Symmetry in Escher's Drawings

MATH3001 Project in Mathematics

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Contents

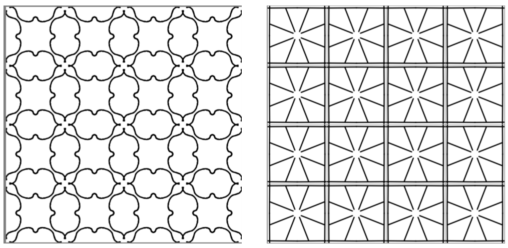
1 Introduction

Any figures referencing [4] have been generated using the engine on this website.

Mathematically, the two patterns in Figure 1 are the same. We will study what it actually means to say two patterns are the same.

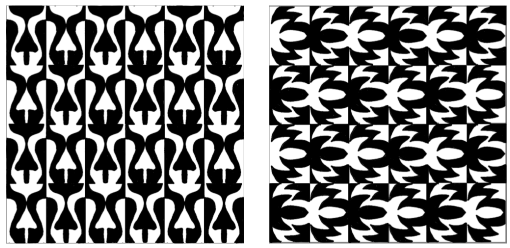
Definition 1.1. *A* ***wallpaper pattern*** *is a two dimensional repeating pattern [[1] p.145].*

We study wallpaper groups, which are groups of symmetries which act on wallpaper patterns. So to say that two patterns are the same is to say that the wallpaper groups acting on these patterns are the same. We will prove that there are actually only 17 different wallpaper groups, so any repeating pattern of the plane is one of 17 different types.



(a) (b)

Figure 1: Two wallpaper patterns [4]



(a) (b)

Figure 2: Two two-colour patterns [4]

We will then go on to consider what it looks like when we introduce colours to our pattern.

Definition 1.2. *A* ***two-colour pattern*** *is a wallpaper pattern coloured in two colours.*

With two-colour patterns, we allow symmetries to permute the colours. The two patterns in Figure 2 are the same, even though they may look different.

To study wallpaper groups and two-colour groups, we firstly study isometries of the plane, which are functions preserving Euclidean distance. We then use these isometries to define the wallpaper groups, and prove there are only seventeen. Then we move onto two-colour symmetries and two-colour groups

1.1 Ethics

Intellectual property is anything which someone has created, including inventions, designs or art [7]. Sometimes intellectual property may be protected by the law in order to prevent people profiting from someone else's work. However there is a fine line, as it would be hoped we could use someone else's work to research and develop our own findings. It is important however that when using work in your research which is not your own, or that has been inspired by someone else, you always give credit where it is due to avoid passing off someone else's work as your own. Hence in writing this report I have had to ensure I reference everything I use which I did not come up with entirely myself, and ensure I have been adding enough of my own ideas to say that the work is my own.

2 Isometries

2.1 Translations and Orthogonal Transformations

Arguments in this section are developed from [[1] pp.45-46].

Definition 2.1. $t : \mathbb{R}^2 \to \mathbb{R}^2$ *is a* ***translation*** *if* $\exists \boldsymbol{v} \in \mathbb{R}^2, \forall \boldsymbol{x} \in \mathbb{R}^2$, $t(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{v}$.

Definition 2.2. *An* ***orthogonal matrix*** *is one who's transpose is its inverse,* $A^T A = I$. *An* ***orthogonal transformation*** *is a function* $f_A : \mathbb{R}^2 \to \mathbb{R}^2$, $f_A(\boldsymbol{x}) = \boldsymbol{x}A^T$ *where* A *is an orthogonal matrix.*

Theorem 2.3. *The set of orthogonal matrices* O_2 *is a group under matrix multiplication.*

Proof. We check group axioms:

- Closure: Let $A, B \in O_2$. $(AB)^T(AB) = B^T A^T AB = B^T IB = B^T B = I$. So $AB \in O_2$.
- Associativity: Matrix multiplication is associative
- Identity: The identity matrix is orthogonal $I^T I = II = I$.
- Inverse: Let $A \in O_2$. $A^{-1} = A^T$. Note $I = A^T A = (A^T)(A^T)^T$. So $A^T \in O_2, A^{-1} \in O_2$.

So O_2 is a group. $\square$

Theorem 2.4. *Orthogonal transformations preserve the dot product.*

Proof. Note $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}\mathbf{y}^T$. Let $A \in O_2$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

$$f_A(\mathbf{x}) \cdot f_A(\mathbf{y}) = \mathbf{x}A^T \cdot \mathbf{y}A^T = (\mathbf{x}A^T)(\mathbf{y}A^T)^T = \mathbf{x}A^T A\mathbf{y}^T = \mathbf{x}\mathbf{y}^T = \mathbf{x} \cdot \mathbf{y}$$

□

Remark. *Orthogonal transformations preserve the origin. Let $A \in O_2$. $f_A(\boldsymbol{0}) = \boldsymbol{0}A^T = \boldsymbol{0}$.*

Let A be an orthogonal matrix. Since $A^T A = I$, $det(A^T)det(A) = det(I)$, $det(A)^2 = 1$, so $det(A) = \pm 1$.

All orthogonal matrices are of one of these two forms: $A_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$, $B_\varphi = \begin{bmatrix} cos\varphi & sin\varphi \\ sin\varphi & -cos\varphi \end{bmatrix}$.

2.2 Isometries

Arguments in this section are developed from [[1] pp.136-139].

Definition 2.5. *An **isometry** is a function that preserves distance. That is to say $g : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry if $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$,*

$$\|g(\boldsymbol{x}) - g(\boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\|$$

Theorem 2.6. *The isometries of the plane form a group, namely the **Euclidean group**, E_2, under the group operation of composition of functions.*

Proof. We check group axioms:

- Identity: The identity map $id : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \mapsto \mathbf{x}$ is an isometry as

$$\|id(\mathbf{x}) - id(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$$

  So $id \in E_2$.
- Associativity: Composition of functions is associative.
- Closure: Let $g, h \in E_2$:

$$\|g(h(\mathbf{x})) - g(h(\mathbf{y}))\| = \|h(\mathbf{x}) - h(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$$

  as both g and h are isometries themselves. So $g \circ h \in E_2$.
- Inverses: Each $g \in E_2$ is a bijection [[1] p.136] and so an inverse exists.

$$\|g^{-1}(\mathbf{x}) - g^{-1}(\mathbf{y})\| = \|g(g^{-1}(\mathbf{x})) - g(g^{-1}(\mathbf{y}))\| = \|\mathbf{x} - \mathbf{y}\|$$

  So $g^{-1} \in E_2$ as required.

□

Theorem 2.7. *Translations and orthogonal transformations are isometries.*

Proof. Let $t : \mathbb{R}^2 \to \mathbb{R}^2$ be a translation. So $\exists \mathbf{v} \in \mathbb{R}^2, \forall \mathbf{x} \in \mathbb{R}^2$, $t(\mathbf{x}) = \mathbf{x} + \mathbf{v}$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$

$$\|t(\mathbf{x}) - t(\mathbf{y})\| = \|(\mathbf{x} + \mathbf{v}) - (\mathbf{y} + \mathbf{v})\| = \|\mathbf{x} - \mathbf{y}\|$$

So translations are isometries.
Let $f_A : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal transformation. So $A \in O_2$. $\forall \mathbf{x} \in \mathbb{R}^2$, $f_A(\mathbf{x}) = \mathbf{x}A^T$. Note $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. So

$$\|f_A(\mathbf{x})\| = \sqrt{f_A(\mathbf{x}) \cdot f_A(\mathbf{x})} = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \|\mathbf{x}\|$$

Where the second equality is given by Theorem 2.4.
Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $\|f_A(\mathbf{x}) - f_A(\mathbf{y})\| = \|\mathbf{x}A^T - \mathbf{y}A^T\| = \|(\mathbf{x} - \mathbf{y})A^T\| = \|f_A(\mathbf{x} - \mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$, so orthogonal transformations are isometries.

□

Theorem 2.8. *A general element of E_2 is either a rotation about the origin followed by a translation, or a reflection in a line through the origin followed by a translation.*

Proof. For a proof of this statement, see [1] p.137. □

Theorem 2.9. *The set of translations T is a subgroup of E_2 .*

Proof. We use the subgroup criterion. As $id(\mathbf{x}) = \mathbf{0} + \mathbf{x}$, $id \in T$, so T is non-empty. Let $\tau, \tau_1 \in T$. Then $\forall \mathbf{x} \in \mathbb{R}_2$, $\tau(\mathbf{x}) = \mathbf{u} + \mathbf{x}$, $\tau_1(\mathbf{x}) = \mathbf{v} + \mathbf{x}$, for some $\mathbf{u}, \mathbf{v} \in \mathbb{R}_2$. Now

$$\tau{\tau_1}^{-1}(\mathbf{x}) = \tau((-\mathbf{v}) + \mathbf{x}) = \mathbf{u} + (-\mathbf{v} + \mathbf{x}) = (\mathbf{u} - \mathbf{v}) + \mathbf{x}$$

So $\tau{\tau_1}^{-1} \in T$, T is a subgroup of E_2. $\square$

Theorem 2.10. *The set of orthogonal transformations O is a subgroup of E_2.*

Proof. We use the subgroup criterion. We know from Theorem 2.3, $I \in O_2$, so $f_I \in O$. $\forall \mathbf{x} \in \mathbb{R}^2$,
$f_I(\mathbf{x}) = \mathbf{x}I = \mathbf{x} = id(\mathbf{x})$. So $id \in O$, O is non-empty.
Let $f_A, f_B \in O$, so $A, B \in O_2$. Let $\mathbf{x} \in \mathbb{R}^2$. Note $B^{-1} = B^T$.

$$(f_A \circ (f_B)^{-1})(\mathbf{x}) = f_A(f_{B^{-1}}(\mathbf{x})) = f_A(f_{B^T}(\mathbf{x})) = f_A(\mathbf{x}B) = \mathbf{x}BA^T = \mathbf{x}(AB^T)^T = f_{AB^T}(\mathbf{x})$$

As O_2 is a group by Theorem 2.3, $B^T \in O_2$, $AB^T \in O_2$. So $(f_A \circ (f_B)^{-1}) \in O$. Hence O is a subgroup of E_2 by the subgroup criterion. $\square$

Remark. *The elements of O are rotations about the origin and reflections in lines through the origin, and hence by Theorem 2.8 we can see that $E_2 = TO$. Any element $g \in E_2$ can be written $g = \tau f$, $f \in O$, $\tau \in T$.*

Theorem 2.11. *The only element in the intersection of T and O is the identity transformation. .*

Proof. All elements of T other than the identity move the origin, but all elements of O preserve the origin. $\square$

Theorem 2.12. *Each $g \in E_2$ can be written in a unique way as $g = \tau f$, with $\tau \in T$, $f \in O_2$.*

Proof. Say $g = \tau f = \tau' f'$, with $\tau, \tau' \in T$, $f, f' \in O$.

$$\tau f = \tau' f' \Rightarrow (\tau')^{-1}\tau f f^{-1} = (\tau')^{-1}\tau' f' f^{-1} \Rightarrow (\tau')^{-1}\tau = f' f^{-1}$$

But $(\tau')^{-1}\tau \in T$, $f' f^{-1} \in O$, So we must have $(\tau')^{-1}\tau = f' f^{-1} \in T \cap O$. Theorem 2.11 says the only element in $T \cap O$ is the identity, so

$$(\tau')^{-1}\tau = id \Rightarrow \tau' = \tau$$
$$f' f^{-1} = id \Rightarrow f' = f$$

So our expression of $g = \tau f$ is unique. $\square$

Theorem 2.13. *T is a normal subgroup of E_2.*

Proof. Show $\forall \tau \in T$, $f \in E_2$, $f\tau f^{-1} \in T$. Say $\tau(\mathbf{0}) = \mathbf{v}$, $\forall \mathbf{x} \in \mathbb{R}^2$,

$$f\tau f^{-1}(\mathbf{x}) = f(\mathbf{v} + f^{-1}(\mathbf{x})) = f(\mathbf{v}) + f(f^{-1}(\mathbf{x})) = f(\mathbf{v}) + \mathbf{x}$$

So the conjugate $f\tau f^{-1}$ is translation by the vector $f(\mathbf{v})$. So T is a normal subgroup. $\square$

Definition 2.14. *If $g = \tau f \in E_2$ and f is a rotation, g is called a **direct isometry**. If f is a reflection, g is called an **opposite isometry**.*

3 Ordered Pairs

Arguments in this section are developed from [[1] pp.139-140].

Definition 3.1. *Each isometry $g = \tau f \in E_2$ with $\tau \in T$, $f \in O$ can be written as an **ordered pair** $(\boldsymbol{v}, M)$, where $\boldsymbol{v} = \tau(\boldsymbol{0}) \in \mathbb{R}^2$, and $M \in O_2$ is the orthogonal matrix which $\forall \boldsymbol{x} \in \mathbb{R}^2$, $f(\boldsymbol{x}) = \boldsymbol{x}M^T$.*
Say g acts on a vector $\boldsymbol{x}$ and is represented by the ordered pair $(\boldsymbol{v}, M)$. Then $g(\boldsymbol{x}) = \boldsymbol{v} + f_M(\boldsymbol{x}) = \boldsymbol{v} + \boldsymbol{x}M^T$

If an isometry g corresponds to an ordered pair $(\mathbf{v}, M)$, write $g \simeq (\mathbf{v}, M)$.

In order to work with ordered pairs, it makes sense to consider how composition of isometries and inverse isometries would look when viewing them as ordered pairs.

Theorem 3.2. *If we have isometries $g \simeq (\boldsymbol{v}, M)$, $h \simeq (\boldsymbol{v}', M')$, their product under composition of functions is $g \circ h \simeq (\boldsymbol{v} + f_M(\boldsymbol{v}'), MM')$.*

Proof. $\forall \mathbf{x} \in \mathbb{R}^2$,

$$(g \circ h)(\mathbf{x}) = g(h(\mathbf{x})) = g(\mathbf{v}' + \mathbf{x}M'^T) = \mathbf{v} + (\mathbf{v}' + \mathbf{x}M'^T)M^T = \mathbf{v} + \mathbf{v}'M^T + \mathbf{x}M'^T M^T = \mathbf{v} + f_M(\mathbf{v}') + \mathbf{x}(MM')^T$$

comparing to Definition 3.1, we can see $g \circ h \simeq (\mathbf{v} + f_M(\mathbf{v}'), MM')$. □

Theorem 3.3. *For an isometry $g \simeq (\boldsymbol{v}, M)$, its inverse isometry is given by $g' \simeq (-{f_M}^{-1}(\boldsymbol{v}), M^{-1})$.*

Proof. We need to show that $g \circ g' = id$, and that $g' \circ g = id$. Using Theorem 3.2,

$$g \circ g' \simeq (\mathbf{v}, M)(-{f_M}^{-1}(\mathbf{v}), M^{-1}) = (\mathbf{v} + f_M(-{f_M}^{-1}(\mathbf{v})), MM^{-1}) = (\mathbf{v} - \mathbf{v}, I) = (\mathbf{0}, I) \simeq id$$
$$g' \circ g \simeq (-{f_M}^{-1}(\mathbf{v}), M^{-1})(\mathbf{v}, M) = (-{f_M}^{-1}(\mathbf{v}) + f_{M^{-1}}(\mathbf{v}), M^{-1}M) = (-\mathbf{v}{M^{-1}}^T + \mathbf{v}{M^{-1}}^T, I) = (\mathbf{0}, I) \simeq id$$

So g' is the inverse to g, as required. □

3.1 Possible Cases

Consider the below matrices:

$A_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$, $B_\varphi = \begin{bmatrix} cos\varphi & sin\varphi \\ sin\varphi & -cos\varphi \end{bmatrix}$.

