b_{1,1},...,b_{1,n}),...,(b_{t,1},...,b_{t,n})$, in C_t^n we have $(b_{1,1},...,b_{1,n})\in\Gamma_{a_1+r,a_2,\ldots,a_t}$, ..., $(b_{t,1},...,b_{t,n})\in\Gamma_{a_1,a_2,\ldots,a_t+r}$.

Example 6.8. In our example, our line free set found from B would be,

$$A_B=\{(1,1,1,1)\}\cup\{(1,1,1,3),(1,1,3,1),(1,3,1,1),(3,1,1,1)\}\cup...$$

$$\cup\{(2,3,3,3),(3,2,3,3),(3,3,2,3),(3,3,3,2)\}$$

In summary, A_B contains all the elements with four 1's, all the elements with three 1's and one 3 etc.

The number of elements in $\Gamma_{a_1,\ldots,a_t}$ is equal to $\frac{n!}{a_1!a_2!...a_t!}$ and so we get the following lower bound:

$$c_{n,t}\geq|A_B|=\sum_{(a_1,a_2,\ldots,a_t)\in B}\frac{n!}{a_1!a_2!...a_t!}$$

where B is a Fujimura set of $\triangle_{n,t}$. In our example above A_B has 44 elements. We therefore know that $c_{4,3}\geq 44$ as we have found a set of size 44 which is line free. However, if we can build a Fujimura set such that A_B is larger then we can improve upon this bound.

Our aim is now to find ways to build Fujimura sets of $\triangle_{n,3}$ such that A_B is as large as possible. Therefore we get the best lower bounds for $c_{n,3}$ that we can get from this method.

Polymath [20] start by looking at the sets,

$$B_{j,n} = \{(a,b,c) \in \triangle_{n,3} : a + 2b \neq j \pmod 3\}$$

for $j = 0, 1, 2$. We shall use this starting point also but we shall include our own examples and explanations. Below is an example of the set $B_{0,4}$.

Example 6.9. When $n = 4$ and $j = 0$ we get the set,

$$B_{0,4} = \{(4,0,0), (3,1,0), (2,0,2), (2,1,1), (1,0,3), (1,3,0), (1,2,1), (0,4,0),$$
$$(0,1,3), (0,2,2)\}$$

For example $(4,0,0)$ is in the set $B_{0,4}$ because $a+2b = 4+2\times 0 = 4 = 1 \neq 0 \pmod 3$.

We first observe that if a simplex, $(a+r,b,c), (a,b+r,c)$ and $(a,b,c+r)$ lies in $B_{j,n}$ then we have,

$$a + r + 2b \neq j \pmod 3$$

$$a + 2b + 2r \neq j \pmod 3$$

$$a + 2b \neq j \pmod 3$$

therefore r must be a multiple of 3. So if $n < 3$, there can be no simplices in $B_{j,n}$ and therefore it is a Fujimura set. Also, if $n = 3$ then $B_{0,3}$ is a Fujimura set as the only simplex that could be in $B_{0,3}$ is the one where $r = 3$ but the elements $(3,0,0), (0,3,0)$ and $(0,0,3)$ are not in $B_{0,3}$.

We shall make a further two important observations about the sets $B_{j,n}$. These are given in [20] but no proof was given so we shall give our own proof for them.

Proposition 6.10. *When n is not a multiple of 3, the sets $B_{j,n}$ are rotations of each other. Furthermore, they each give sets $A_{B_{j,n}}$ of size* 2.3^{n-1}.

Proof. If $(a,b,c) \in B_{0,n}$ then by definition $a + 2b \neq 0 \pmod 3$. We shall look at which sets the rotations of (a,b,c) are in. As n is not a multiple of 3, we have two possible cases either $n = 1 \pmod 3$ or $n = 2 \pmod 3$. We shall look at each case separately.

1) $n = 1 \pmod 3$:
For (c,a,b) we know $c + 2a = n - b - a + 2a$ because $n = a + b + c$. So $c+2a = n+a-b = n+a+2b \pmod 3$. Hence, as $a+2b = 1$ or $2 \pmod 3$, we have $c + 2a = n + a + 2b = 2$ or $0 \pmod 3$. Therefore, $(c,a,b) \in B_{1,n}$.

Next for (b,c,a), we have $b+2c = b+2n-2a-2b = 2n-b-2a = 2n+a+2b \pmod 3$. As $a+2b = 1$ or $2 \pmod 3$, we have $b+2c = 2n+a+2b = 0$ or $1 \pmod 3$. Hence, $(b,c,a) \in B_{2,n}$.
Therefore, if $n = 1 \pmod 3$ then the sets are rotations of each other.

2) $n = 2 \pmod 3$:
For (c,a,b) we know $c+2a = n-b-a+2a$ because $n = a+b+c$. So $c+2a = n+a-b = n+a+2b \pmod 3$. Hence, as $a+2b = 1$ or $2 \pmod 3$, we have $c+2a = n+a+2b = 0$ or $1 \pmod 3$. Therefore, $(c,a,b) \in B_{2,n}$.
Next for (b,c,a), we have $b+2c = b+2n-2a-2b = 2n-b-2a = 2n+a+2b \pmod 3$. As $a+2b = 1$ or $2 \pmod 3$, we have $b+2c = 2n+a+2b = 2$ or $0 \pmod 3$. Hence, $(b,c,a) \in B_{1,n}$.
Therefore, if $n = 2 \pmod 3$ then the sets are rotations of each other.

Finally, the sets $B_{j,n}$ are rotations of each other therefore the corresponding $A_{B_{j,n}}$ will have 2.3^{n-1} number of elements. This is because there are 3^n elements in C_3^n and each $x \in \triangle_{n,3}$ is clearly in two of the three sets $B_{j,n}$ and as these sets are rotations we therefore have 2.3^{n-1} elements in each of $A_{B_{j,n}}$. $\square$

We shall provide an example to demonstrate this proposition.

Example 6.11. For $n = 4$, we have $n = 1 \pmod 3$ so if $(a,b,c) \in B_{0,4}$ then $(c,a,b) \in B_{1,4}$ and $(b,c,a) \in B_{2,4}$ Therefore the $B_{j,4}$ for $j = 0,1,2$ are as follows:

$$B_{0,4} = \{(4,0,0),(3,1,0),(2,0,2),(2,1,1),(1,0,3),(1,3,0),(1,2,1),(0,4,0),$$
$$(0,1,3),(0,2,2)\}$$
$$B_{1,4} = \{(0,4,0),(0,3,1),(2,2,0),(1,2,1),(3,1,0),(0,1,3),(1,1,2),(0,0,4),$$
$$(3,0,1),(2,0,2)\}$$
$$B_{2,4} = \{(0,0,4),(1,0,3),(0,2,2),(1,1,2),(0,3,1),(3,0,1),(2,1,1),(4,0,0),$$
$$(1,3,0),(2,2,0)\}$$

The next proposition concerns the cases when n is a multiple of 3.