Here A_θ is the matrix for rotating through angle θ about the origin, and B_φ is the matrix for reflecting in a line through the origin subtending angle $\varphi/2$ with the x axis.

Let l, m be the lines in Figure 3.

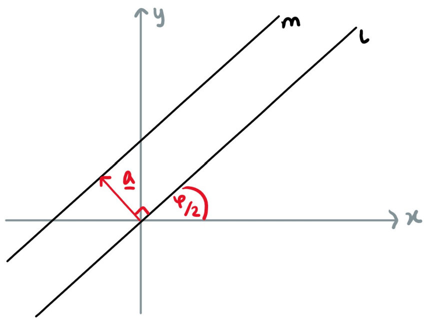


Figure 3: Lines l, m

We find the ordered pairs for each type of isometry:

Case a Translation by vector $\mathbf{v}$ is represented by the ordered pair $(\mathbf{v}, I)$.

Case b Rotation anticlockwise through angle θ about the origin is $(\mathbf{0}, A_\theta)$.

Case c Reflection in the line l in Figure 3 is $(\mathbf{0}, B_\varphi)$.

Cases a, b, c follow directly from the definitions of the matrices I, A_θ and B_φ.

Case d Rotation anticlockwise through angle θ about $\mathbf{c}$ is $(\mathbf{c} - f_{A_\theta}(\mathbf{c}), A_\theta)$.

Explanation: To use matrix A_θ we need rotation about the origin, so
1) We take $\mathbf{x}$ to $\mathbf{x}$-$\mathbf{c}$, so that the centre of rotation is now the origin.
2) Then matrix A_θ acts on this point to rotate through angle θ. Now we are at point $\mathbf{x}'$ (see Figure 4a), or $f_{A_\theta}(\mathbf{x}\text{-}\mathbf{c})$.
3) Then we need to shift this back to where we started, so we need to add vector $\mathbf{c}$, and we reach point $\mathbf{x}''$ (see Figure 4a).
So overall we have carried out the isometry

$$f_{A_\theta}(\mathbf{x}\text{-}\mathbf{c}) + \mathbf{c} = f_{A_\theta}(\mathbf{x}) - f_{A_\theta}(\mathbf{c}) + \mathbf{c} = \mathbf{c} - f_{A_\theta}(\mathbf{c}) + f_{A_\theta}(\mathbf{x})$$

So we get ordered pair $(\mathbf{c} - f_{A_\theta}(\mathbf{c}), A_\theta)$.

Case e Reflection in the line m in Figure 3 is $(2\mathbf{a}, B_\varphi)$.

Explanation: To use matrix B_φ to represent reflection in a line, we need the line to go through the origin.
1) We want to map line m to line l, so we shift our whole picture by -$\mathbf{a}$, taking $\mathbf{x}$ to $\mathbf{x}' = \mathbf{x} - \mathbf{a}$.
2) We then reflect in the line l, which is the same as having matrix B_φ act on our point $\mathbf{x}'$, to take it to $\mathbf{x}'' = f_{B_\varphi}(\mathbf{x} - \mathbf{a})$. Notice as $\mathbf{a}$ is orthogonal to our line, reflecting vector $\mathbf{a}$ in the line l gives -$\mathbf{a}$, $f_{B_\varphi}(\mathbf{a}) = -\mathbf{a}$.

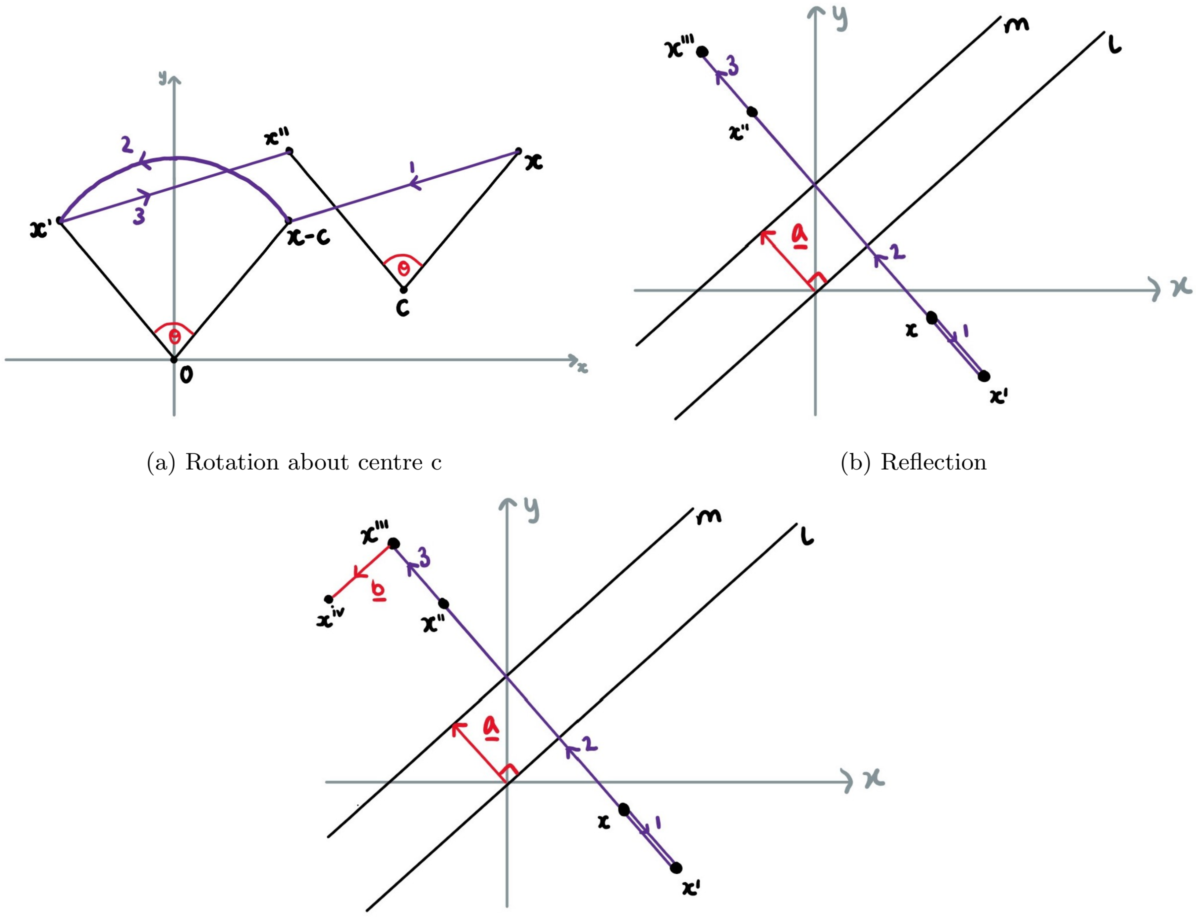
3) We then need to shift our picture back by $\mathbf{a}$ again (to correct step 1) to reach $\mathbf{x}'''$ (see Figure 4b).
So the isometry we have carried out is:

$$\mathbf{a} + f_{B_\varphi}(\mathbf{x} - \mathbf{a}) = \mathbf{a} + f_{B_\varphi}(\mathbf{x}) - f_{B_\varphi}(\mathbf{a}) = \mathbf{a} - -\mathbf{a} + f_{B_\varphi}(\mathbf{x}) = 2\mathbf{a} + f_{B_\varphi}(\mathbf{x})$$

So our required ordered pair is $(2\mathbf{a}, B_\varphi)$.

Case f Glide Reflection (reflection in the line m (see Figure 3) followed by translation by vector $\mathbf{b}$ parallel to the line) is $(2\mathbf{a} + \mathbf{b}, B_\varphi)$.

Explanation: For a glide reflection in the line m, we take the same steps as we would for a reflection to reach the point $\mathbf{x}'''$ (see Figure 4c), but then we shift by a vector $\mathbf{b}$ parallel to the line m to reach $\mathbf{x}^{iv}$ (see Figure 4c). Note $f_{B_\varphi}(\mathbf{b}) = \mathbf{b}$, and we require $\mathbf{b} \neq \mathbf{0}$, as if we had $\mathbf{b} = \mathbf{0}$, we would just have case e.
So we have ordered pair $(2\mathbf{a} + \mathbf{b}, B_\varphi)$.



(a) Rotation about centre c

(b) Reflection

(c) Glide reflection

Figure 4: Standard ordered pair cases

Example 3.1.1. Express anticlockwise rotation through $\pi/6$ about the point (-1,-1) as an ordered pair ($\mathbf{v}$,M) with $\mathbf{v} \in \mathbb{R}_2$, $M \in O_2$. (Question from [1], p.143)

Solution. Clearly our centre of rotation is $\mathbf{c} = (-1, -1)$. Our matrix needs to be that representing rotation about the origin by angle $\pi/6$, so $A_{\pi/6} = \begin{bmatrix} cos(\pi/6) & -sin(\pi/6) \\ sin(\pi/6) & cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

This is case d, so our ordered pair is $(\mathbf{c} - f_{A_{\pi/6}}(\mathbf{c}), A_{\pi/6})$.

$$\begin{aligned} \mathbf{c} - f_{A_{\pi/6}}(\mathbf{c}) &= \mathbf{c} - \mathbf{c}A^t_{\pi/6} \\ &= (-1, -1) - (-1, -1) \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ &= (-1, -1) - (-\frac{\sqrt{3}}{2} + \frac{1}{2}, -\frac{1}{2} - \frac{\sqrt{3}}{2}) \\ &= (-\frac{3}{2} + \frac{\sqrt{3}}{2}, -\frac{1}{2} + \frac{\sqrt{3}}{2}) \end{aligned}$$

So the ordered pair for anticlockwise rotation through $\pi/6$ about the point (-1,-1) is

$$((-\frac{3}{2}+\frac{\sqrt{3}}{2},-\frac{1}{2}+\frac{\sqrt{3}}{2}),\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix})$$

4 Going Backwards from An Ordered Pair

Arguments in this section are based from [[1], p.140-141].

Say we are given an ordered pair $g \simeq (\mathbf{v}, M)$ and we want to find the isometry it represents. Our first step is to analyse our matrix. $\forall g \in E_2$, if $g \simeq (\mathbf{v}, M)$, M is an orthogonal matrix. Recall all orthogonal matrices are one of the two forms: $A_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$, $B_\varphi = \begin{bmatrix} cos\varphi & sin\varphi \\ sin\varphi & -cos\varphi \end{bmatrix}$.
So if $det(M) = 1$, then $M = A_\theta$ for some $\theta \in [0, 2\pi)$, and $(\mathbf{v}, M)$ represents a direct isometry. If $det(M) = -1$, then B_φ for some $\varphi \in [0, 2\pi)$, and $(\mathbf{v}, M)$ represents an opposite isometry.

4.1 Direct Isometries

4.1.1 Translation

If M is the identity matrix and $g \simeq (\mathbf{v}, M)$,

$$g(\mathbf{x}) = \mathbf{v} + \mathbf{x}M^T = \mathbf{v} + \mathbf{x}I = \mathbf{v} + \mathbf{x}$$

So g is translation by vector $\mathbf{v}$.

4.1.2 Rotation

If M is not the identity but $detM = 1$, $(\mathbf{v}, M)$ represents a rotation. We need to find the angle we are rotating by and the centre of rotation. Compare matrix M to the matrix A_θ, and use the inverse trigonometric functions to find θ. So the isometry is an anticlockwise rotation through angle θ.
Looking at case d in the Section 3.1, compare to see

$$\mathbf{v} = \mathbf{c} - f_M(\mathbf{c}) = \mathbf{c} - \mathbf{c}M^T = \mathbf{c}(I - M^T)$$
$$\mathbf{c} = \mathbf{v}(I - M^T)^{-1}$$

So use $\mathbf{c} = \mathbf{v}(I - M^T)^{-1}$ to find the centre of rotation.

So a direct isometry is either a translation or rotation.

4.2 Opposite Isometries

If $f_M(\mathbf{v}) = -\mathbf{v}$, see Section 4.2.1. Otherwise, see Section 4.2.2.

4.2.1 Reflection

If $f_M(\mathbf{v}) = -\mathbf{v}$, then our vector $\mathbf{v}$ is orthogonal to our mirror line, and so we can just shift our mirror line to see g is just reflection in a new line.
To find the gradient of our mirror line, we use the fact $\mathbf{v}$ is orthogonal to the mirror line. So if $\mathbf{v} = (v_1, v_2)$, the mirror line is parallel to $(-v_2, v_1)$.
Lets call this line parallel to $(-v_2, v_1)$ l (see Figure 5a). So the isometry $(\mathbf{v}, M)$ is equivalent to reflecting in the line l and then translating by a vector $\mathbf{v}$ orthogonal to l. This is the same as reflecting in the line that is shifted by vector $\frac{\mathbf{v}}{2}$, line m in Figure 5a.

4.2.2 Glide Reflection

We have that $f_M(\mathbf{v}) \neq -\mathbf{v}$. $\mathbf{v}$ is not orthogonal to the mirror line, so we can't just shift the mirror line. Therefore the isometry is a glide reflection. We want to decompose this into reflection in a line and then translation by a vector parallel to the line. We want to find the glide line and the vector that we translate by.
Set $\mathbf{w} = \mathbf{v} - f_M(\mathbf{v})$.

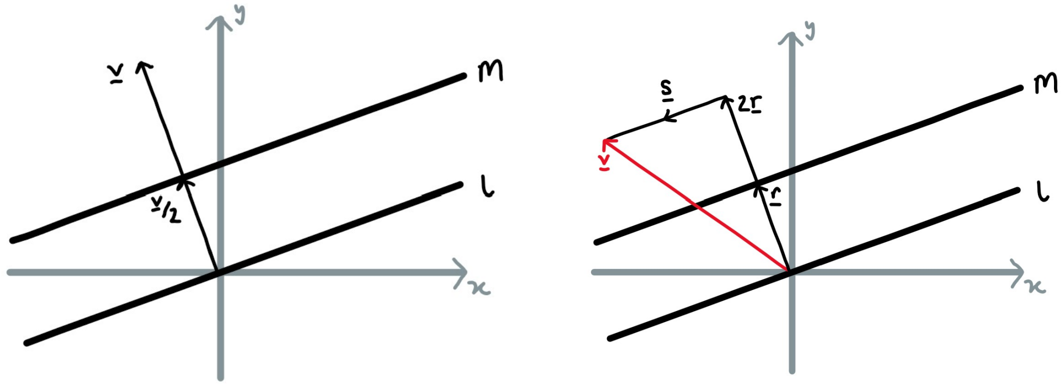
$$f_M(\mathbf{w}) = f_M(\mathbf{v} - f_M(\mathbf{v})) = f_M(\mathbf{v}) - f_M(f_M(\mathbf{v})) = f_M(\mathbf{v}) - \mathbf{v} = -\mathbf{w}$$

So $\mathbf{w}$ is orthogonal to our mirror line. We want to find the full component of $\mathbf{v}$ that is orthogonal to our mirror line,

to do that we resolve $\mathbf{v}$ along $\mathbf{w}$, and call this component of $\mathbf{v}$ $2\mathbf{r}$.
Where α is the angle between $\mathbf{w}$ and $\mathbf{v}$,

$$2\mathbf{r} = ||\ \mathbf{v}\ ||^2 cos(\alpha)\mathbf{w} = ||\ \mathbf{v}\ ||^2 (\frac{\mathbf{v}\cdot\mathbf{w}}{||\ \mathbf{v}\ ||^2||\ \mathbf{w}\ ||^2})\mathbf{w} = (\frac{\mathbf{v}\cdot\mathbf{w}}{||\ \mathbf{w}\ ||^2})\mathbf{w}$$

So $2\mathbf{r}$ is the component of our vector $\mathbf{v}$ that is orthogonal to the mirror line, this implies $f_M(2\mathbf{r}) = -2\mathbf{r}$. We saw in Section 4.2.1 that reflecting in a mirror line l (see Figure 5b) and then translating by $2\mathbf{r}$, orthogonal to l, is equivalent to reflecting in a mirror line parallel to l, shifted by $\mathbf{r}$. For a visual description of this, see Figure 5b.
Now we still have some of $\mathbf{v}$ left to translate by. Let $\mathbf{s} = \mathbf{v} - 2\mathbf{r}$ be the part of $\mathbf{v}$ that is "left over".
As $2\mathbf{r}$ is the component of $\mathbf{v}$ that is orthogonal to the mirror line, $\mathbf{s} = \mathbf{v} - 2\mathbf{r}$ must be parallel to the mirror line. Hence $f_M(\mathbf{s}) = \mathbf{s}$.
So to carry out our isometry, we first reflect in the line parallel to $\mathbf{s}$ passing through $\mathbf{r}$, and then translate by vector $\mathbf{s}$.



(a) Shifting the mirror line

(b) Decomposing the glide reflection

Figure 5: Opposite isometries

4.3 Example

Example 4.3.1. Show that $g \simeq ((-14, 3), \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix})$ represents a glide reflection. Find the axis of the glide and the amount by which points are translated along this axis. (Question ammended from [1] p.143).

Solution. $\mathbf{v} = (-14, 3)$, $M = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$. $det(M) = -1$, so g is an opposite isometry.

$$g_M(\mathbf{v}) = (-14, 3) \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix} = (-6, -13)$$

As $g_M(\mathbf{v}) \neq -\mathbf{v}$, g is a glide reflection.

$$\begin{aligned} \mathbf{w} &= \mathbf{v} - g_M(\mathbf{v}) = (-8, 16) \\ \mathbf{v}\cdot\mathbf{w} &= (-14, 3)\cdot(-8, 16) = 160 \\ ||\ \mathbf{w}\ ||^2 &= 320 \\ 2\mathbf{r} &= (\frac{\mathbf{v}\cdot\mathbf{w}}{||\ \mathbf{w}\ ||^2})\mathbf{w} = (-4, 8) \\ \mathbf{r} &= (-2, 4) \\ \mathbf{s} &= \mathbf{v} - 2\mathbf{r} = (-10, -5) \end{aligned}$$

g is reflection in the line parallel to $\mathbf{s}$ passing through $\mathbf{r}$ followed by translation by vector $\mathbf{s}$.

So g is reflection in the line $y = \frac{1}{2}x + 5$ followed by translation by $(-10, -5)$.

5 Translation Subgroups, Point Groups and Lattices

5.1 Translation Subgroups and Point Groups

Arguments in this section are developed from [[1] p. 145-147].

In order to classify the wallpaper groups we must firstly strictly define what they are.

Theorem 5.1. *Let $\pi : E_2 \mapsto O_2$, $\pi(g) = M$ for $g \in E_2$, $g \simeq (\boldsymbol{v}, M)$. The map π is a homomorphism.*

Proof. Let $g, g' \in E_2$, $g \simeq (\mathbf{v}, M)$, $g' \simeq (\mathbf{v}', M')$.

$$\pi(gg') \simeq \pi((\mathbf{v}, M)(\mathbf{v}', M')) = \pi(\mathbf{v} + f_M(\mathbf{v}'), MM') = MM' = \pi(\mathbf{v}, M)\pi(\mathbf{v}', M') \simeq \pi(g)\pi(g')$$

So π is a homomorphism. □

Theorem 5.2. *The kernel of π as defined in Theorem 5.1 is the set of translations in E_2.*

Proof.

$$ker(\pi) = \{g \in E_2 : \pi(g) = I\} = \{g \in E_2 : g \simeq (\mathbf{v}, I),\ \forall \mathbf{v} \in \mathbb{R}^2\} = T$$

□

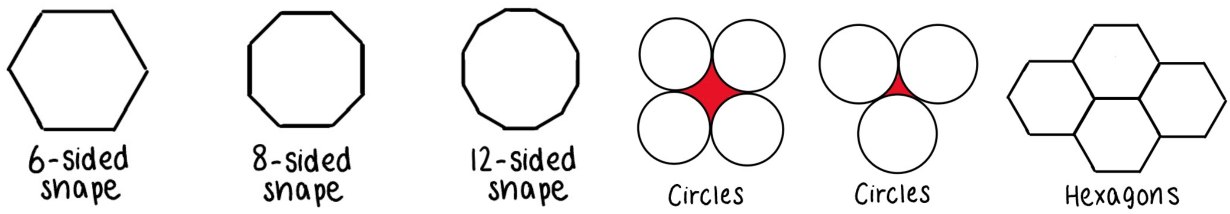
Definition 5.3. *For G a subgroup of E_2, we write $H = G \cap T$ and call H the* ***translation subgroup*** *of G. We take $J = \pi(G)$ and call J the* ***point group*** *of G.*

Note the translation subgroup is a subgroup as it is the intersection of two subgroups of E_2. The point group is a group as the image of a homomorphism is a subgroup of the target group.

Definition 5.4. *A subgroup G of E_2 is a* ***wallpaper group*** *if its translation subgroup is generated by two independent translations and its point group is finite.*

Explanation. We require the translation subgroup to be generated by two independent vectors as we want to fill the whole plane, a two-dimensional space. If we had just one vector we would have a one-dimensional frieze pattern, and more than two linearly independent vectors would generate a higher dimensional space.

We envision our wallpaper group as a shape with certain symmetries which is then translated by a linear combination of our two independent vectors, in order to fill the whole plane.
We require the point group to be finite. We see in Figure 6a n-sided shapes with order n rotational symmetry. Clearly, as the order of rotational symmetry of a shape increases, the shape begins to look more circular. A shape with infinite rotational symmetry is a circle. The same holds for reflectional symmetry. A shape with infinite mirrors is a circle.
Figure 6b shows why we cannot have a wallpaper group made up from a repeating pattern of circles. There is no way to fit the circles together without having the red gaps in between. See for contrast the way we can repeat our pattern with the hexagons to fill the whole plane. So we cannot have a wallpaper pattern generated from repeating circles, hence why in the definition of a wallpaper group, we impose the condition that the point group must be finite.



(a) n-sided shapes (b) Repeating patterns of circles, hexagons

Figure 6: Justification for definition 5.4

5.2 Lattices

Arguments in this section are developed from [[1] p. 148-150].

Now that we have defined our translation subgroup and our point group, we turn to looking at the lattice.

Definition 5.5. *The* ***lattice*** *L of a wallpaper group G is the orbit of the origin in $\mathbb{R}^2$ under the action of translations in H, the translation subgroup of G.*

Theorem 5.6. *For a given wallpaper group G, its lattice L is a subgroup of $\mathbb{R}^2$.*

Proof. Recall H is the translation subgroup of G. Let $\gamma : H \mapsto \mathbb{R}^2$, $\gamma(f) = \mathbf{v}$, if $f \simeq (\mathbf{v}, I)$.
We claim γ is a homorphism. Let $f, g \in H$, $f \simeq (\mathbf{v}, I)$, $g \simeq (\mathbf{w}, I)$ so $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$.

$$\gamma(f \circ g) \simeq \gamma((\mathbf{v}, I)(\mathbf{w}, I)) = \gamma(\mathbf{v} + \mathbf{w}, I) = \mathbf{v} + \mathbf{w} = \gamma(\mathbf{v}, I) + \gamma(\mathbf{w}, I) \simeq \gamma(f) + \gamma(g)$$

So γ is a homomorphism. The image of a homomorphism is a subgroup of the target group. The set of vectors $\mathbf{v} \in \mathbb{R}^2$ such that $(\mathbf{v}, I) \simeq f \in H$ is clearly L. So L is a subgroup of $\mathbb{R}^2$. □

Definition 5.7. *We define a vector* $\mathbf{a}$ *as the vector of minimum length in L. We choose a second vector* $\mathbf{b}$*, which is any vector in L that is not parallel to* $\mathbf{a}$*, with length also as small as possible [1].*

Note **a** and **b** in Definition 5.7 are certainly well-defined, L definitely contains two vectors which are not parallel as the translation subgroup H is generated by two linearly independent vectors.

Theorem 5.8. *The vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *as defined in Definition 5.7 form an* $\mathbb{R}^2$*-basis of the lattice L. [1].*

Proof. Clearly as in their definition we specified that **a** and **b** are non-parallel, they are not multiples of eachother, and hence are $\mathbb{R}^2$-linearly independent.
As $\mathbf{a}, \mathbf{b} \in L$ and L is a subgroup of $\mathbb{R}^2$, $\forall m, n \in \mathbb{Z}$, $m\mathbf{a} + n\mathbf{b} \in L$.
We now need to show L is spanned by **a** and **b**. Assume towards a contradiction this is not true, say there exists $\mathbf{x} \in L$ such that **x** is not a linear combination of **a** and **b**.
$\exists m, n \in \mathbb{Z}$ such that $\mathbf{c} = m\mathbf{a} + n\mathbf{b}$ is the point in the span of **a** and **b** that is closest to **x**. As **x** is not in the span of **a** and **b**, the vector $\mathbf{x} - \mathbf{c}$ is non zero, and is not a multiple of **a** or **b**, or a linear combination of the two. As **c** is the point in the lattice closest to **x**, $\|\mathbf{x} - \mathbf{c}\| \leq \|\mathbf{b}\|$ as otherwise we would have points in the lattice closer to **x**. As $\mathbf{x}, \mathbf{c} \in L$, $(\mathbf{x} - \mathbf{c}) \in L$. By definition of **a**, $\|\mathbf{x} - \mathbf{c}\| \geq \|\mathbf{a}\|$. But as $(\mathbf{x} - \mathbf{c})$ is not a multiple of **a**, it is skew to **a**, contradicting our choice of **b** as the smallest vector skew to **a**. So we have reached a contradiction. $\forall \mathbf{x} \in L$, $\exists m, n \in \mathbb{Z}$ such that $\mathbf{x} = m\mathbf{a} + n\mathbf{b}$, so L is spanned by **a** and **b**.
Hence **a** and **b** form an $\mathbb{R}^2$-basis of L as required.

□

5.2.1 Classification of Lattices

We want a way to classify our lattices, so we think about a way we can picture shapes determined by vectors **a** and **b**. Note by definition we always have $\|\mathbf{a}\| \leq \|\mathbf{b}\|$.
We require $\|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a} + \mathbf{b}\|$. If this is not already the case, replace **b** by -**b**.
Note $\|\mathbf{b}\| \leq \|\mathbf{a} - \mathbf{b}\|$ as otherwise this would contradict the choice of **b** as the smallest vector in the lattice skew to **a**.
So $\|\mathbf{a}\| \leq \|\mathbf{b}\| \leq \|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a} + \mathbf{b}\|$

From our vectors **a**, **b**, we can classify lattices into five different types, depending on whether the above $\leq$ signs are $=$ or $<$. Generally $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ are the lengths of the sides of our shape. $\|\mathbf{a} - \mathbf{b}\|$ and $\|\mathbf{a} + \mathbf{b}\|$ are the lengths of the diagonals. So we can use properties of basic parallelograms to view the shape determined by **a** and **b**. We have the below five possibilities, see Figure 7 for pictures [1]:

1. **Oblique**: $\|\mathbf{a}\| < \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$
2. **Rectangle**: $\|\mathbf{a}\| < \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} + \mathbf{b}\|$
3. **Centred Rectangle**: $\|\mathbf{a}\| < \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$
4. **Square**: $\|\mathbf{a}\| = \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} + \mathbf{b}\|$
5. **Hexagonal**: $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$

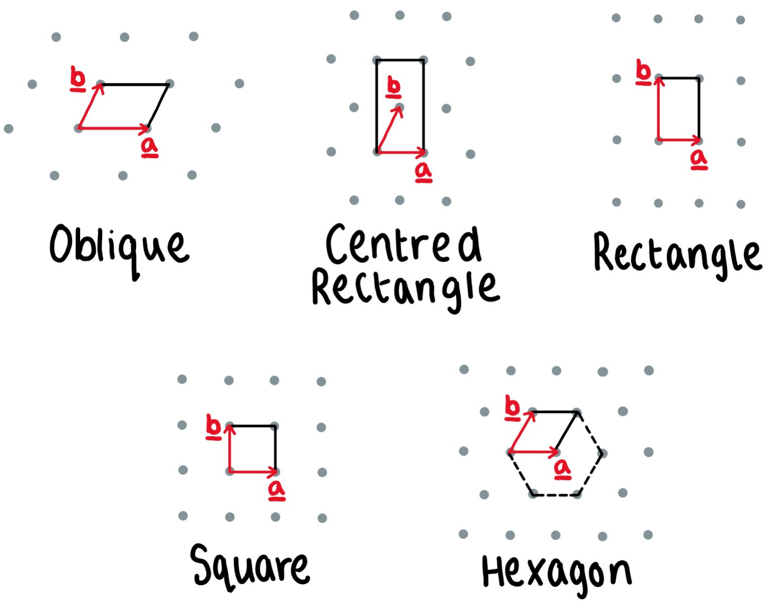


Figure 7: Types of lattice [1]

At first glance it may look like above we missed out a few cases. However, we will prove that we have not.

Theorem 5.9. *With $\boldsymbol{a}$ and $\boldsymbol{b}$ as defined in Definition 5.7, we cannot have $\|\boldsymbol{b}\| = \|\boldsymbol{a}+\boldsymbol{b}\|$.*

Proof. Assume towards a contradiction we do have $\|\mathbf{b}\| = \|\mathbf{a}+\mathbf{b}\|$. Using the triangle inequality,

$$\begin{aligned} \|\mathbf{a}+\mathbf{b}\| &\leq \|\mathbf{a}\| + \|\mathbf{b}\| \\ &\leq \|\mathbf{a}\| + \|\mathbf{a}+\mathbf{b}\| \end{aligned}$$

But $\|\mathbf{a}\| \geq 0$, so we get $\|\mathbf{a}\| = 0$. But by definition $\mathbf{a}$ is a non-zero vector, a contradiction. $\square$

So the above disregards cases $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\| = \|\mathbf{a}+\mathbf{b}\|$ and $\|\mathbf{a}\| < \|\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\| = \|\mathbf{a}+\mathbf{b}\|$.

So the only case not yet considered is $\|\mathbf{a}\| = \|\mathbf{b}\| < \|\mathbf{a}-\mathbf{b}\| < \|\mathbf{a}+\mathbf{b}\|$. The basic shape is a rhombus whose diagonals intersect at right angles. This can be viewed as a centred rectangle lattice with the sides of the rectangle being vectors **a-b** and **a+b** [1]. So this is not a unique lattice. See Figure 8 for a visual depiction of this.

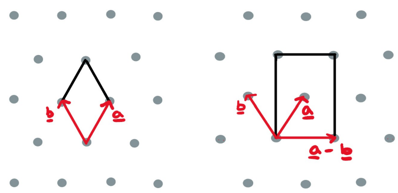


Figure 8: Discounting case $\|\mathbf{a}\| = \|\mathbf{b}\| < \|\mathbf{a}-\mathbf{b}\| < \|\mathbf{a}+\mathbf{b}\|$

The hexagon lattice is sometimes drawn in different ways by different people. Both of these have been demonstrated in Figure 7. The solid lines show the shape most commonly used, while the dotted lines display why this is called the hexagon lattice.