Proposition 6.12. *When n is a multiple of 3, $B_{1,n}$ and $B_{2,n}$ are reflections of each other but $B_{0,n}$ is not in any way equivalent to $B_{1,n}$ and $B_{2,n}$.*

Proof. If $(a,b,c) \in B_{1,n}$ then we know $a+2b \neq 1 \pmod 3$. We also know $n = 0 \pmod 3$ as n is a multiple of 3 and we know that $a+b+c=n$. We can use these facts to show that the reflection (c,b,a) of (a,b,c) is in the set $B_{2,n}$. We have,

$$c+2b = n-b-a+2b = n+b-a = n+b+2a = n+(a+2b)+(a-b) = (a+2b)+(a+2b) \pmod 3$$

So if $a+2b = 0 \pmod 3$, then $c+2b = 0 \pmod 3$ and if $a+2b = 2 \pmod 3$, then $c+2b = 1 \pmod 3$. Therefore $(c,b,a) \in B_{2,n}$ if $(a,b,c) \in B_{1,n}$. Hence $B_{1,n}$ and $B_{2,n}$ are reflections of each other.

$B_{0,n}$ is not equivalent to the other two sets as for example $B_{0,n}$ does not contain any of the the corners of $\triangle_{n,3}$ (the corners are $(n,0,0)$, $(0,n,0)$ and $(0,0,n)$). □

We shall again provide an example to demonstrate this proposition.

Example 6.13. For $n=3$, the sets $B_{j,3}$ are as follows:

$$B_{0,3} = \{(2,1,0),(2,0,1),(1,2,0),(1,0,2),(0,1,2),(0,2,1)\}$$

$$B_{1,3} = \{(3,0,0),(2,0,1),(1,2,0),(1,1,1),(0,3,0),(0,0,3),(0,1,2)\}$$

$$B_{2,3} = \{(0,0,3),(1,0,2),(0,2,1),(1,1,1),(0,3,0),(3,0,0),(2,1,0)\}$$

We can clearly see in this example that $B_{1,3}$ and $B_{2,3}$ are reflections of each other but $B_{0,3}$ has no relationship with the other two sets.

So when n is not a multiple of 3 the sets $A_{B_{j,n}}$ for $j = 0,1,2$ are all the same size so we can work with any of the three sets. However that is not the case when n is a multiple of 3. We need to work out if it is best to work with $B_{0,n}$ or one of the other sets. Polymath [20] tell us that in fact the set $A_{B_{0,n}}$ is slightly larger than $A_{B_{1,n}}$ and $A_{B_{2,n}}$, when n is a multiple of 3. Polymath [20] however give no explanation of this fact so we shall provide one. We can see this by looking at table 4 which shows how many elements, with the first number in the triple as indicated in the left hand column, that is in $B_{j,n}$ for $j = 0,1,2$.

The results follows a pattern as we go down the table. Firstly, $B_{1,n}$ and $B_{2,n}$ have one more element than $B_{0,n}$, then $B_{0,n}$ has one more element than the other two and then they all have the same number of elements. This then repeats again. So $B_{1,n}$ and $B_{2,n}$ will always have the same number of elements but $B_{0,n}$ will have one less, one more or the same number of elements than the other two. However, as $B_{0,n}$ does not contain $(n,0,0)$, $(0,n,0)$ or $(0,0,n)$

(a,b,c)	$B_{0,n}$	$B_{1,n}$	$B_{2,n}$
$(n,-,-)$	0	1	1
$(n-1,-,-)$	2	1	1
$(n-2,-,-)$	2	2	2
$(n-3,-,-)$	2	3	3
$(n-4,-,-)$	4	3	3
$(n-5,-,-)$	4	4	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4: A table showing how many elements of a certain form, given in the first column, that are in the sets $B_{0,n}$, $B_{1,n}$ and $B_{2,n}$.

but they are contained in both the other sets, $A_{B_{0,n}}$ will be slightly larger than $A_{B_{1,n}}$ and $A_{B_{2,n}}$. We shall therefore work with $B_{0,n}$ from now on.

As we discussed earlier $B_{0,n}$ is a Fujimura set for $n \leq 3$, hence we can find the following lower bounds:

$$B_{0,1} = \{(1,0,0),(0,1,0)\} \Rightarrow |A_{B_{0,1}}| = 2 \Rightarrow c_{1,3} \geq 2$$

$$B_{0,2} = \{(2,0,0),(1,0,1),(0,2,0),(0,1,1)\} \Rightarrow |A_{B_{0,2}}| = 6 \Rightarrow c_{2,3} \geq 2$$

$$B_{0,3} = \{(2,1,0),(2,0,1),(1,2,0),(1,0,2),(0,1,2),(0,2,1)\} \Rightarrow |A_{B_{0,3}}| = 18$$
$$\Rightarrow c_{3,3} \geq 18$$

For $n > 3$, $B_{0,n}$ is not a Fujimura set, we can however remove points from the set to ensure there are no three points of the form $(a+r,b,c),(a,b+r,c)$ and $(a,b,c+r)$ and therefore making the new set a Fujimura set.

For $3 < n \leq 6$ the only simplices in $B_{0,n}$ is when $r = 3$. This is because we know that r has to be a multiple of 3 and $r \leq n$. So for $3 < n < 6$, $r = 3$ is the only possible r and when $n = 6$ if $r = 6$ then our simplex would be $(6,0,0),(0,6,0),(0,0,6)$ but none of these three elements are in $B_{0,6}$.

For $3 < n \leq 6$, we shall manually find all the simplices in $B_{0,n}$ and remove one point from each in such a way that we maximise the size of the line free set. Polymath [20] give the resulting sets from doing this but provide no further information and have some mistakes in their results.

6.1.1 Lower bounds for $c_{n,3}$ for $4 \leq n \leq 6$

We shall start with $n = 4$.

6.1.1.1 $n = 4$

For $n = 4$ the only possible simplices have $r = 3$, so a simplex is of the form $(a + 3, b, c)$, $(a, b + 3, c)$ and $(a, b, c + 3)$, where $a + b + c = 4 - 3 = 1$. We have three different possible simplices that could be in $B_{0,4}$, which are:

$$(4, 0, 0), (1, 3, 0), (1, 0, 3)$$

$$(3, 1, 0), (0, 4, 0), (0, 1, 3)$$

$$(3, 0, 1), (0, 3, 1), (0, 0, 4)$$

All the elements in the first two lines are in $B_{0,4}$ so we shall remove one from each. From the first simplex we shall remove $(4, 0, 0)$ as this only gives us one element in the line free set. Similarly, for the second simplex we shall remove $(0, 4, 0)$. The third possible simplex is not in $B_{0,4}$. Our Fujimura set is therefore,

$$B = B_{0,4} \backslash \{(4, 0, 0), (0, 4, 0)\} = \{(3, 1, 0), (2, 0, 2), (2, 1, 1), (1, 0, 3), (1, 3, 0),$$

$$(1, 2, 1), (0, 1, 3), (0, 2, 2)\}$$

Our resulting line free subset of C_3^4 is $A_B = \cup_{(a,b,c)\in B} \Gamma_{(a,b,c)}$ (we remember that here $\Gamma_{(a,b,c)}$ is the set of all elements in C_3^4 with exactly a 1's, b 2's and c 3's). The size of A_B is equal to $\sum_{(a,b,c)\in B} \frac{n!}{a!b!c!} = 52$. Hence, $c_{4,3} \geq 52$.

6.1.1.2 $n = 5$

For $n = 5$, the possible simplices in $B_{0,5}$ have the form $(a+3, b, c)$, $(a, b+3, c)$ and $(a, b, c + 3)$ where $a + b + c = 5 - 3 = 2$. There are six different possible combinations of a, b and c so we have six possible simplices that could be in $B_{0,5}$. These are,

$$(5, 0, 0), (2, 3, 0), (2, 0, 3)$$

$$(3, 2, 0), (0, 5, 0), (0, 2, 3)$$

$$(3, 0, 2), (0, 3, 2), (0, 0, 5)$$

$$(4, 1, 0), (1, 4, 0), (1, 1, 3)$$

$$(3, 1, 1), (0, 4, 1), (0, 1, 4)$$

$$(4, 0, 1), (1, 3, 1), (1, 0, 4)$$

The third and fourth of the above simplices are not in $B_{0,5}$ so we do not need to worry about them. As for the $n = 4$ case we shall remove one element from each simplex in such a way we maximise the line free set. For the first simplex we remove $(5,0,0)$, for the second we shall remove $(0,5,0)$, for the fifth simplex we shall remove $(0,4,1)$ (note we could have removed $(0,1,4)$ instead, it will make no difference) and finally for the sixth simplex we shall remove $(4,0,1)$ (again we could of removed $(1,0,4)$ instead). So our Fujimura set is as follows,

$$B = B_{0,5} \backslash \{(5,0,0),(0,5,0),(0,4,1),(4,0,1)\} = \{(3,2,0),(3,1,1),(2,3,0),$$
$$(2,0,3),(2,1,2),(1,0,4),(1,3,1),(1,2,2),(0,1,4),(0,2,3)\}$$

Our resulting line free subset of C_3^5 is $A_B = \cup_{(a,b,c)\in B}\Gamma_{(a,b,c)}$ ($\Gamma_{(a,b,c)}$ is the set of all elements in C_3^5 with exactly a 1's, b 2's and c 3's). The size of A_B is equal to $\sum_{(a,b,c)\in B} \frac{n!}{a!b!c!} = 150$. Hence, $c_{5,3} \geq 150$.

6.1.1.3 $n = 6$

For $n = 6$, the only simplices that could be in $B_{0,6}$ are those of the form $(a+3,b,c)$, $(a,b+3,c)$ and $(a,b,c+3)$ where $a+b+c = 6-3 = 3$. There are ten different possible a, b and c that sum to 3. However we can immediately see that if one of a, b or c is equal to 3 and the other two are zero then the resulting simplex will not be in $B_{0,6}$. We then have seven possible simplices left, which are:

$$(5,1,0),(2,4,0),(2,1,3)$$
$$(5,0,1),(2,3,1),(2,0,4)$$
$$(4,2,0),(1,5,0),(1,2,3)$$
$$(4,0,2),(1,3,2),(1,0,5)$$
$$(3,2,1),(0,5,1),(0,2,4)$$
$$(3,1,2),(0,4,2),(0,1,5)$$
$$(4,1,1),(1,4,1),(1,1,4)$$

The last one is not in $B_{0,6}$ so we do not need to worry about it. We shall again remove one from each simplex. We shall remove $(5,1,0)$, $(5,0,1)$, $(1,5,0)$, $(1,0,5)$, $(0,5,1)$ and $(0,1,5)$. There was a slight mistake in Polymath's [20] paper as they missed $(5,0,1)$ out. If we remove these elements from $B_{0,6}$ we get a Fujimura set which is,

$$B = B_{0,6} \backslash \{(5,1,0),(5,0,1),(1,5,0),(1,0,5),(0,5,1),(0,1,5)\} = \{(4,2,0),$$
$$(4,0,2),(3,2,1),(3,1,2),(2,4,0),(2,0,4),(2,3,1),(2,1,3),(1,2,3),(1,3,2),$$
$$(0,4,2),(0,2,4)\}$$

Our resulting line free subset of C_3^6 is $A_B = \cup_{(a,b,c)\in B}\Gamma_{(a,b,c)}$. The size of A_B is equal to $\sum_{(a,b,c)\in B} \frac{n!}{a!b!c!} = 450$. Hence, $c_{6,3} \geq 450$.

6.1.2 Lower bounds for $c_{n,3}$ for $n > 6$

In summary, we have the following lower bounds for $c_{n,3}$ so far,

$$c_{1,3} \geq 2; c_{2,3} \geq 6; c_{3,3} \geq 18; c_{4,3} \geq 52; c_{5,3} \geq 150; c_{6,3} \geq 450$$

We can carry on finding the simplices that are in $B_{0,n}$ by hand and removing one element from each simplex (the one that contributes the smallest number of elements in the line free set). However, this is very time consuming especially as n gets larger. We shall therefore use Python to do this for us. First of all we tell Python to calculate the set $B_{0,n}$, then find all simplices in $B_{0,n}$ and remove one element from each simplex in such a way we maximise A_B (our line free subset in C_3^n). We shall then get Python to calculate the size of the line free set A_B. The code for this is given in appendix A of the paper and the results from this method is shown in the table 5.

n	Lower bound for $c_{n,3}$	n	Lower bound for $c_{n,3}$
1	2	11	95832
2	6	12	287496
3	18	13	834834
4	52	14	2445300
5	150	15	7335900
6	450	16	21359052
7	1302	17	62748972
8	3780	18	188246916
9	11340	19	549430752
10	32844	20	1617908292

Table 5: Lower bounds for $c_{n,3}$

For $1 \leq n \leq 9$ the lower bounds in the above table are the best known lower bounds for $c_{n,3}$. Polymath do publish better lower bounds for n greater than 9 in [20] but they provide little information on how they came to these results. We shall try improve upon and extend our results in table 5 further.

Polymath [20] point out a simplification they found when n is a multiple of 3 and also for when n is not a multiple of 3. We shall discuss these simplifications and give an explanation to why they are true, which Polymath do not provide.