5.3 Crystallographic Restriction

Theorem 5.10. *The point group J acts on the lattice L [[1] p.149].*

Proof. This proof is developed from [[1] p.149-151]. We want to show that for $M \in J$, $\mathbf{x} \in L$, $f_M(\mathbf{x}) \in L$, i.e. any matrix from the point group sends a point in the lattice to another point in the lattice.
So let $\tau \simeq (\mathbf{x}, I) \in H$, and $g \simeq (\mathbf{v}, M) \in G$.
Taking the same map from Theorem 5.1, we have $\pi(g) = M$.
Recall $ker(\pi) = H$, so as H is the kernel of a homomorphism, H is a normal subgroup of G by the First Isomorphism Theorem. By the definition of a normal subgroup, $\forall g \in G$, $\forall \tau \in H$, $g\tau g^{-1} \in H$. Note, using Theorems 3.2 and 3.3,

$$\begin{aligned} g\tau g^{-1} &= (\mathbf{v}, M)(\mathbf{x}, I)(-{f_M}^{-1}(\mathbf{v}), M^{-1}) \\ &= (\mathbf{v}, M)(\mathbf{x} - {f_M}^{-1}(\mathbf{v}), M^{-1}) \\ &= (\mathbf{v} + f_M(\mathbf{x} - {f_M}^{-1}(\mathbf{v})), MM^{-1}) \\ &= (\mathbf{v} + f_M(\mathbf{x}) - f_M({f_M}^{-1}(\mathbf{v})), I) \\ &= (\mathbf{v} + f_M(\mathbf{x}) - \mathbf{v}, I) \\ &= (f_M(\mathbf{x}), I) \end{aligned}$$

So $(f_M(\mathbf{x}), I) \in H$, so $f_M(\mathbf{x}) \in L$. Hence any matrix sends a point in the lattice to another point in the lattice.
J acts on the lattice L. $\square$

Corollary 5.11. *The only possible rotations in a wallpaper group have order 2,3,4 or 6. This is often known as the **'Crystallographic Restriction'** [[2] p.213].*

Proof. The idea for this proof was taken from [[2] p.213]. Firstly we have the vector $\mathbf{a}$, as in Definiton 5.7, the vector of minimum length in the lattice L. Let τ be the translation by vector $\mathbf{a}$, so $\tau \simeq (\mathbf{a}, I)$. We also consider a rotation about the origin by angle θ, so $g \simeq (\mathbf{0}, A_\theta)$, inheriting matrix A_θ from Section 3.1.
By Theorem 5.10, g sends a point on the lattice to another point on the lattice. So any composition of g and τ sends the origin to a point on the lattice.
Say $\mathbf{a} = (a_1, a_2)$, so $\|\mathbf{a}\| = \sqrt{{a_1}^2 + {a_2}^2}$

Claim: We cannot have rotation of order greater than 6 in a wallpaper group.
Proof of Claim: Suppose towards a contradiction we have rotation of order greater than 6 in our wallpaper group. Consider

$$(\tau^{-1} \circ g \circ \tau)(\mathbf{0}) = (\tau^{-1} \circ g)(\mathbf{a}) = \tau^{-1}(f_{A_\theta}(\mathbf{a})) = f_{A_\theta}(\mathbf{a}) - \mathbf{a}$$

So $f_{A_\theta}(\mathbf{a}) - \mathbf{a}$ is a point on the lattice.
Note

$$\begin{aligned}
f_{A_\theta}(\mathbf{a}) - \mathbf{a} &= \mathbf{a}{A_\theta}^t - \mathbf{a} = (a_1, a_2)\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} - (a_1, a_2) \\
&= (a_1 cos\theta - a_2 sin\theta, a_1 sin\theta + a_2 cos\theta) - (a_1, a_2) \\
&= (a_1(cos\theta - 1) - a_2 sin\theta, a_1 sin\theta + a_2(cos\theta - 1)) \\
\|f_{A_\theta}(\mathbf{a}) - \mathbf{a}\| &= \sqrt{(a_1(cos\theta - 1) - a_2 sin\theta)^2 + (a_1 sin\theta + a_2(cos\theta - 1))^2} \\
&= \sqrt{{a_1}^2(cos\theta - 1)^2 + {a_2}^2 sin^2\theta + a_1 sin^2\theta + {a_2}^2(cos\theta - 1)^2} \\
&= \sqrt{({a_1}^2 + {a_2}^2)((cos\theta - 1)^2 + sin^2\theta)} \\
&= \sqrt{({a_1}^2 + {a_2}^2)(2 - 2cos\theta)} \\
&= \sqrt{(4sin^2\frac{\theta}{2})({a_1}^2 + {a_2}^2)} \; * \\
&= 2sin(\frac{\theta}{2}) \cdot \|\mathbf{a}\|
\end{aligned}$$

Where on line $*$ we have used double angle formula $cos\theta = 1 - 2sin^2(\frac{\theta}{2})$.
So $\|f_{A_\theta}(\mathbf{a}) - \mathbf{a}\| = 2sin(\frac{\theta}{2}) \cdot \|\mathbf{a}\|$. As we have rotation of order greater than 6, we have $\theta < 2\pi/6$. Then $2sin(\frac{\theta}{2}) < 1$. So $\|f_{A_\theta}(\mathbf{a}) - \mathbf{a}\| < \|\mathbf{a}\|$, contradicting our initial choice of $\mathbf{a}$. So in a wallpaper group we cannot have rotation of order greater than 6. $\square$

We have narrowed down our choices for order of rotation to 2,3,4,5,6.

Claim: In a wallpaper group, we cannot have rotation of order 5.
Proof of Claim: Assume towards a contradiction we have rotation of order 5 in a wallpaper group. This is generated by rotation by angle $\frac{2\pi}{5}$.
We still use the above g and τ, with $\theta = \frac{2\pi}{5}$. We consider

$$(\tau \circ g \circ g \circ \tau)(\mathbf{0}) = (\tau \circ g \circ g)(\mathbf{a}) = (\tau \circ g)(f_{A_\theta}(\mathbf{a})) = \tau({f_{A_\theta}}^2(\mathbf{a})) = {f_{A_\theta}}^2(\mathbf{a}) + \mathbf{a}$$

So ${f_{A_\theta}}^2(\mathbf{a}) + \mathbf{a}$ is a point on the lattice.
Note

$$\begin{aligned}
{f_{A_\theta}}^2(\mathbf{a}) &= (a_1, a_2)\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \\
&= (a_1, a_2)\begin{bmatrix} cos^2\theta - sin^2\theta & 2cos\theta sin\theta \\ -2cos\theta sin\theta & cos^2\theta - sin^2\theta \end{bmatrix} \\
&= (a_1(cos^2\theta - sin^2\theta) - 2a_2(cos\theta sin\theta), 2a_1(cos\theta sin\theta) + a_2(cos^2\theta - sin^2\theta)) \\
{f_{A_\theta}}^2(\mathbf{a}) + \mathbf{a} &= (a_1(cos^2\theta - sin^2\theta + 1) - 2a_2(cos\theta sin\theta), 2a_1(cos\theta sin\theta) + a_2(cos^2\theta - sin^2\theta + 1)) \\
\|{f_{A_\theta}}^2(\mathbf{a}) + \mathbf{a}\| &= \sqrt{(a_1(cos^2\theta - sin^2\theta + 1) - 2a_2(cos\theta sin\theta))^2 + (2a_1(cos\theta sin\theta) + a_2(cos^2\theta - sin^2\theta + 1))^2} \\
&= \sqrt{{a_1}^2(cos^2\theta - sin^2\theta + 1)^2 + 4{a_2}^2 cos^2\theta sin^2\theta + 4{a_1}^2 cos^2\theta sin^2\theta + {a_2}^2(cos^2\theta - sin^2\theta + 1)^2} \\
&= \sqrt{({a_1}^2 + {a_2}^2)((cos^2\theta - sin^2\theta + 1)^2 + 4cos^2\theta sin^2\theta)} \\
&= \sqrt{({a_1}^2 + {a_2}^2)((2cos^2\theta)^2 + 4cos^2\theta(1 - cos^2\theta))} \; * \\
&= \sqrt{({a_1}^2 + {a_2}^2)(4cos^2\theta)} \\
&= 2cos\theta \cdot \|\mathbf{a}\|
\end{aligned}$$

On line $*$ we have used trig identity $cos^2\theta + sin^2\theta = 1$
So $\|{f_{A_\theta}}^2(\mathbf{a}) + \mathbf{a}\| = 2cos\theta \cdot \|\mathbf{a}\|$. When $\theta = \frac{2\pi}{5}$, $2cos\theta < 1$, so we get $\|{f_{A_\theta}}^2(\mathbf{a}) + \mathbf{a}\| < \|\mathbf{a}\|$, contradicting our choice of $\mathbf{a}$ as the smallest length vector on the lattice. So we cannot have rotations of order 5. $\square$

So our only allowed rotations are of order 2,3,4,6.

$\square$

Corollary 5.12. *The rotations in a wallpaper group are generated by rotations through one of the angles* $\pi, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}$ *[[1] p.152].*

Proof. This follows directly from Corollary 5.11. The rotations in a wallpaper group are of order 2,3,4,6, and a rotation of order p is generated by rotation by angle $\frac{2\pi}{p}$. So the rotations in a wallpaper group are generated by rotations through one of the angles $\frac{2\pi}{2} = \pi, \frac{2\pi}{3}, \frac{2\pi}{4} = \frac{\pi}{2}, \frac{2\pi}{6} = \frac{\pi}{3}$. □

Theorem 5.13. *An isomorphism between two wallpaper groups preserves the type of isometry, i.e. it sends translations to translations, reflections to reflections, rotations to rotations and glide reflections to glide reflections. [[1] p. 152]*

Proof. For a proof of this theorem, refer to [[1] p. 152-153]. □

Corollary 5.14. *If two wallpaper groups are isomorphic then their point groups are isomorphic [[1] p.153].*

Proof. For a proof of this theorem, refer to [[1] p. 153]. □

The converse to Corollary 5.14 is not necessarily true.

6 Wallpaper Groups

We now want to define each of the seventeen wallpaper groups. We will then show no two groups are isomorphic.

Definition 6.1. *For a wallpaper pattern, we define the* ***unit cell*** *as the minimal part of the repeating pattern that will produce the whole pattern by translations alone [[8] p.681].*

6.1 Outline of Method

The arguments in this section are developed from [[1] p.155-156].

We consider each lattice L in turn. When we consider a lattice L, the unit cell of our wallpaper group is the shape drawn in Figure 7. We find the set of orthogonal transformations which preserve the lattice and call this O_L. We inherit the notation from the Section 5 with the lattice being the set of points of the form $m\mathbf{a} + n\mathbf{b}, m, n \in \mathbb{Z}$, and $\mathbf{a}, \mathbf{b}$ as in Definition 5.7. See Table 1 for O_L for each lattice.

Lattice	Orthogonal transformations preserving lattice	Lattice type
Oblique	$O_L = \{I, -I\}$	Primitive
Rectangle	$O_L = \{I, -I, B_0, B_\pi\}$	Primitive
Centred Rectangle	$O_L = \{I, -I, B_0, B_\pi\}$	Centred
Square	O_L is generated by $A_{\frac{\pi}{2}}$ and B_0	Primitive
Hexagonal	O_L is generated by $A_{\frac{\pi}{3}}$ and B_0	Primitive

Table 1: Properties of lattices

For any wallpaper group G which has L as it's lattice, its point group J must be a subgroup of O_L, and its translation subgroup is $H = \{m\mathbf{a} + n\mathbf{b} : m, n \in \mathbb{Z}\}$. So for each lattice, we analyse each subgroup of O_L in turn, and find the wallpaper group G with point group J and translation subgroup H. (We need not consider cases equivalent to those we have already found).
We find the ordered pair form of all isometries in G and use the arguments in Section 4 to analyse these ordered pairs. For any reflections or glide reflections we find the mirror/glide lines, and for rotations we find the centres of rotation.

Note when analysing a lattice L we need not consider the point groups which we have already looked at with previous lattices, unless these point groups can generate new isometries in G.

6.2 Notation

The notation in this section is inherited from [[1] p.155-156].

We call each wallpaper group a name made up from various symbols describing the group.
The first letter is regarding the lattice of our group. We have two kinds of lattice, primitive and centred. A lattice is primitive if it contains no lattice points in its interior (e.g. the oblique lattice from Figure 7). It is centred otherwise (e.g. centred rectangular lattice from Figure 7). In the name of the wallpaper group, we have a p if the lattice is primitive, and a c if it is centred.
Next in the name of our wallpaper group we have a number, 1,2,3,4,6, pertaining to the highest order of rotation in

our group. Finally we have an m if our wallpaper group has a reflection, and a g if it has a glide reflection. A_θ and B_φ are matrices as defined in Section 3.1.

6.3 An Example: p4mm

The lattice of G is square. This is a primitive lattice.
The group of orthogonal transformations preserving this lattice are generated by $A_{\frac{\pi}{2}}$ and B_0. So $O_L = \{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}, B_0, B_{\frac{\pi}{2}}, -B_0, B_{\frac{3\pi}{2}}\}$.

For this wallpaper group, say $J = O_L$ and B_0 can be realised by a reflection in G.

For ease, we choose the origin to be the centre of order 4 rotation. So $(\mathbf{0}, A_{\frac{\pi}{2}}) \in G$.
Note as G is a group,

$$(\mathbf{0}, A_{\frac{\pi}{2}})(\mathbf{0}, A_{\frac{\pi}{2}}) = (\mathbf{0}, -I) \in G \qquad (\mathbf{0}, A_{\frac{\pi}{2}})(\mathbf{0}, -I) = (\mathbf{0}, A_{\frac{3\pi}{2}}) \in G$$

G contains all translations of the form $(m\mathbf{a} + n\mathbf{b}, I)$, $m, n \in \mathbb{Z}$.

6.3.1 Centres of Rotation

Half-turns in G are of the form $(m\mathbf{a} + n\mathbf{b}, I)(\mathbf{0}, -I) = (m\mathbf{a} + n\mathbf{b}, -I)$, $m, n \in \mathbb{Z}$.
We want to find the centres for this order 2 rotation. If $g \simeq (m\mathbf{a} + n\mathbf{b}, -I)$, then as per our argument in Section 4, if $\mathbf{c}$ is our center of rotation,

$$m\mathbf{a} + n\mathbf{b} = \mathbf{c} - f_{-I}(\mathbf{c}) = \mathbf{c} - \mathbf{c}(-I)^T = \mathbf{c} - (-\mathbf{c}) = 2\mathbf{c}$$
$$\mathbf{c} = \frac{1}{2}m\mathbf{a} + \frac{1}{2}n\mathbf{b}$$

So we have half turns around all points $\frac{1}{2}m\mathbf{a} + \frac{1}{2}n\mathbf{b}$, $\forall m, n \in \mathbb{Z}$. This means we have half turns around all lattice points and all points halfway between lattice points.

Note a centre of order 4 rotation is also a center of order 2 rotation, so the centres of order 4 rotation are also of the form $\mathbf{c} = \frac{1}{2}l\mathbf{a} + \frac{1}{2}k\mathbf{b}$ for some $k, l \in \mathbb{Z}$.
Also as $(\mathbf{0}, A_{\frac{\pi}{2}}) \in G$, order 4 rotations in G have the form $(q\mathbf{a} + r\mathbf{b}, A_{\frac{\pi}{2}}), \forall q, r \in \mathbb{Z}$.
Let $\mathbf{c}$ be the center of an order 4 rotation, $\exists l, k \in \mathbb{Z}, \mathbf{c} = \frac{1}{2}l\mathbf{a} + \frac{1}{2}k\mathbf{b}$. $\forall q, r \in \mathbb{Z}$:

$$q\mathbf{a} + r\mathbf{b} = \mathbf{c} - f_{A_{\frac{\pi}{2}}}(\mathbf{c}) = \frac{1}{2}l\mathbf{a} + \frac{1}{2}k\mathbf{b} - (\frac{1}{2}l\mathbf{b} - \frac{1}{2}k\mathbf{a}) = (\frac{1}{2}l + \frac{1}{2}k)\mathbf{a} + (\frac{1}{2}k - \frac{1}{2}l)\mathbf{b}$$

So we have order 4 rotations when $\frac{1}{2}l + \frac{1}{2}k \in \mathbb{Z}$.

So G has order 2 rotations at $\frac{1}{2}m\mathbf{a} + \frac{1}{2}n\mathbf{b}$, and if $\frac{1}{2}m + \frac{1}{2}n \in \mathbb{Z}$, this point is a centre of an order 4 rotation.

6.3.2 Reflections and Glides

So far we have found the centres of all rotations in G, we now want to look at the reflections and glides in G.

Let $(q\mathbf{a} + r\mathbf{b}, B_0) \in G$. Let $\mathbf{v} = q\mathbf{a} + r\mathbf{b}$. Recall we said B_0 realises a reflection so $\mathbf{v} + f_{B_0}(\mathbf{v}) = 0$.

$$q\mathbf{a} + r\mathbf{b} + f_{B_0}(q\mathbf{a} + r\mathbf{b}) = \mathbf{0}$$
$$q\mathbf{a} + r\mathbf{b} = -q\mathbf{a} + r\mathbf{b}$$

So $q = 0$, $(r\mathbf{b}, B_0) \in G$. Recall $(\mathbf{0}, A_{\frac{\pi}{2}}) \in G$. So $(\mathbf{0}, A_{\frac{\pi}{2}})(r\mathbf{b}, B_0) = (r\mathbf{a}, B_{\frac{\pi}{2}}) \in G$.
$(r\mathbf{a}, B_{\frac{\pi}{2}})(r\mathbf{a}, B_{\frac{\pi}{2}}) = (r\mathbf{a} + r\mathbf{b}, I) \in G$. So we see $r \in \mathbb{Z}$. So $(r\mathbf{b}, B_0) \in G$, which implies $(\mathbf{0}, B_0) \in G$.

Following from this we see

$$(\mathbf{0}, A_{\frac{\pi}{2}})(\mathbf{0}, B_0) = (\mathbf{0}, B_{\frac{\pi}{2}}) \in G \qquad (\mathbf{0}, A_{\frac{\pi}{2}})(\mathbf{0}, B_{\frac{\pi}{2}}) = (\mathbf{0}, B_\pi) \in G \qquad (\mathbf{0}, A_{\frac{\pi}{2}})(\mathbf{0}, B_\pi) = (\mathbf{0}, B_{\frac{3\pi}{2}}) \in G$$

So the origin is the intersection of the horizontal, vertical and diagonal mirrors.

Now we want to find the horizontal mirrors in G. $(m\mathbf{a} + n\mathbf{b}, I)(0, B_0) = (m\mathbf{a} + n\mathbf{b}, B_0) \in G$, $\forall m, n \in \mathbb{Z}$. Following our argument in Section 4, $\mathbf{v} = m\mathbf{a} + n\mathbf{b}$, $m, n \in \mathbb{Z}$, and as G contains horizontal reflections, $f_{B_0}(\mathbf{v}) = -\mathbf{v}$.
$f_{B_0}(m\mathbf{a} + n\mathbf{b}) = m\mathbf{a} - n\mathbf{b}$ and we have the condition $m\mathbf{a} - n\mathbf{b} = -(m\mathbf{a} + n\mathbf{b})$. This implies $m = 0$, and $n \in \mathbb{Z}$. So $\mathbf{v} = n\mathbf{b}$, $\forall n \in \mathbb{Z}$. Our mirrors are now lines parallel with the x axis, shifted by $\frac{1}{2}\mathbf{v} = \frac{1}{2}n\mathbf{b}$, $\forall n \in \mathbb{Z}$. So we have reflections in horizontal mirrors either passing through lattice points or lying halfway between lattice points.

Adapting this above argument to consider vertical reflections, we see G has reflections in vertical mirrors either passing through lattice points or lying halfway between lattice points.

Now we want to analyse how $B_{\frac{\pi}{2}}$ and $B_{\frac{3\pi}{2}}$ are realised in G.

Note $(\mathbf{0}, B_{\frac{\pi}{2}}), (m\mathbf{a}+n\mathbf{b}, I) \in G$ so $(m\mathbf{a}+n\mathbf{b}, B_{\frac{\pi}{2}}) \in G$. So $\mathbf{v} = m\mathbf{a}+n\mathbf{b}$.
In the below recall $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, so $\mathbf{a}\cdot\mathbf{b} = 0$. Also this is the square lattice, so $|| \mathbf{a} ||^2 = || \mathbf{b} ||^2$.

$$
\begin{aligned}
\mathbf{w} &= \mathbf{v} - f_{B_{\frac{\pi}{2}}}(\mathbf{v}) = m\mathbf{a}+n\mathbf{b} - (m\mathbf{b}+n\mathbf{a}) = (m-n)(\mathbf{a}-\mathbf{b}) \\
\mathbf{v}\cdot\mathbf{w} &= (m-n)(m\mathbf{a}+n\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\
&= (m-n)(m|| \mathbf{a} ||^2 - m(\mathbf{a}\cdot\mathbf{b}) + n(\mathbf{a}\cdot\mathbf{b}) - n|| \mathbf{b} ||^2) \\
&= (m-n)(m|| \mathbf{a} ||^2 - n|| \mathbf{b} ||^2) \\
&= (m-n)^2|| \mathbf{a} ||^2 \\
|| \mathbf{w} ||^2 &= (m-n)^2((\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})) \\
&= (m-n)^2(|| \mathbf{a} ||^2 - 2(\mathbf{a}\cdot\mathbf{b}) + || \mathbf{b} ||^2) \\
&= 2(m-n)^2|| \mathbf{a} ||^2 \\
2\mathbf{r} &= (\frac{\mathbf{v}\cdot\mathbf{w}}{|| \mathbf{w} ||^2})\mathbf{w} \\
&= \frac{(m-n)^2|| \mathbf{a} ||^2}{2(m-n)^2|| \mathbf{a} ||^2}(m-n)(\mathbf{a}-\mathbf{b}) \\
&= \frac{1}{2}(m-n)(\mathbf{a}-\mathbf{b}) \\
\mathbf{s} &= \mathbf{v} - 2\mathbf{r} = m\mathbf{a}+n\mathbf{b} - \frac{1}{2}(m-n)(\mathbf{a}-\mathbf{b}) = (\frac{1}{2}m + \frac{1}{2}n)(\mathbf{a}+\mathbf{b})
\end{aligned}
$$

This is a reflection if $f_{B_{\frac{\pi}{2}}}(\mathbf{v}) = -\mathbf{v}$, $n\mathbf{a}+m\mathbf{b} = -(m\mathbf{a}+n\mathbf{b})$. So if $n = -m$ we have a reflection through $2\mathbf{r} = m(\mathbf{a}-\mathbf{b})$. These are the diagonal lines through lattice points (subtending angle $\frac{\pi}{4}$ with the horizontal axis).

If $f_{B_{\frac{\pi}{2}}}(\mathbf{v}) \neq -\mathbf{v}$, then we have glide reflections, with the glide lines parallel to the above but passing through points $\frac{1}{2}(m-n)(\mathbf{a}-\mathbf{b})$. So there are glide lines subtending angle $\frac{\pi}{4}$ with the horizontal axis passing through lattice points and halfway between lattice points. To carry out a glide reflection we reflect in one of those lines and then translate by vector $(\frac{1}{2}m + \frac{1}{2}n)(\mathbf{a}+\mathbf{b})$.

We follow a similar argument for $B_{\frac{3\pi}{4}}$ to see G has mirror lines subtending angle $\frac{3\pi}{4}$ with the horizontal axis passing through lattice points. G also has glide lines subtending angle $\frac{3\pi}{4}$ with the horizontal axis passing through lattice points and halfway between lattice points. To carry out these glides reflect in the glide lines and translate by vector $(\frac{1}{2}m - \frac{1}{2}n)(\mathbf{a}-\mathbf{b})$.

6.3.3 p4mm Summary

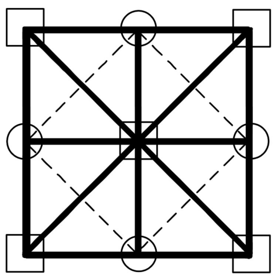


Figure 9: p4mm

So overall, our wallpaper group G contains order 2 rotations at $\frac{1}{2}m\mathbf{a}+\frac{1}{2}n\mathbf{b}$, and if $\frac{1}{2}m+\frac{1}{2}n \in \mathbb{Z}$, this point is a centre of an order 4 rotation. We have horizontal and vertical mirrors through lattice points and halfway between lattice points. We have mirrors through lattice points on both diagonals. So there are mirror lines subtending angle $\frac{\pi}{4}$ with the horizontal axis passing through lattice points and mirror lines subtending angle $\frac{3\pi}{4}$ with the horizontal axis passing through lattice points. There are glide lines subtending angle $\frac{\pi}{4}$ with the horizontal axis passing through lattice points and halfway between lattice points, and glide lines subtending angle $\frac{3\pi}{4}$ with the horizontal axis passing through lattice points and halfway between lattice points

As the wallpaper group's highest order of rotation is 4, and it has mirrors lying in two directions, we call the wallpaper group p4mm. See Figure 9 for the unit cell with centers of rotations, mirrors and glides labelled.

6.4 All 17 Groups

The information in this section is based from [[1] p.156-161].

A description of all wallpaper groups is laid out in Table 2 and Figure 10.
See Table 2 for each group, the lattice it is based on, its point group and any additional stipulations. The mirror lines, glide lines and centres of rotation can be seen in Figure 10.
See Figure 10 for diagrams of unit cells of each of the 17 wallpaper groups. We draw a thick line for a mirror and a broken line for a glide line. At the centres of rotation we draw a ○, ▲, □, ● to represent rotation of order 2,3,4,6 respectively.

Wallpaper group	Lattice	Point group	Additional stipulations
p1	Oblique	$J = \{I\}$	
p2	Oblique	$J = \{I, -I\}$	
pm	Rectangle	$J = \{I, B_0\}$	B_0 can be realised by a reflection in G
pg	Rectangle	$J = \{I, B_0\}$	B_0 can't be realised by a reflection in G
p2mm	Rectangle	$J = \{I, -I, B_0, B_\pi\}$	B_0 and B_π can be realised by reflections in G
p2mg	Rectangle	$J = \{I, -I, B_0, B_\pi\}$	B_0 can be realised by a reflection in G but B_π can't be realised by a reflection in G
p2gg	Rectangle	$J = \{I, -I, B_0, B_\pi\}$	B_0 and B_π can't be realised by reflections in G
cm	Centred Rectangle	$J = \{I, B_0\}$	B_0 can be realised by a reflection in G
c2mm	Centred Rectangle	$J = \{I, -I, B_0, B_\pi\}$	B_0 and B_π can be realised by reflections in G
p4	Square	J is generated by $A_{\frac{\pi}{2}}$	
p4mm	Square	J is generated by $A_{\frac{\pi}{2}}$ and B_0	B_0 can be realised by a reflection in G
p4gm	Square	J is generated by $A_{\frac{\pi}{2}}$ and B_0	B_0 can't be realised by a reflection in G
p3	Hexagon	J is generated by $A_{\frac{2\pi}{3}}$	
p3m1	Hexagon	J is generated by $A_{\frac{2\pi}{3}}$ and B_0	B_0 can be realised by a reflection in G
p31m	Hexagon	J is generated by $A_{\frac{2\pi}{3}}$ and $B_{\frac{\pi}{3}}$	$B_{\frac{\pi}{3}}$ can be realised by a reflection in G
p6	Hexagon	J is generated by $A_{\frac{\pi}{3}}$	
p6mm	Hexagon	J is generated by $A_{\frac{\pi}{3}}$ and B_0	B_0 can be realised by a reflection in G

Table 2: Wallpaper groups

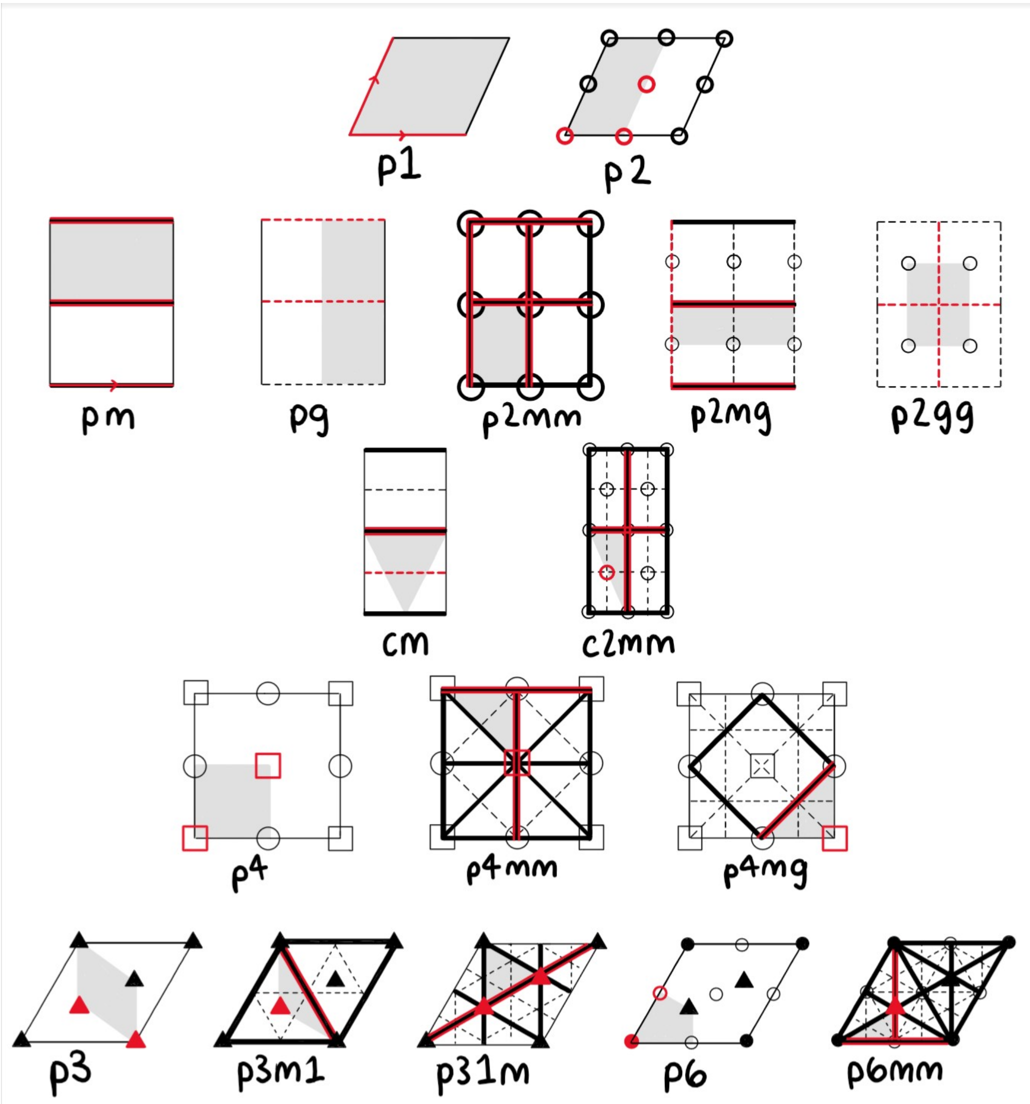


Figure 10: 17 Wallpaper groups with generating regions shaded

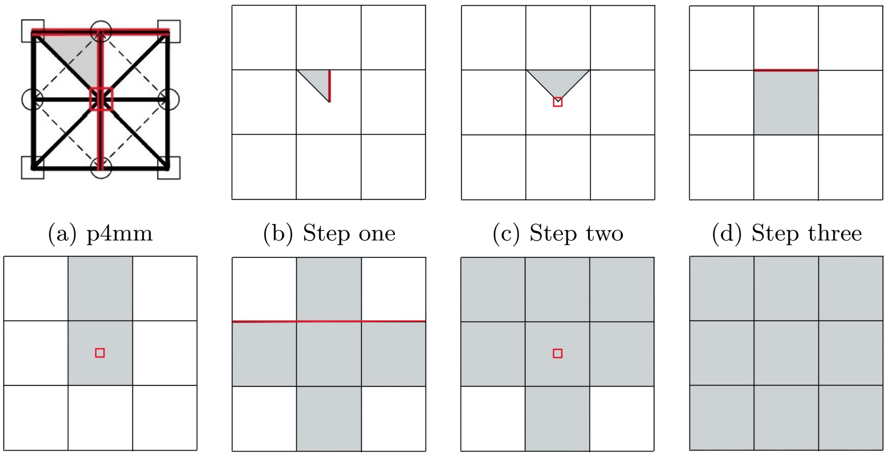
6.5 Generating Regions and Isometries

Definition 6.2. *For each wallpaper group, the* ***generating region*** *is the minimum area that when transformed by the isometries in the wallpaper group, creates the complete design. The generating region will comprise part of the unit cell [[8] p.683].*