6.1.2.1 A simplification when n is a multiple of 3

When n is a multiple of 3, if we know $B_{0,n}$ we can quickly find the set $B_{0,n+3}$. The set $B_{0,n+3}$ contains all the elements in $B_{0,n}$ but with one added to each (i.e. if (a,b,c) is in $B_{0,n}$ then $(a+1,b+1,c+1)$ is in $B_{0,n+3}$) and then new elements with a zero in. Also, if a set F contains no simplices then the set $G = \{(a+1,b+1,c+1) : (a,b,c) \in F\}$ also contains no simplices. To see this we shall look for a contradiction. Assume there exists a simplex $(a+r,b,c)$, $(a,b+r,c)$ and $(a,b,c+r)$ in the set G then $(a+r-1,b-1,c-1)$, $(a-1,b+r-1,c-1)$ and $(a-1,b-1,c+r-1)$ is a simplex in the set F, a contradiction.

We shall use these facts to find which elements of $B_{0,n}$ for $n = 9, 12, 15, 18$ (Polymath do not include any information at all for $n = 18$) we can use to make a Fujimura set which should hopefully improve upon our previous lower bounds in table 5.

$\mathbf{n = 9}$

Firstly, let us look back on the work we did for $n = 6$. Notice that the elements we removed from $B_{0,6}$ to create a Fujimura set are $(5,1,0)$ and all of it's permutations. We also note the elements in the Fujimura set are $(4,2,0)$ and $(3,2,1)$ and permutations. If we add one to $(4,2,0)$ and $(3,2,1)$ we get $(5,3,1)$ and $(4,3,2)$. The set of all the permutations of $(5,3,1)$ and $(4,3,2)$ is a subset of $B_{0,9}$ and it contains no simplices (as shown above).

To build a Fujimura set for $n = 9$, as well as adding one to all elements of the Fujimura set for $n = 6$, we can add new elements that contain a zero from $B_{0,9}$. We have $(0,8,1)$, $(0,7,2)$, $(0,5,4)$ and permutations in $B_{0,9}$ which contain a zero. We just need to decide which ones we can add to the set of all the permutations of $(5,3,1)$ and $(4,3,2)$ so that the set contains no simplices and is such that the line free set is as large as possible. We cannot have $(0,5,4)$ and its permutations alongside $(0,7,2)$ and its permutations as you can form simplices between some combination of permutations of $(0,5,4)$, $(0,7,2)$ and $(4,3,2)$. For example $(0,5,4)$, $(3,2,4)$ and $(0,2,7)$ is a simplex. Also we cannot have $(0,5,4)$ and its permutations alongside $(0,8,1)$ and its permutations as you can form simplices between some combination of permutations of $(0,5,4)$, $(0,8,1)$ and $(5,3,1)$. For example $(0,5,4)$, $(0,8,1)$ and $(3,5,1)$ is a simplex. We can however have $(0,8,1)$, $(0,7,2)$ and permutations together alongside $(4,3,2)$ and $(5,3,1)$ and permutations. We however, get a larger resulting line free set if we include $(0,5,4)$ and permutations with all the permutations of $(5,3,1)$ and $(4,3,2)$. Therefore our Fujimura set is

$$B = \{(5,3,1), (4,3,2), (0,5,4) \text{ and permutations}\}$$

Note this is $B_{0,9}$ with $(6,2,1)$, $(0,8,1)$, $(0,7,2)$ and permutations removed. The size of the line free subset, A_B, of C_3^9 is equal to 11340 (which is the same as our lower bound found before).

$\mathbf{n = 12}$

We can use the same method to find which elements of $B_{0,12}$ form a Fujimura set in hope that we may improve our previous lower bound for $n = 12$. So we add one onto each element in the Fujimura set found above for $n = 9$. We get $(6, 4, 2)$, $(5, 4, 3)$, $(1, 6, 5)$ and permutations. We then need to include elements of $B_{0,12}$ with zeros in. These are $(0, 11, 1)$, $(0, 10, 2)$, $(0, 8, 4)$, $(0, 7, 5)$ and permutations. We just need to find which ones we can add to the set of $(6, 4, 2)$, $(5, 4, 3)$, $(1, 6, 5)$ and permutations such that there are no simplices in the set and such that the resulting line free set is as large as possible. The elements, $(0, 7, 5)$ and permutations would give 4752 elements in the line free set which is more than the sum of what the other three and permutations would give. Therefore, we shall include $(0, 7, 5)$ and permutations in our Fujimura set. We can also add $(0, 10, 2)$ and permutations to $(0, 7, 5)$, $(6, 4, 2)$, $(5, 4, 3)$, $(1, 6, 5)$ and permutations without forming a simplex but we cannot add either of the other two. Our resulting Fujimura set is,

$$B = \{(6, 4, 2), (5, 4, 3), (1, 6, 5), (0, 7, 5), (0, 10, 2) \text{ and permutations}\}$$

The size of the line free subset A_B of C_3^{12} is 287892. So $c_{12,3} \leq 287892$, this improves upon our previous bound by 396.

$\mathbf{n = 15}$

For $n = 15$ we shall add one to all the elements in the Fujimura set found for $n = 12$ above. We get $(7, 5, 3)$, $(6, 5, 4)$, $(2, 7, 6)$, $(1, 8, 6)$, $(1, 11, 3)$ and permutations. We now need to choose which of the elements in $B_{0,15}$ with zeros in we shall include also in our Fujimura set. Our options are $(0, 14, 1)$, $(0, 13, 2)$, $(0, 11, 4)$, $(0, 10, 5)$, $(0, 8, 7)$ and permutations. The elements, $(0, 8, 7)$ and permutations will contribute 38610 elements to the line free set which is again greater than all the others put together. So we shall definitely include $(0, 8, 7)$ and permutations in our Fujimura set. The only other elements with a zero in that we could add to $(7, 5, 3)$, $(6, 5, 4)$, $(2, 7, 6)$, $(1, 8, 6)$, $(1, 11, 3)$, $(0, 8, 7)$ and permutations to ensure there are no simplices are $(0, 11, 4)$ and its permutations. Our resulting Fujimura set is therefore,

$$B = \{(7, 5, 3), (6, 5, 4), (2, 7, 6), (1, 8, 6), (1, 11, 3), (0, 8, 7), (0, 11, 4)$$
$$\text{and permutations}\}$$

The size of the line free subset, A_B, of C_3^{15} is 7376850. So $c_{15,3} \leq 7376850$, this improves our previous bound by 40950.

$\mathbf{n = 18}$

Finally, for $n = 18$ we shall add one to all the elements in the Fujimura set found for $n = 15$ above. We get $(8, 6, 4)$, $(7, 6, 5)$, $(3, 8, 7)$, $(2, 9, 7)$, $(2, 12, 4)$, $(1, 9, 8)$, $(1, 12, 5)$ and permutations. We now need to choose which of $(0, 17, 1)$, $(0, 16, 2)$, $(0, 14, 4)$, $(0, 13, 5)$, $(0, 11, 7)$ and $(0, 10, 8)$ we shall add to create a Fujimura set such that the resulting line free set is as large as possible. Again the number of elements $(0, 10, 8)$ and its permutations will give for our line free set is more than the sum of all the others put together. So we shall definitely include $(0, 10, 8)$. If we have the set of all permutations of $(8, 6, 4)$, $(7, 6, 5)$, $(3, 8, 7)$, $(2, 9, 7)$, $(2, 12, 4)$, $(1, 9, 8)$, $(1, 12, 5)$ and $(0, 10, 8)$ then we can also include $(0, 13, 5)$ and $(0, 16, 2)$ and still keep the Fujimura property. Our Fujimura set is,

$$B = \{(8,6,4),(7,6,5),(3,8,7),(2,9,7),(2,12,4),(1,9,8),(1,12,5),(0,10,8),$$
$$(0,13,5),(0,16,2) \text{ and permutations}\}$$

The size of the line free subset, A_B, of C_3^{18} is 190638306. So $c_{18,3} \leq 190638306$, this improves our previous bound by 2391390.

Generalisation

We can generalise this simplification. When we go from $n = 3m$ to $n = 3(m+1)$ we add one onto all elements and then include new elements with zeros in. Polymath [20, p 10], give a general formula for this. If $n = 3m$, then the Fujimura set is given by the following elements and all their permutations:

$$(-7+m,-3+m,10+m),(-7+m,m,7+m),(-7+m,3+m,4+m),$$
$$(-6+m,-4+m,10+m),(-6+m,-1+m,7+m),(-6+m,2+m,4+m),$$
$$(-5+m,-1+m,6+m),(-5+m,2+m,3+m),(-4+m,-2+m,6+m),$$
$$(-4+m,1+m,3+m),(-3+m,1+m,2+m),(-2+m,m,2+m),$$
$$(-1+m,m,1+m)$$

And for $x \geq 0$ and $y = 0, 1$

$$(-8-y-2x+m,-6+y-2x+m,14+4x+m),(-8-y-2x+m,-3+y-2x+m,11+4x+m),$$
$$(-8-y-2x+m,y+x+m,8+x+m),(-8-2x+m,3+x+m,5+x+m)$$

This then gives us a Fujimura set for $n = 3m$ (ignoring any that have a negative number in). For larger n these Fujimura sets can be time consuming

n	Lower bound for $c_{n,3}$	n	Lower bound for $c_{n,3}$
3	18	33	2.38842074523E+15
6	450	36	6.33857586458E+16
9	11340	39	1.68533359471E+18
12	287892	42	4.48787358958E+19
15	7376850	45	1.19656848219E+21
18	190638306	48	3.19362786072E+22
21	4962044826	51	8.53115912807E+23
24	1.29909551418E+11	54	2.28060881535E+25
27	3.41696938401E+12	57	6.10050759146E+26
30	9.02072683291E+13	60	1.63273507586E+28

Table 6: Lower bounds (found from the simplification discussed) for $c_{n,3}$ when n is a multiple of 3.

to create so we shall use Python to create the Fujimura set from the above generalisation and then find the size of the corresponding line free set. Table 6 gives the set of results of lower bounds for $c_{n,3}$ (n a multiple of 3) which were found using the Python code as described above and which can be seen in the appendix.

These are the best lower bounds known for $n = 3, 6, 9, 12, 21, 24$. Up to date values can be found from a link on [18].

6.1.2.2 A simplification when n is not a multiple of 3

When n is not a multiple of three, say $n = 3m - 1$ or $n = 3m - 2$ for some m, if we first find a lower bound for $n = 3m$ we can then easily find a lower bound for $n = 3m - 1$ or $n = 3m - 2$. Polymath [20] briefly mention this simplification but give little further information and explanations. We shall start by looking at the $n = 3m - 1$ case.

<u>n = 3m − 1</u>

Let B be a Fujimura set for $n = 3m$. In the line free set A_B if we take all the elements that start with a 1 and remove this 1 from them, we then have a set of elements in C_3^{3m-1}. Furthermore, as A_B is a line free set our new set in C_3^{3m-1} is line free also. We saw earlier that when n is a multiple of 3 our Fujimura set contains elements (a, b, c) and all their permutations, therefore the number of elements in A_B that begin with a 1 is exactly $\frac{1}{3}$. So $c_{3m-1,3} \geq \frac{c_{3m,3}}{3}$. We shall see a quick example for when $m = 1$.

Example 6.14. We found earlier that for $n = 6$ our Fujimura set, B, is the set, $(4, 2, 0)$, $(3, 2, 1)$ and permutations. We found the set A_B has 450 elements. If we take all the elements in A_B that begin with a one (for example $(1, 1, 1, 1, 2, 2)$) and remove the first one (for example $(1, 1, 1, 1, 2, 2)$ becomes $(1, 1, 1, 2, 2)$) then we have a line free set in C_3^5. We have one third as many elements in this new set for C_3^5 than we do for A_B hence it has 150 elements. We can also find the corresponding Fujimura set for $n = 5$ which is $C = \{(a, b, c) : (a + 1, b, c) \in B, a \geq 0\}$.

<u>$\mathbf{n = 3m - 2}$</u>

We can do a similar thing for when $n = 3m - 2$. If B is a Fujimura set for $n = 3m$. If we take all the elements in the line free set A_B that start with 1,2, then we get a line free subset of C_3^{3m-2} (because A_B is a line free set). Roughly $\frac{1}{9}$ of the elements in A_B begin with 1,2 because $\frac{1}{3}$ of A_B begin with a 1 and $\frac{1}{3}$ have a 2 in the second position so there are roughly $\frac{1}{9}$ as many. Hence we get a lower bound for $c_{3m-2,3}$ that is roughly $\frac{1}{9}$ the size of $c_{3m,3}$.

6.1.2.3 Resulting lower bounds from these simplifications

Using these two simplifications means we can improve upon some of our lower bounds from earlier and we can also use these to get some new lower bounds too. We can alter our Python code that created our lower bounds for $c_{3m,3}$ to also give us our new bounds from these simplifications. This code can be seen in the appendix. Table 7 gives us the lower bounds for $1 \leq n \leq 20$ found from this method. See appendix for a larger table of results.

n	Lower bound for $c_{n,3}$	n	Lower bound for $c_{n,3}$
1	2	11	95964
2	6	12	287892
3	18	13	837850
4	52	14	2458950
5	150	15	7376850
6	450	16	21564380
7	1302	17	63546102
8	3780	18	190638306
9	11340	19	559502880
10	32864	20	1654014942

Table 7: Lower bounds for $c_{n,3}$ found from the simplifications discussed.

This is as far as we shall go in regards to improving these lower bounds but table 8 shows the best known lower bounds known so far which are taken from [18] (a fuller table can be found in the appendix).

n	Lower bound for $c_{n,3}$	n	Lower bound for $c_{n,3}$
1	2	11	96338
2	6	12	287892
3	18	13	854139
4	52	14	2537821
5	150	15	7528835
6	450	16	22517082
7	1302	17	66944301
8	3780	18	198629224
9	11340	19	593911730
10	32864	20	1766894722

Table 8: Best known lower bounds for $c_{n,3}$

The rest of the paper will now focus on finding upper bounds in hope of finding a few actual values for $c_{n,3}$.