Definition 6.3. *For a wallpaper group and its generating region, the set of* ***generating isometries*** *is a set (of minimal cardinality) of isometries in the wallpaper group, which when acting on the generating region can fill the whole plane [[8] p.683].*

Note that the generating region and the generating isometries are certainly not unique, we have many choices. However for a given wallpaper group, two generating regions must have the same area, and two sets of generating isometries must have the same cardinality, as otherwise this would contradict minimality.
The generating regions and generating isometries of each wallpaper group are highlighted in Figure 10. The generating regions have been shaded in grey. The generating isometries have been drawn in red. Where a mirror is a generating isometry, it has been drawn as two red lines to emphasise. Some of these generating isometries and regions have been used from [[8] p.683], while others have been ammended.

Example 6.5.1. We will show that the generating region and generating isometries for wallpaper group p4mm in Figure 10 do in fact fill the plane. See Figure 11, where in each step the next isometry enacted is denoted in red. Clearly if we continue on we will fill the whole plane.



(a) p4mm (b) Step one (c) Step two (d) Step three

(e) Step four (f) Step five (g) Step six (h) Step seven

Figure 11: Generating region and isometries for p4mm

6.6 Proving No Two Groups are Isomorphic

The arguments in this section are based from [[1] p.161-162].

See Table 2 which has the point group listed for each wallpaper group. See Table 3 for a list of each wallpaper group along with a standard group isomorphic to its point group. D_n is the group of symmetries of a regular n-sided polygon.

Wallpaper group	Group isomorphic to point group
p1	$\{I\}$
p2	$(\mathbb{Z}_2, +)$
pm	$(\mathbb{Z}_2, +)$
pg	$(\mathbb{Z}_2, +)$
p2mm	$(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$
p2mg	$(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$
p2gg	$(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$
cm	$(\mathbb{Z}_2, +)$
c2mm	$(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$

Wallpaper group	Group isomorphic to point group
p4	$(\mathbb{Z}_4, +)$
p4mm	$(D_4, \circ)$
p4gm	$(D_4, \circ)$
p3	$(\mathbb{Z}_3, +)$
p3m1	$(D_3, \circ)$
p31m	$(D_3, \circ)$
p6	$(\mathbb{Z}_6, +)$
p6mm	$(D_6, \circ)$

Table 3: Wallpaper groups and point groups [1]

By Corollary 5.14, groups cannot be isomorphic if their point groups are not isomorphic, so we only need to show that groups with the same entry in the second column in Table 3 are not isomorphic.
So it immediately follows that none of p1, p4, p3, p6 and p6mm are isomorphic to each other or any other wallpaper group.

Theorem 6.4. *No two of p2, pm, pg or cm are isomorphic.*

Proof. p2 does not have mirror or glide lines so cannot be isomorphic to pm, pg or cm.
pg does not contain a reflection so cannot be isomorphic to pm or cm.
For all glides in pm, its reflection part and its translation part both belong to pm themselves. However in cm, there are glides who's reflection parts do not lie in cm themselves. So pm is not isomorphic to cm. □

Theorem 6.5. *No two of p2mm, p2mg, p2gg or c2mm are isomorphic.*

Proof. p2gg does not contain any reflections so cannot be isomorphic to p2mm, p2mg or c2mm.
p2mm contains all constituent parts of its glides so p2mm is not isomorphic to p2mg or c2mm.
Mirrors in p2mg are all horizontal, so the composition of two reflections is always a translation. However, c2mm contains horizontal and vertical mirrors, so the product of two reflections in c2mm could be a half-turn. So p2mg is not isomorphic to c2mm. □

Theorem 6.6. *p4mm and p4mg are not isomorphic.*

Proof. Each rotation of order 4 in p4mm can be written as the product of two reflections in p4mm. For example, rotation by angle $\frac{\pi}{4}$ about the origin can be found by the composition of two reflections in lines through the origin. $(\mathbf{0}, B_\pi)(\mathbf{0}, B_{\frac{\pi}{2}}) = (\mathbf{0}, A_{\frac{\pi}{2}})$, where both reflections themselves are in p4mm. However, $(\mathbf{0}, A_{\frac{\pi}{2}})$ cannot be realised as the product of two reflections in p4gm, so p4mm is not isomorphic to p4gm. □

Theorem 6.7. *p3m1 is not isomorphic to p31m.*

Proof. In p31m each rotation of order 3 can be realised as the composition of two reflections, for example $(\mathbf{0}, A_{\frac{\pi}{3}}) = (\mathbf{0}, B_{\frac{\pi}{3}})(\mathbf{0}, B_0)$. However $(\mathbf{0}, A_{\frac{\pi}{2}})$ cannot be realised as the composition of two reflections in p3m1. So p31m is not isomorphic to p3m1. □

6.7 Classifying Wallpaper Groups

The arguments in this section are based from [6].

Say we are given a wallpaper pattern, and we want to find the wallpaper group which acts on the pattern.
We classify the pattern by considering the isometries which preserve it. Looking at Figure 10 a good way to divide the set of wallpaper groups into smaller sets is by finding the highest order rotation. We then use Table 4 to figure out exactly which wallpaper group we are dealing with, by considering the mirrors and glide lines of the pattern.
So to classify a wallpaper pattern answer questions from Table 4 to find the exact wallpaper group.
See Section 8 for an example of classifying a wallpaper pattern.

Highest order rotation	Are there mirrors?	Are there mirrors lying in two directions?	Are there glide lines that aren't mirror lines?	Do all centres of rotation lie on mirror lines?	Wallpaper group
0	No		No		p1
0	No		Yes		pg
0	Yes		No		pm
0	Yes		Yes		cm
2	No	No	No		p2
2	No	No	Yes		p2gg
2	Yes	No	Yes		p2mg
2	Yes	Yes	No		p2mm
2	Yes	Yes	Yes		c2mm
3	No			No	p3
3	Yes			No	p3m1
3	Yes			Yes	p31m
4	No			No	p4
4	Yes			Yes	p4mm
4	Yes			No	p4mg
6	No				p6
6	Yes				p6

Table 4: Classifying a wallpaper pattern

7 Two-Colour Groups

7.1 Two-Colour Symmetries

If we introduce colours into our patterns, we can analyse the symmetries in three different ways. Firstly we could just consider all colours to be the same, which would keep the symmetries of the original pattern the same. We could consider the colours as entirely different, which could reduce the symmetries of our pattern, or for example require our translation vectors to be larger. We may consider colours to be opposite, allowing symmetries which permute colours, but think of these as different isometries to the ones preserving colours [3].
We will investigate this 'opposite' type of symmetry when we have two colours. This has many names, including antisymmetry, two-colour symmetry or black-and-white symmetry.

Definition 7.1. *Say we have a pattern in two colours, call these colours a and b.*
Type (i) *A symmetry which sends all a coloured segments to a coloured segments and all b coloured segments to b coloured segments, i.e the symmetry preserves colours. This is normal symmetry.*
Type (ii) *A symmetry which sends all a coloured segments to b coloured segments and all b coloured segments to a coloured segments, i.e the symmetry reverses colours. This is called an **antisymmetry**.*
*If we have a symmetry satisfying one of the above, we call it a **two-colour symmetry**. [[8] p.685]*

Theorem 7.2. *Let K={i,ii} be the set of the two types of symmetry. This is a group under the composition of functions.*

Proof. Say g, g' are type (i) symmetries and h and h' are type (ii) symmetries. g and g' both preserve colours so $g \circ g'$ clearly preserves colours, and is a type (i) symmetry. $h \circ h'$ first reverses colours then reverses them back, so is a type (i) symmetry. $g \circ h$ first reverses colours then preserves them, so overall reverses colours. So $g \circ h$ is a type (ii) symmetry. Similarly $h \circ g$ first preserves colours and then reverses them, so is a type (ii) symmetry.
See the Cayley Table for K in Table 5.

·	i	ii
i	i	ii
ii	ii	i

Table 5: Cayley table for K={i,ii}

- Closure: Looking at Table 5, K is closed under composition of two-colour symmetries. i.e. the composition of 2 two-colour symmetries is a two-colour symmetry.
- Associativity: Checking case by case, we can see $\forall x, y, z \in K$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- Identity: Clearly i is the identity from looking at the Cayley Table.
- Inverse: Again looking at Table 5, $(i)^{-1} = i$, $(ii)^{-1} = ii$.

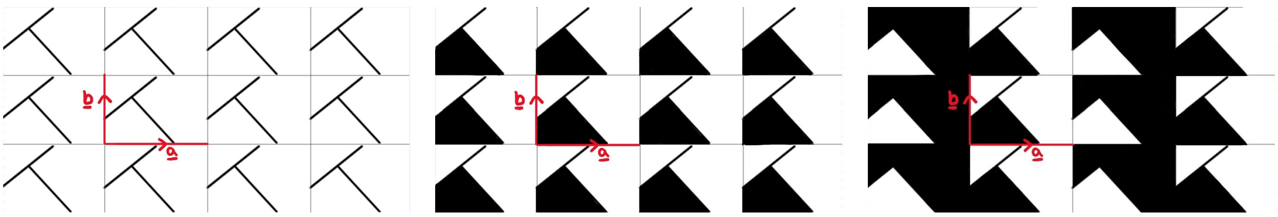
□

Remark. *Notice type (i) and (ii) symmetries induce permutations of colours, and if we label our colours a and b, it is clear to see $K \cong Sym(\{a, b\})$.*

Definition 7.3. *For a given wallpaper group G, the group of all two-colour symmetries is given by the direct product group $G \times K$. A general element of this is (g, x), where $g \in G$ is an isometry of the wallpaper group, and $x \in K$ represents the type of isometry. So (g, i) is the two-colour symmetry where g preserves colours, and (g, ii) is the two-colour symmetry where g reverses colours.*

Remark. *Note if $g \in G$ and $g \simeq (\boldsymbol{v}, M)$ in ordered pair form, sometimes we write $(g, x) \simeq (\boldsymbol{v}, M, x)$, where $(g, x) \in G \times K$.*

Example 7.1.1. To visualise two-colour symmetries, see Figure 12. Figure 12a is a usual wallpaper pattern. As this has no rotations, mirrors or glides, this is a pattern on which the **p1** group acts. The generating isometries for this group are translations by **a** and **b** as labelled. The ordered pair form for these isometries are $(\mathbf{a}, I)$ and $(\mathbf{b}, I)$ respectively. Figures 12b and 12c are two-colour patterns of this original wallpaper pattern. See in 12b translation by vector **a** preserves colours, and so this two-colour symmetry can be represented by tuple $(\mathbf{a}, I, i)$. In 12c translation by vector **a** reverses colours, and so this two-colour symmetry can be represented by tuple $(\mathbf{a}, I, ii)$. In both of these figures translation by vector **b** preserves colours.



(a) A p1 wallpaper pattern (b) Two-colour: $(\mathbf{b}, I, i), (\mathbf{a}, I, i)$ (c) Two-colour: $(\mathbf{b}, I, i), (\mathbf{a}, I, ii)$

Figure 12: Two-colour symmetries

7.2 Two-Colour Groups

Definition 7.4. *For a given wallpaper group G, a* ***two-colour group*** *G' is a subgroup of the direct product group $G \times K$, where $\forall g \in G$, we assign it a type, so either $(g, i) \in G'$ or $(g, ii) \in G'$. Note as a symmetry cannot both preserve colours and reverse colours, we can't have both $(g, i) \in G'$ and $(g, ii) \in G'$.*

Theorem 7.5. *The identity cannot be a type (ii) symmetry.*

Proof. The identity has no effect on the plane, so the identity must preserve colours. □

Theorem 7.6. *Order 3 rotation cannot be a type (ii) symmetry rotation [[10] p.419].*

Proof. Say we have a two-colour group G'. Assume towards a contradiction for some $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, $(\mathbf{v}, A_{\frac{2\pi}{3}}, ii) \in G'$. As G' is a group, $(\mathbf{v}, A_{\frac{2\pi}{3}}, ii)(\mathbf{v}, A_{\frac{2\pi}{3}}, ii)(\mathbf{v}, A_{\frac{2\pi}{3}}, ii) \in G'$.

$$\begin{aligned}
(\mathbf{v}, A_{\frac{2\pi}{3}}, ii)(\mathbf{v}, A_{\frac{2\pi}{3}}, ii)(\mathbf{v}, A_{\frac{2\pi}{3}}, ii) &= (\mathbf{v} + f_{A_{\frac{2\pi}{3}}}(\mathbf{v}), A_{\frac{4\pi}{3}}, i)(\mathbf{v}, A_{\frac{2\pi}{3}}, ii) \\
&= (\mathbf{v} + f_{A_{\frac{2\pi}{3}}}(\mathbf{v}) + f_{A_{\frac{4\pi}{3}}}(\mathbf{v}), I, ii) \\
&= ((v_1, v_2) + (v_1, v_2)\begin{bmatrix} cos\frac{2\pi}{3} & sin\frac{2\pi}{3} \\ -sin\frac{2\pi}{3} & cos\frac{2\pi}{3} \end{bmatrix} + (v_1, v_2)\begin{bmatrix} cos\frac{4\pi}{3} & sin\frac{4\pi}{3} \\ -sin\frac{4\pi}{3} & cos\frac{4\pi}{3} \end{bmatrix}, I, ii) \\
&= ((v_1, v_2) + (v_1, v_2)\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} + (v_1, v_2)\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, I, ii) \\
&= ((v_1, v_2) + (-\frac{1}{2}v_1 + \frac{\sqrt{3}}{2}v_2, -\frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2) + (-\frac{1}{2}v_1 - \frac{\sqrt{3}}{2}v_2, \frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2), I, ii) \\
&= (\mathbf{0}, I, ii) \simeq (id, ii) \in G'
\end{aligned}$$