6.2 Upper bounds and exact values for $c_{n,3}$

We can straight away give the exact values for $n = 0, 1$, both these values are trivial. For $n = 0$ we clearly have $c_{0,3} = 1$. For $n = 1$, there are three elements in C_3^1 which are (1), (2) and (3). If we take any two of these we have a line free set, but all three of them together form a line. Hence $c_{1,3} = 2$.

The next theorem will allow us to deduce the next few upper bounds. This is taken from [20] but we shall provide our own proof for this.

Theorem 6.15. *For all n,*

$$c_{n+1,3} \leq 3c_{n,3}$$

Proof. We can write C_3^{n+1} as the union of three copies of C_3^n as follows

$$C_3^{n+1} = \{(x_1, x_2, ..., x_n, 1) : (x_1, x_2, ..., x_n) \in C_3^n\} \cup \{(x_1, x_2, ..., x_n, 2) : (x_1, x_2, ..., x_n) \in C_3^n\}$$
$$\cup \{(x_1, x_2, ..., x_n, 3) : (x_1, x_2, ..., x_n) \in C_3^n\}$$

Therefore, we get $c_{n+1,3} \leq 3c_{n,3}$. □

Now using theorem 6.15. and the fact that $c_{1,3} = 2$ we can find an upper bound for $c_{2,3}$. We get

$$c_{2,3} \leq 3c_{1,3} = 6$$

as we found the lower bound $c_{2,3} \geq 6$ in the last section we can deduce that $c_{2,3} = 6$.

Similarly for $n = 3$ using theorem 6.15. we get,

$$c_{3,3} \leq 3c_{2,3} = 18$$

and in the last section we found that $c_{3,3} \geq 18$, hence we know $c_{3,3} = 18$. We can keep doing this, for example $c_{4,3} \leq 3c_{3,3} = 54$ but we will get no good enough bounds to give us any more exact values.

In the rest of this paper we shall establish the exact values for when $n = 4$, give a brief account how we would deduce the exact value when $n = 5$ and then from this how we find the exact value for $n = 6$. We shall start with $n = 4$.

6.2.1 Upper bound for $c_{4,3}$

We follow the method from [20] in order to find the exact value for $c_{4,3}$ but we shall add a lot of extra explanations. In order to get the best upper bound we can get for $c_{4,3}$, we first need to find all the line free subsets of C_3^2 which are of size 6 ($c_{2,3} = 6$).

Lemma 6.16. *There are only four different line free subsets of size 6 of C_3^2 which are as follows,*

$$x = A_{B_{2,2}} = \{(1,2),(1,3),(2,1),(2,2),(3,1),(3,3)\}$$

$$y = A_{B_{2,1}} = \{(1,1),(1,2),(2,1),(2,3),(3,2),(3,3)\}$$

$$z = A_{B_{2,0}} = \{(1,1),(1,3),(2,2),(2,3),(3,1),(3,2)\}$$

$$w = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$$

Proof. We have 9 elements in C_3^2, and 84 subsets of size 6. We find the above line free sets by checking through all 84 subsets of size 6. We however know from the previous section that the sets $A_{B_{2,n}}$ for $n = 0, 1, 2$ will be line free subsets for C_3^2. □

We can now use this to find line free subsets of C_3^3 which are close to the maximal size of $c_{3,3} = 18$. We first note that we can split C_3^3 into three subsets. The three subsets are, all the elements of the form $(1, *, *)$, all the elements of the form $(2, *, *)$ and all the elements of the form $(3, *, *)$, where

$(*,*) \in C_3^2$. In the same way we can split a subset $D \subseteq C_3^3$ into three subsets D_1, D_2 and D_3 (where D_i is the $(*,*)$ part of all the elements of the form $(i,*,*) \in D$) of C_3^2. Let us see a quick example of this notation.

Example 6.17. Let $D = \{(1,1,2),(2,2,2),(1,3,2),(3,2,1)\} \in C_3^3$. Then our corresponding subsets of C_3^2 are $D_1 = \{(1,2),(3,2)\}$, $D_2 = \{(2,2)\}$ and $D_3 = \{(2,1)\}$.

We follow with a quick lemma.

Lemma 6.18. *If $D \in C_3^3$ is line free then the subsets D_1, D_2 and D_3 are line free also. The converse however is not true.*

Proof. We shall look for a contradiction. Suppose there is some line in D_1. If we add a one to the beginning of every element in D_1 we get a subset of D. This new subset clearly has a line also, so this would mean that D would contain a line which is a contradiction. We can do the same for D_2 and D_3.

If all of D_i for $i = 1,2,3$ are line free then this does not necessarily mean that D is line free. It would be possible to have a line in D where each element of the line comes from a different D_i. □

If D is a subset of C_3^3 we can write $D = D_1D_2D_3 = \{(1,*,*) : (*,*) \in D_1\} \cup \{(2,*,*) : (*,*) \in D_1\} \cup \{(3,*,*) : (*,*) \in D_1\}$. Let us look at a quick example of this.

Example 6.19. If x, y and z are as in lemma 6.16. then we can write

$$zxy = \{(1,1,1),(1,1,3),(1,2,2),(1,2,3),(1,3,1),(1,3,2)\} \cup \{(2,1,2),(2,1,3),$$
$$(2,2,1),(2,2,2),(2,3,1),(2,3,3)\} \cup \{(3,1,1),(3,1,2),(3,2,1),(3,2,3),(3,3,2),$$
$$(3,3,3)\}$$

We note that $A_{B_{0,3}} = xyz$, $A_{B_{1,3}} = zxy$ and $A_{B_{2,3}} = yzx$ (in [20], Polymath get $A_{B_{1,3}}$ and $A_{B_{2,3}}$ the wrong way round). We are now ready to find the subsets of C_3^3 which are close to the maximal size 18.

Lemma 6.20. *The only 17-element line free subsets of C_3^3 are xyz with 1 point removed or yzx or zxy with $(1,1,1)$, $(2,2,2)$ or $(3,3,3)$ removed.*

Proof. If we have a 17 element line free subset of C_3^3, call this set D, as discussed above we can split it into 3 sets that are also line free ($D = D_1D_2D_3$). As $c_{2,3} = 6$ our only option is that two of D_1, D_2 and D_3 has six elements and the other has five ($17 = 6+6+5$). We know that C_3^2 only has four

6-element subsets which are x, y, z and w. Hence two of D_1, D_2 and D_3 must be from x, y, z or w.

If both of the six element D_i are the same (x, y, z or w) then the five element set must be in the complement in order to not create a line. But the complement of x, y, z and w only has 3 elements each, so we can not have two the same.

If one of the six element D_i is w and the other is x, y or z, then the set with five elements in is in the complement of the two sets. The set w shares 4 elements in common with each of x, y and z so we shall have to use the remaining five elements in C_3^2. However, you will then get a diagonal as $(1,1)$, $(2,2)$ and $(3,3)$ is not in w, so our two six element D_i's must be a combination of x, y and z.

By symmetry we may assume that the two six-element D_i's are x and y, so the five element one is z with a point removed. There are six different possibilities for which of D_1, D_2 and D_3 are x, y and z. If we have them in the orders xzy, yxz or zyx then there are too many lines that we need to remove. Therefore the remaining orders are xyz, yzx and zyx. Both of yzx and zyx contain the diagonal so we remove one of $(1,1,1)$, $(2,2,2)$ or $(3,3,3)$ from it. The set xyz does not contain a line so we can remove any point from this. □

Polymath [20] provide no proof at all for the next lemma so we shall fill this gap in.

Lemma 6.21. *The only 18 element line free subset of C_3^3 is xyz.*

Proof. If D is an 18-element subset of C_3^3 then we can split it into three sets $D = D_1D_2D_3$ where D_1, D_2 and D_3 are subsets of C_3^2. If D is line free then so are D_1, D_2 and D_3. Hence as $c_{2,3} = 6$ all of D_1, D_2 and D_3 must be of size 6 $(18 = 6+6+6)$. Therefore D_1, D_2 and D_3 must be made from x, y, z and w. As in the last lemma we can not have two of the same or one with a w in. So they must be a combination of x, y and z. The only combination of x, y and z that give no line is xyz. Therefore xyz is the only line free subset of C_3^3 that has size 18. □

We now have everything we need to find the best possible upper bound for $c_{4,3}$.

Lemma 6.22.

$$c_{4,3} \leq 52$$

Proof. Let A be a line free set in C_3^4. Similar to the $n = 3$ case we can split A into three line free sets, $A = A_1A_2A_3$ where A_1, A_2 and A_3 are in C_3^3. If two of A_i are of size 18 then by the previous lemma they must both be of the form xyz. Then the third A_i must be in the complement of xyz, but as there are 27 elements in C_3^3 and 18 elements in xyz there are only 9 possible elements left for the third A_i. Then we have, $18 + 18 + 9 = 45$ which is not what we want. So at most one of the A_i can have size 18. So $18+17+17 = 52$ gives us the best bound as we can not have any bigger without having two or more 18 element sets. □

Combining this upper bound with the lower bound we found in the last section we now know that $c_{4,3} = 52$.

6.2.2 Upper bound for $c_{5,3}$

We shall only give a brief description on how Polymath [20] prove that $c_{5,3} \leq 150$. Firstly they find all the line free subsets of C_3^4 that are close to the maximum 52. From the previous section we know that,

$$A_{B_{0,4}} \backslash \{(1,1,1,1),(2,2,2,2)\}$$

$$A_{B_{1,4}} \backslash \{(3,3,3,3),(2,2,2,2)\}$$

$$A_{B_{2,4}} \backslash \{(1,1,1,1),(3,3,3,3)\}$$

are line free subsets of C_3^4 with 52 elements and it can be shown that these are the only line free subsets with 52 elements.

Furthermore, the only line free subsets of C_3^4 that have 51 elements are these three sets above with one point removed from each.

The only 50 element line free subsets of C_3^4 are formed by removing two points from each of the three sets above or is one of the permutations of the set

$$X = \Gamma_{(3,1,0)} \cup \Gamma_{(3,0,1)} \cup \Gamma_{(2,2,0)} \cup \Gamma_{(2,0,2)} \cup \Gamma_{(1,1,2)} \cup \Gamma_{(1,2,1)} \cup \Gamma_{(0,2,2)}$$

It is enough to show that the only 50 element line free subsets of C_3^4 are as described above as then the other two claims follow.

This can then be used to show that there is no line free subset of C_3^5 of size 151 and therefore $c_{5,3} \leq 150$. We then can deduce that $c_{5,3} = 150$ as in the previous section we found the lower bound $c_{5,3} \geq 150$.

6.2.3 Upper bound for $c_{6,3}$

We can now use theorem 6.15. and the fact that $c_{5,3} = 150$ to find an upper bound for $c_{6,3}$. We have,

$$c_{6,3} \leq 3c_{5,3} = 3.150 = 450$$

Therefore, we have $c_{6,3} = 450$ as we know $c_{6,3} \geq 450$ from the previous section.

6.3 Summary and further research

In this chapter we have only just touched on density Hales-Jewett numbers. We showed how to find the best known lower bounds (to date) of $c_{n,3}$ for $1 \leq n \leq 10$ and $n = 12, 21, 22, 23, 24, 25$ (the rest of our lower bounds can be improved upon slightly). We found these lower bounds by following Polymath's [20] method of finding Fujimura sets, B, such that the resulting line free set $A_B = \cup_{a \in B} \Gamma_a$ is as large as possible. In order to find the larger line free sets, A_B, we found simplifications and used Python.

We then, with Polymath's [20] guidance, found upper bounds of $c_{n,3}$ for $0 \leq n \leq 6$ that then led us to find the following exact values,

$$\begin{aligned} c_{0,3} &= 1 \\ c_{1,3} &= 2 \\ c_{2,3} &= 6 \\ c_{3,3} &= 18 \\ c_{4,3} &= 52 \\ c_{5,3} &= 150 \\ c_{6,3} &= 450 \end{aligned}$$

and in fact these are the only known values for $c_{n,3}$.

There are other upper and lower bounds we can find for $c_{n,t}$, including the following asymptotic lower bound that Polymath prove in [20],

$$c_{n,t} \geq t^n(-O(\sqrt[l]{logn}))$$

where l is the largest integer that satisfies $2t > 2^l$. All these bounds that we have discussed improve upon the bounds that were previously known as a result of the proofs of the Density Hales-Jewett theorem. For example from Polymath's [19] proof we get the following bound,

$$c_{n,3} << 3^n m^{-0.5}$$

where if n can be written as an exponential tower of 2's (for example $65536 = 2^{2^{2^2}}$) with m 2's then the above bound holds.

A lot more work can be done to improve upon the bounds known for $c_{n,t}$, especially for $t > 3$. Hopefully in the future more precise bounds (and even maybe more exact values) will start to appear.