This contradicts Theorem 7.5, so we cannot have an order three rotation as a type (ii) symmetry. □

Remark. *By Theorems 7.5 and 7.6, for any two-colour group G', $(\mathbf{0}, I, ii) \notin G'$, and $\forall \boldsymbol{v} \in \mathbb{R}^2, (\boldsymbol{v}, A_{\frac{2\pi}{3}}, ii), (\boldsymbol{v}, A_{\frac{4\pi}{3}}, ii) \notin G'$.*

Theorem 7.7. *Say we have G a wallpaper group, and $(g, x) \in G \times K$, so $g \in G, x \in K$. Then $(g, x)(g, x) = (g \circ g, i)$. So the composition of a two-colour symmetry with itself is always a type (i) isometry.*

Proof. Say $x = i$. Clearly $(g, i)(g, i) = (g \circ g, i \cdot i) = (g \circ g, i)$
Say $x = ii$. Clearly $(g, ii)(g, ii) = (g \circ g, ii \cdot ii) = (g \circ g, i)$. □

Theorem 7.8. *Say we have a wallpaper group G, and 2 two-colour symmetries, $(g_1, x_1), (g_2, x_2) \in G \times K$. Then $(g_1, x_1)(g_2, x_2)$ and $(g_2, x_1)(g_1, x_2)$ have the same type. That is to say, $\forall x_1, x_2 \in K$, $x_1 x_2 = x_2 x_1$.*

Proof. This is the same as saying K is abelian, and this is clear to see from Table 5. □

Theorem 7.9. *Translations in any two-colour groups based on the hexagonal lattice must be type (i).*

Proof. Looking at Figure 10, see for a wallpaper group G based on the hexagon lattice, $f \simeq (\mathbf{a}, A_{\frac{\pi}{3}})$, $g \simeq (2\mathbf{a}, A_{\frac{\pi}{3}}) \in G$, where f and g are rotations by angle $\frac{\pi}{3}$ about centres $\frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ and $\frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$ respectively. Now as order 3 rotation must be a type (i) symmetry by Theorem 7.6, for any two-colour group $G' \leq G \times K$, $(f, i), (g, i) \in G'$.
Using Theorem 3.2, $(\mathbf{a}, I) \simeq g \circ f \circ f$, $(\mathbf{b}, I) \simeq g \circ g \circ f$. Hence in a two-colour group G', $(\mathbf{a}, I, i)$, $(\mathbf{b}, I, i) \in G'$, so as translation by $\mathbf{a}$ and $\mathbf{b}$ are both compositions of type (i) symmetries, they are type (i) symmetries.
Hence as translation by $\mathbf{a}$ and $\mathbf{b}$ are type (i), all translations in a two-colour group based on the hexagon lattice are type (i). □

7.2.1 Finding the Two-Colour Groups

To find all two-colour groups, consider each of the 17 wallpaper groups in turn. This method is based on [[8] p.686-687].

1. For a given wallpaper group, use Figure 10 to determine the generating isometries for the wallpaper group.
2. We need to consider all combinations where each generating isometry can be either type (i) or type (ii). Remember that order 3 rotations cannot be type (ii) isometries. The groups generated by the two-colour generating isometries are two-colour groups.
3. Once we have found the two-colour groups, we need to find if any of these are isomorphic to one another.
4. The two-colour group where all generating isometries are type (i) is isomorphic to the original wallpaper group, so we do not count this as a two-colour group.

For example, if we have a wallpaper group G with generating isometries f and g, we need to consider four two-colour groups, $G_1 = \langle (g, i), (f, i) \rangle$, $G_2 = \langle (g, i), (f, ii) \rangle$, $G_3 = \langle (g, ii), (f, i) \rangle$, $G_4 = \langle (g, ii), (f, ii) \rangle$. We then need to find if any of these are isomorphic to each other. Clearly G_1 will be isomorphic to the original wallpaper group G.

Note if two wallpaper groups G_1 and G_2 are not isomorphic, then 2 two-colour groups based on these wallpaper groups, say $G_1' \leq G_1 \times K$, $G_2' \leq G_2 \times G = K$ cannot be isomorphic. This is due to the fact they have different types of symmetries, and an isomorphism must send translations to translations, rotations to rotations etc by Theorem 5.13.

We impose the condition that isomorphisms between two-colour groups must send type (i) symmetries to type (i) symmetries, and type (ii) symmetries to type (ii) symmetries.

7.3 Two-Colour Notation

Unfortunately there is no globally accepted notation for two-colour groups like there is for the wallpaper groups. This report will use its own notation, mainly taken from the notation described in [[9] p.212].

7.3.1 Prefixes

Similarly to with wallpaper groups, the notation will firstly describe the lattice.
Where the prefix in the wallpaper group was either p or c, we now have different prefixes depending on how the translations by **a** and **b** permute colours. We start with the lattice of the wallpaper group we have built our two-colour group from.

- **Oblique**: If all translations preserve colours, we keep prefix p. Otherwise, take prefix p_b'.
- **Rectangle**: If all translations preserve colours, we keep prefix p. If translation by either **a** or **b** reverses colours, the prefix is now p_b'. If both translations by **a** and **b** reverse colours, the prefix is now p_c'.
- **Centred Rectangular**: If all translations preserve colours, our prefix is still c. If a translation reverses colours, our prefix is c'.
- **Square**: If translations preserve colours, the prefix is p. If any translations reverse colours, we have p_c' prefix.
- **Hexagon**: Recall from Theorem 7.9, all translations preserve colours so the prefix for the two-colour group is always p.

7.3.2 Following Isometries

In wallpaper group notation the prefix is followed by a 1,2,3,4,6 for rotations, an m for mirrors and a g for glide reflections.
Let w be one of these symbols representing a kind of isometry in the wallpaper group. If, in our two-colour group all symmetries of kind w are type (i), the two-colour group name has symbol $1w$. If some but not all symmetries of kind w are type (ii), the two colour group has symbol w. If all symmetries of type w are type (ii) in the two-colour group, the two-colour group has symbol w'.

For example, if we had a two-colour group based on p4, where both translations preserved colours but all order 4 rotations reverse colours, the name of the two-colour group would be $p4'$.

We have examples of naming two-colour groups in Section 7.4.

7.4 Two-Colour Group Examples

7.4.1 p2 Two-Colour Groups

We consider the wallpaper group p2. See in Figure 13a a depiction of the unit cell with the generating isometries labelled f, g, h.

$$\begin{aligned} f &\simeq (2m\mathbf{a} + 2n\mathbf{b}, -I) \\ g &\simeq ((2p+1)\mathbf{a} + 2q\mathbf{b}, -I) \\ h &\simeq ((2r+1)\mathbf{a} + (2s+1)\mathbf{b}, -I) \end{aligned}$$

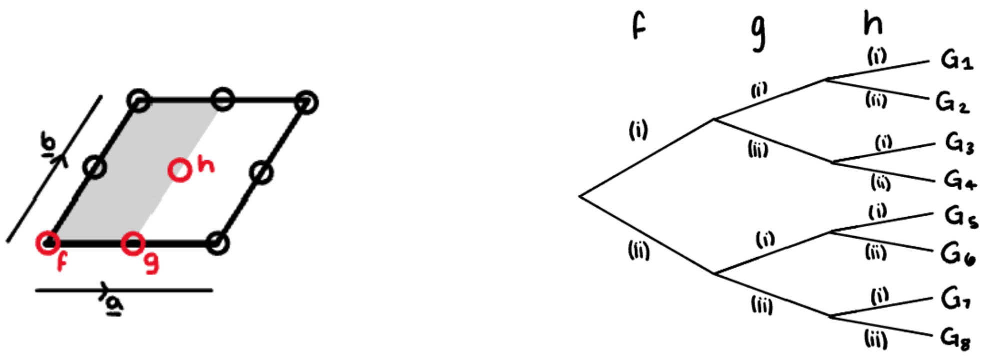
Where $m, n, p, q, r, s \in \mathbb{Z}$.
We want to find the general form of all isometries in p2, and then consider in our two-colour groups whether these will be type (i) or type (ii).

$$\begin{aligned} g \circ f &\simeq ((2p+1)\mathbf{a} + 2q\mathbf{b}, -I)(2m\mathbf{a} + 2n\mathbf{b}, -I) = ((2k+1)\mathbf{a} + 2l\mathbf{b}, I) \\ h \circ f &\simeq ((2r+1)\mathbf{a} + (2s+1)\mathbf{b}, -I)(2m\mathbf{a} + 2n\mathbf{b}, -I) = ((2t+1)\mathbf{a} + (2u+1)\mathbf{b}, I) \\ h \circ g &\simeq ((2r+1)\mathbf{a} + (2s+1)\mathbf{b}, -I)((2p+1)\mathbf{a} + 2q\mathbf{b}, -I) = (2v\mathbf{a} + (2w+1)\mathbf{b}, I) \\ h \circ f \circ g &\simeq ((2r+1)\mathbf{a} + (2s+1)\mathbf{b}, -I)(2m\mathbf{a} + 2n\mathbf{b}, -I)((2p+1)\mathbf{a} + 2q\mathbf{b}, -I) \\ &= ((2t+1)\mathbf{a} + (2u+1)\mathbf{b}, I)((2p+1)\mathbf{a} + 2q\mathbf{b}, -I) = (2x\mathbf{a} + (2y+1)\mathbf{b}, -I) \end{aligned}$$

Where $k, l, t, u, v, w, x, y \in \mathbb{Z}$.
It may look like we have missed the isometry $(2c\mathbf{a} + 2d\mathbf{b}, I)$. However note $(2c\mathbf{a} + 2d\mathbf{b}, I) \simeq (h \circ f) \circ (h \circ f)$, which by Theorem 7.7 will always be type (i), so does not need considering.
So the are the symmetries we need to consider when looking at two-colour groups are f, g, h, $g \circ f$, $h \circ f$, $h \circ g$, $h \circ f \circ g$.
We need to consider every combination of f, g, h when they can be type (i) or type (ii). To ensure we don't miss any combinations, we use a decision tree. See this in Figure 13b. So

$$G_1 = \langle (f, i), (g, i), (h, i) \rangle \quad G_2 = \langle (f, i), (g, i), (h, ii) \rangle$$

and similarly for the other two-colour groups found from p2.



(a) p2 unit cell

(b) All two-colour groups which need considering for p2

Figure 13: Finding two-colour groups for p2

Table 6 shows each isometry in p2, along with its type in each Two-Colour Group.

	Rotations type				Translations type			Total number of ii symmetries	
Two-colour group	f	g	h	$h \circ f \circ g$	$g \circ f$	$h \circ f$	$h \circ g$	Rotations	Translations
G_1	i	i	i	i	i	i	i	0	0
G_2	i	i	ii	ii	i	ii	ii	2	2
G_3	i	ii	i	ii	ii	i	ii	2	2
G_4	i	ii	ii	i	ii	ii	i	2	2
G_5	ii	i	i	ii	ii	ii	i	2	2
G_6	ii	i	ii	i	ii	i	ii	2	2
G_7	ii	ii	i	i	i	ii	ii	2	2
G_8	ii	ii	ii	ii	i	i	i	4	0

Table 6: Type of symmetries for each two-colour group of p2

Looking at table 6 it is clear to see that G_1 is not isomorphic to any of the other two-colour groups for p2, as it has no type (ii) symmetries. G_1 is in fact isomorphic to the wallpaper group p2 as it doesn't have any colour reversing symmetries.
Also, clearly as G_8 doesn't have any colour reversing translations, it is not isomorphic to any of $G_2 - G_7$. This is two-colour group $p2'$ as all rotations are colour reversing.
In general form, groups $G_2 - G_7$ have two type (i) rotations, two type (ii) rotations, one type (i) translation and two type (ii) rotations. As all general isometries are of the same form, these groups are isomorphic.
See that for these groups $G_2 - G_7$, the generating isometries were not always the same, with the generators of G_2, G_3 and G_5 consisiting of one type (ii) rotation and two type (i) rotations, while the generators of G_4, G_6 and G_7 were two type (ii) rotations and one type (i) rotation. So at first glance we may not have thought that these groups would all be isomorphic but in fact they are. This shows why we need to look at more than just the generators to determine if groups are isomorphic.
These two-colour groups all have some colour reversing translations, so the prefix is p'_b. They have some colour reversing order two rotations but not all so these groups are all two-colour group p'_b2.
So p2 has 2 unique two-colour groups, $p2'$, p'_b2.

7.4.2 pm Two-Colour Groups

We consider the wallpaper group pm. See Figure 14a for a depiction of the unit cell with the generating isometries labelled f, g, t.

$$\begin{aligned} f &\simeq (2n\mathbf{b}, B_0) \\ g &\simeq ((2k+1)\mathbf{b}, B_0) \\ t &\simeq ((2m+1)\mathbf{a}, I) \end{aligned}$$

Where $m, n, k \in \mathbb{Z}$.
We need to consider isometries of the form f, g, t or

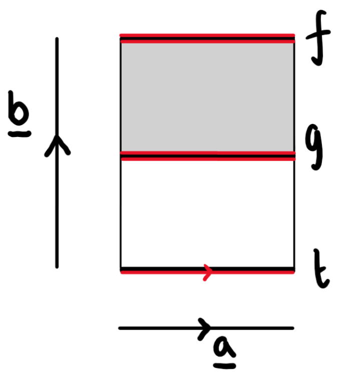
$$\begin{aligned} f \circ t &\simeq (2n\mathbf{b}, B_0)((2m+1)\mathbf{a}, I) = ((2m+1)\mathbf{a} + 2n\mathbf{b}, B_0) \\ g \circ t &\simeq ((2k+1)\mathbf{b}, B_0)((2m+1)\mathbf{a}, I) = ((2m+1)\mathbf{a} + (2k+1)\mathbf{b}, B_0) \\ g \circ f &\simeq ((2k+1)\mathbf{b}, B_0)(2n\mathbf{b}, B_0) = ((2l+1)\mathbf{b}, I) \\ g \circ f \circ t &\simeq ((2k+1)\mathbf{b}, B_0)(2n\mathbf{b}, B_0)((2m+1)\mathbf{a}, I) \\ &= ((2l+1)\mathbf{b}, B_0)((2m+1)\mathbf{a}, I) = ((2p+1)\mathbf{a} + (2q+1)\mathbf{b}, I) \end{aligned}$$

Where $l, p, q \in \mathbb{Z}$.
Note this does not have the isometries of the form $(2r\mathbf{a}, I)$, $r \in \mathbb{Z}$. This is just $t \circ t$, which by Theorem 7.7 is always type (i) so we do not need to consider this.
Similarly, for $s, u, v \in \mathbb{Z}$, $(2s\mathbf{b}, I) \simeq (g \circ f)(g \circ f)$ and $(2v\mathbf{a} + 2u\mathbf{b}, I) \simeq (g \circ f \circ t) \circ (g \circ f \circ t)$ are always type (i).
As $(2r\mathbf{a}, I)$ and $(2s\mathbf{b}, I)$ are always type (i), we need not consider $(2v\mathbf{a} + (2w+1)\mathbf{b}, I)$ or $((2x+1)\mathbf{a} + 2y\mathbf{b}, I)$, $v, w, x, y \in \mathbb{Z}$ as these will just have the same type as $g \circ f$ and t respectively.