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A Appendix

Python codes

Python code for establishing lower bounds for $c_{n,3}$ given in table 5.

```
import math

# This creates B_0,n.

n=20
B_0n=[]
for i in range(n+1):
    a=i
    for j in range(n-a+1):
        b=j
        c=n-b-a
        if (a+2*b) % 3!=0:
            B_0n.append([a,b,c])

#This finds all simplices in B_0,n and puts the element that
#contributes to the least number in the line free set from each
#simplex into a list.

rem=[]
for i in range(3,n+1,3):
    for j in B_0n:
        d=j[0]
        e=j[1]
        f=j[2]

        if [d-i,e+i,f] and [d-i,e,f+i] in B_0n:
            g=math.factorial(n)/float(math.factorial(d)*
            math.factorial(e)*math.factorial(f))
            h=math.factorial(n)/float(math.factorial(d-i)*
            math.factorial(e+i)*math.factorial(f))
            k=math.factorial(n)/float(math.factorial(d-i)*
            math.factorial(e)*math.factorial(f+i))
            list=[g,h,k]

            if g==min(list):
```

```
                    rem.append(j)

                else:
                    if h==min(list):
                        rem.append([d-i,e+i,f])

                    else:
                        rem.append([d-i,e,f+i])

            else:
                pass

#This part simply removes duplicates in the list.

rem1=[]
for i in rem:
    if i not in rem1:
        rem1.append(i)

#This removes the chosen elements from B_0,n.

for i in rem1:
    B_0n.remove(i)

#Finally, this calculates the size of the resulting line free set A_B.

s=0
for i in B_0n:
    l=i[0]
    m=i[1]
    o=i[2]
    s+=math.factorial(n)/float(math.factorial(l)*
    math.factorial(m)*math.factorial(o))

print s
```

Python code for establishing lower bounds for $c_{n,3}$ when n is a multiple of 3 given in table 6.

```
import math

#This part creates the Fujimura set for n=3m.

for m in range(21):
    Fuji=[]
    l=[[-7,-3,10],[-7,0,7],[-7,3,4],[-6,-4,10],[-6,-1,7],
    [-6,2,4],[-5,-1,6],[-5,2,3],[-4,-2,6],[-4,1,3],[-3,1,2],
    [-2,0,2],[-1,0,1]]
    n=3*m
    print n
    for i in l:
        if i[0]+m >= 0 and i[1]+m >= 0 and i[2]+m >= 0:
            Fuji.append([i[0]+m, i[1]+m, i[2]+m])

    for y in range(2):
        for x in range(((-8+m)/2)+2):

            if -8-y-2*x+m >= 0:
                Fuji.extend([[-8-y-2*x+m,-6+y-2*x+m,14+4*x+m],
                [-8-y-2*x+m,-3+y-2*x+m,11+4*x+m],
                [-8-y-2*x+m, y+x+m,8+x+m]])

            if -8-2*x+m >= 0:
                Fuji.append([-8-2*x+m,3+x+m,5+x+m])

    fuji=[]
    for i in Fuji:
        if i not in fuji:
            fuji.append(i)

#This then calculates the size of the corresponding line free set.

    s=0
    for i in fuji:
        l=i[0]
        m=i[1]
        o=i[2]
```

```
    s+=6*(math.factorial(n)/float(math.factorial(l)*
    math.factorial(m)*math.factorial(o)))

print s
```

Python code for creating the lower bounds given in table 7.

```
import math

a=10
lowerbounds=[]

for m in range(a):

# This finds the lower bound for when n=3m

    Fuji=[]
    l=[[-7,-3,10],[-7,0,7],[-7,3,4],[-6,-4,10],[-6,-1,7],
    [-6,2,4],[-5,-1,6],[-5,2,3],[-4,-2,6],[-4,1,3],[-3,1,2],
    [-2,0,2],[-1,0,1]]
    n=3*m

    for i in l:
        if i[0]+m >= 0 and i[1]+m >= 0 and i[2]+m >= 0:
            Fuji.append([i[0]+m, i[1]+m, i[2]+m])

    for y in range(2):
        for x in range(((-8+m)/2)+2):

            if -8-y-2*x+m >= 0:
                Fuji.extend([[-8-y-2*x+m,-6+y-2*x+m,14+4*x+m],
                [-8-y-2*x+m,-3+y-2*x+m,11+4*x+m],
                [-8-y-2*x+m, y+x+m,8+x+m]])

            if -8-2*x+m >= 0:
                Fuji.append([-8-2*x+m,3+x+m,5+x+m])

    fuji=[]
    for i in Fuji:
        if i not in fuji:
            fuji.append(i)



    s=0
    for i in fuji:
```

```
            l=i[0]
            m=i[1]
            o=i[2]
            s+=6*(math.factorial(n)/float(math.factorial(l)*
            math.factorial(m)*math.factorial(o)))

# This gives the lower bound for n=3m-1

        p=s/float(3)

# This finds the lower bound for when n=3m-2

        fuji2=[]
        for i in fuji:
            r=[i,[i[0],i[2],i[1]],[i[1],i[0],i[2]],[i[1],i[2],i[0]],
            [i[2],i[1],i[0]],[i[2],i[0],i[1]]]
            fuji2.extend(r)

        x=[]
        for j in fuji2:
            if j not in x:
                x.append(j)
        q=0
        for i in x:
            if i[0]!=0 and i[1]!=0:
                q+=(math.factorial(n-2)/float(math.factorial(i[0]-1)*
                math.factorial(i[1]-1)*math.factorial(i[2])))
        lowerbounds.extend([q,p,s])


print lowerbounds
```

B Appendix

Tables of results

Extensions of the results in tables 7 and 8. The third column is taken from [18].

n	Lower bound found from the simplifications in section 6.1.2.	Best known lower bounds for $c_{n,3}$
21	4962044826	4962044826
22	14611830116	14611830116
23	43303183806	43303183806
24	129909551418	129909551418
25	383588257270	383598657870
26	1138989794670	1139029911270
27	3416969384010	3417089733810
28	10111356055886	10112634438206
29	30069089443032	30073742968632
30	90207268329096	90221228905896
31	267400102010680	267491466230000
32	796140248408736	796459952460900
33	2388420745226208	2.39E+15
34	7089752982959760	7.09E+15
35	2.1128586215262948E+16	2.11E+16
36	6.338575864578885E+16	6.34E+16
37	1.8836261183853485E+17	1.89E+17
38	5.61777864904243E+17	5.63E+17
39	1.685333594712729E+18	1.69E+18
40	5.012805842639805E+18	5.02E+18
41	1.4959578631920351E+19	1.50E+19
42	4.487873589576105E+19	4.50E+19
43	1.3358489209432411E+20	1.34E+20
44	3.988561607309349E+20	4.00E+20
45	1.1965684821928045E+21	1.20E+21
46	3.5638649173284304E+21	3.58E+21
47	1.06454262024094E+22	1.07E+22
48	3.1936278607228196E+22	3.21E+22
49	9.516820001579667E+22	9.56E+22
50	2.84371970935659E+23	2.86E+23

n	Lower bound found from the simplifications in section 6.1.2.	Best known lower bounds for $c_{n,3}$
51	8.53115912806977E+23	8.58E+23
52	2.543344848890257E+24	2.56E+24
53	7.602029384516345E+24	7.65E+24
54	2.2806088153549035E+25	2.30E+25
55	6.801586281022731E+25	6.85E+25
56	2.0335025304855415E+26	2.05E+26
57	6.1005075914566246E+26	6.15E+26
58	1.819974629850514E+27	1.84E+27
59	5.442450252878161E+27	5.49E+27
60	1.6327350758634484E+28	1.65E+28