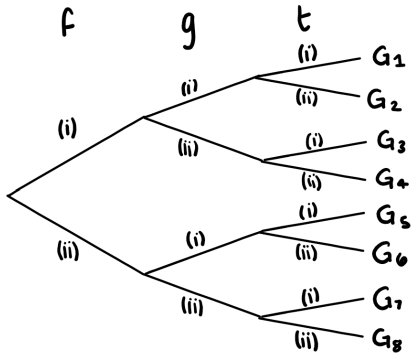
So the symmetries we need to consider when looking at two-colour groups of pm are $f, g, t, g \circ f, f \circ t, g \circ t, g \circ f \circ t$. We need to consider every combination of f,g,h when they can be type (i) or type (ii). To ensure we don't miss any combinations, we use a decision tree. See this in Figure 14b. So

$$G_1 = \langle (f, i), (g, i), (t, i) \rangle \quad G_2 = \langle (f, i), (g, i), (t, ii) \rangle$$

and similarly for the other two-colour groups found from pm.



(a) pm unit cell



(b) All pm two-colour groups which need considering

Figure 14: Finding two-colour groups for pm

Table 7 shows each isometry in pm, along with its type in each two-colour group.

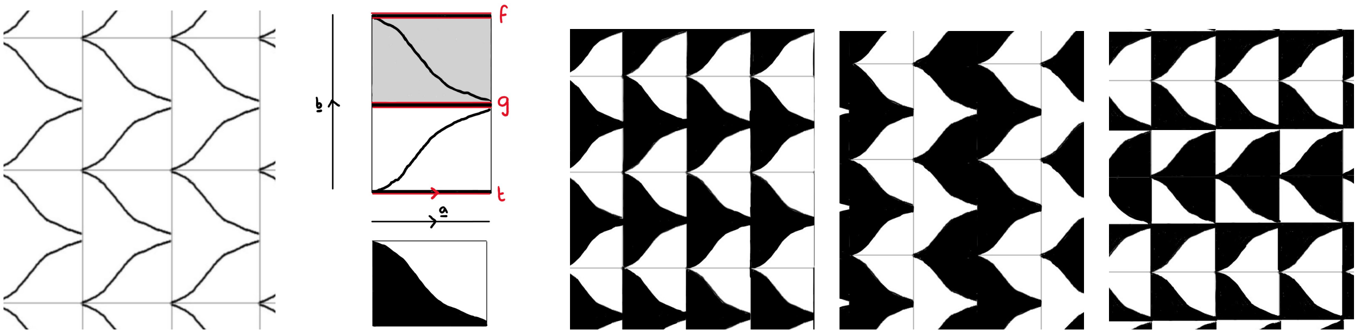
Looking at Table 7 it is clear to see that G_1 is not isomorphic to any of the other two-colour groups for pm as G_1 has no type (ii) symmetries. G_1 is in fact isomorphic to the wallpaper group pm as it doesn't have any colour reversing symmetries. This is therefore not considered a two-colour group.
As G_2 doesn't have any colour-reversing reflections, it is not isomorphic to any other two-colour groups for pm. Translation by $\mathbf{a}$ is type (ii) and translation by $\mathbf{b}$ is type (i) so the prefix is p'_b. There are no colour reversing reflections so this is two-colour group p'_b1m
As G_7 doesn't have any colour-reversing translations, it is not isomorphic to any other two-colour groups for pm. All its mirrors are type (ii), so this is two-colour group pm'
As G_8 doesn't have any colour-reversing glide translations, it is not isomorphic to any other two-colour groups for

	Reflections type		Translations type			Glide reflections type		Total number of ii symmetries		
Two-colour group	f	g	t	$g \circ f$	$g \circ f \circ t$	$f \circ t$	$g \circ t$	Reflections	Translations	Glide Reflections
G_1	i	i	i	i	i	i	i	0	0	0
G_2	i	i	ii	i	ii	ii	ii	0	2	2
G_3	i	ii	i	ii	ii	i	ii	1	2	1
G_4	i	ii	ii	ii	i	ii	i	1	2	1
G_5	ii	i	i	ii	ii	ii	i	1	2	1
G_6	ii	i	ii	ii	i	i	ii	1	2	1
G_7	ii	ii	i	i	i	ii	ii	2	0	2
G_8	ii	ii	ii	i	ii	i	i	2	2	0

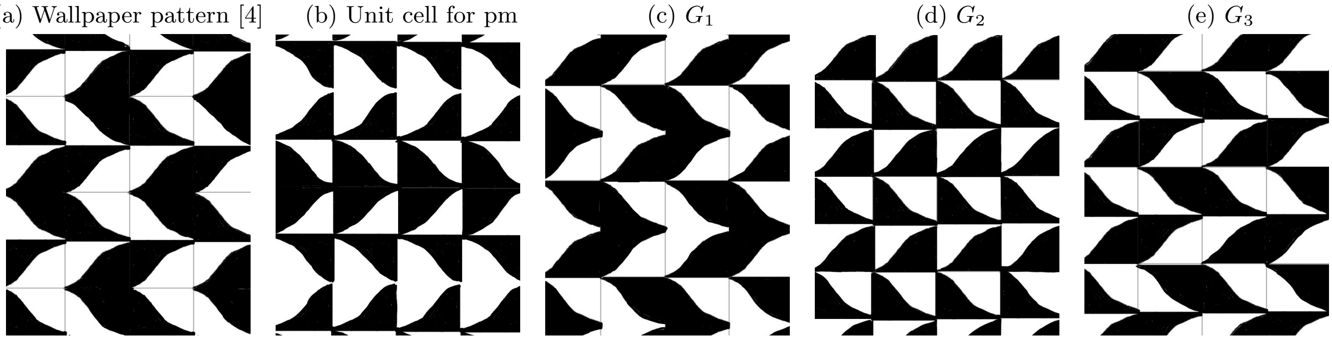
Table 7: Type of symmetries for each two-colour group of pm

pm. Translation by **a** is type (ii) and translation by **b** is type (i) so the prefix is p'_b. All reflections are now colour reversing, so this is two-colour group $p'_b m'$.
Now we need to consider groups G_3, G_4, G_5 and G_6. Notice in both G_3 and G_5 if we decompose the glide reflections into a reflection and a translation, we see that the translation part preserves colours. In G_4 and G_6 if we decompose the glide reflections into a reflection and a translation, we see that the translation part reverses colours. So G_3 and G_5 are isomorphic to each other but not to G_4 or G_6, and G_4 and G_6 are isomorphic to each other.
G_3 and G_5 both have translation by **a** type (i) and translation by **b** type (ii) so the prefix is p'_b. They have some reflections colour reversing but not all so the two-colour group is $p'_b m$.
G_4 and G_6 both have both translation by **a** and translation by **b** type (ii) so the prefix is p'_c. They have some reflections colour reversing but not all so the two-colour group is $p'_c m$.
So from pm, we have 5 unique two-colour groups, $p'_b 1m$, pm', $p'_b m'$, $p'_b m$, $p'_c m$.

We will show examples of these two-colour Groups based from wallpaper group pm. See Figure 15a for a wallpaper pattern on which the pm group acts. We can see this is pm as there are no rotations, mirror lines, and no glides which aren't mirrors.
See Figure 15b for the unit cell for pm with generating isometries drawn in red, along with the generating region shaded. See also the choice for the colouring of the generating region. We will act on this generating region using the isometries for each of the two-colour groups in Table 7 to fill the whole plane, and colour in our whole pattern to get a two-colour pattern.



(a) Wallpaper pattern [4] (b) Unit cell for pm (c) G_1 (d) G_2 (e) G_3



(f) G_4 (g) G_5 (h) G_6 (i) G_7 (j) G_8

Figure 15: Colourings of pm

We see from Figure 15c that this is just the wallpaper group pm acting on this pattern. The figures 15e and 15g make it easy to see that groups G_3 and G_5 are isomorphic. Similarly figures 15f and 15h show that G_4 and G_6 are isomorphic, as they are reflections of eachother.

7.5 Two-Colour Summary

Following the argument in Section 7.2.1, we find the number of two-colour groups for each wallpaper group.

Wallpaper group	Number of two-colour groups
p1	1
p2	2
pm	5
pg	2
p2mm	5
p2mg	5
p2gg	2
cm	3
c2mm	5

Wallpaper group	Number of two-colour groups
p4	2
p4mm	5
p4mg	3
p3	0
p3m1	1
p31m	1
p6	1
p6mm	3

Table 8: Number of two-colour groups for each wallpaper group [[8] p.692]

Recall that order three rotation cannot be a type (ii) symmetry, hence why there are no two-colour groups for p3. The mirrors in p3m1 and p31m allow for two-colour groups [[10] p.419].

We don't count wallpaper groups in our counting of two-colour groups, so we have 17 wallpaper groups, and 46 two-colour groups.

For an example of classifying a two-colour pattern, see Section 8.

8 An Escher Example

We now look to classify Escher's wallpaper pattern in Figure 16. We consider it in three different ways: as a wallpaper pattern ignoring the colours, as a wallpaper pattern considering the colours, and as a two-colour pattern.

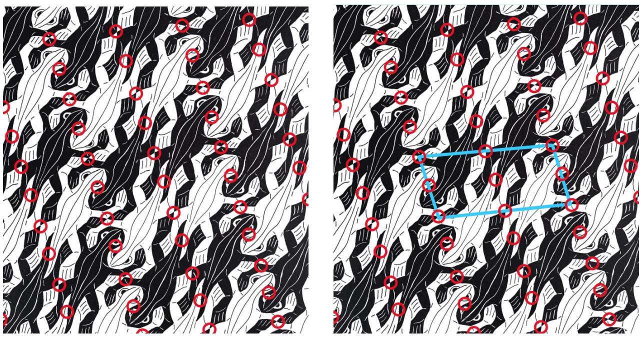
Figure 16: Escher example [[5] p.37]

8.1 Wallpaper Group Ignoring Colours

If we completely disregard colours, we don't mind which colours of lizard the isometries send to each other. Firstly to classify we want to check what the highest order of rotation in the pattern is. This pattern has order 2 rotation about the points in Figure 17a. Looking at Table 4, the next question is whether the pattern has mirrors, to which the answer is clearly no. So we find that if we ignore colours, the wallpaper group acting on the pattern is p2. See the unit cell shown in blue in Figure 17b.

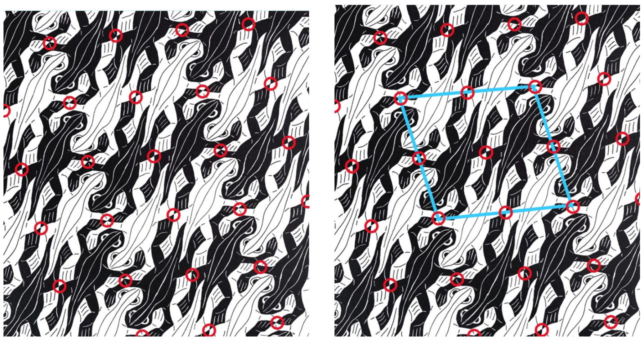
8.2 Wallpaper Group considering Colours

Now we consider colours, so we need only isometries which send black lizards to black lizards and white lizards to white lizards. Again to classify we look at Table 4 and check for the highest order of rotation. The highest order of rotation is 2, see the centers of order 2 rotation shown in Figure 18a. Notice now there are less rotations than there



(a) Centres of rotation (b) Unit cell

Figure 17: Analysis when we ignore colours



(a) Centres of rotation (b) Unit cell

Figure 18: Analysis when we consider colours

were when we were ignoring colours. Again we see that there are no mirrors, so this is again wallpaper group p2. See the unit cell shown in blue in Figure 18b. Notice this is now bigger than the unit cell from when we ignored colours.

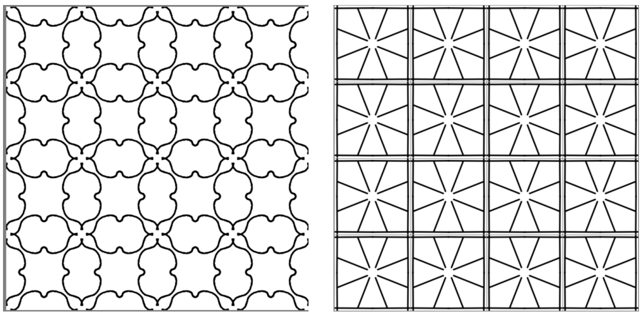
8.3 Two-colour Symmetry Group

Now we want to consider this pattern where we allow two-colour symmetries, where we consider the colours as different but allow symmetries which permute colours.

The first step is to classify the pattern when we completely disregard colours. We have already done this in Section 8.1, and found that the wallpaper group preserving this pattern is p2. Now we can look back at the unit cell in Figure 18b to classify this two-colour pattern. We see one of the original translations preserves colours, while another reverses them, so the prefix for our two-colour group is p_b'. Now looking at rotations, some centres of rotation preserve colours while others reverse them, so we have two-colour group $p_b'2$.

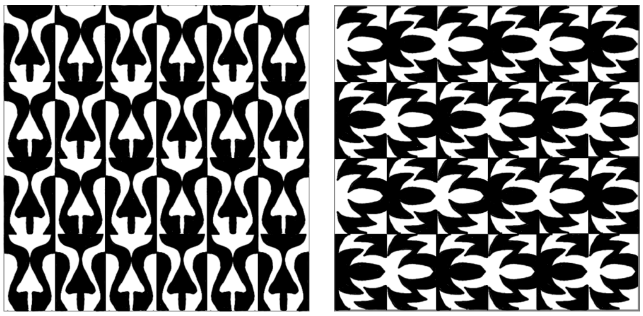
9 Conclusion

We now turn to proving the statements made in the introduction, namely that the two patterns in Figure 19 are the "same", and that the two patterns in Figure 20 are the "same". Now we know that to mean that the wallpaper/two-colour groups acting on the patterns are isomorphic.



(a) (b)

Figure 19: Two wallpaper patterns [4]



(a) (b)

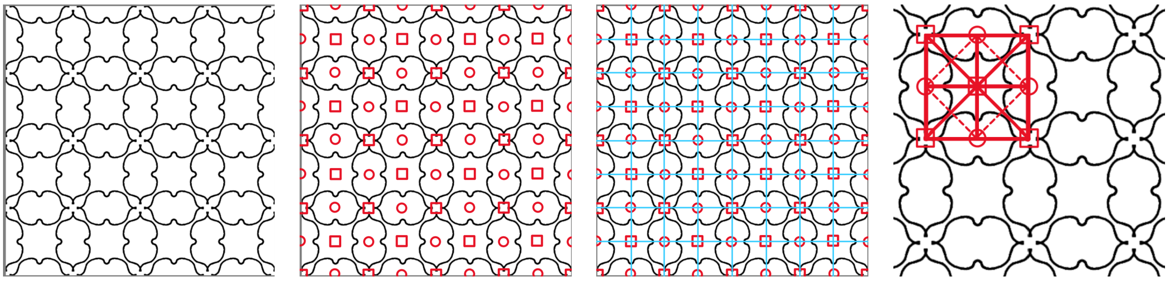
Figure 20: Two two-colour patterns [4]

9.1 Wallpaper Groups

Our aim is to show the same wallpaper group acts on both patterns in Figure 19.
Lets start with the pattern in Figure 21a. We use Table 4 to classify the wallpaper group acting on the pattern. We want to look for any centres of rotation, and find these in 21b. We see the highest order of rotation is 4. The next question is whether there are any mirrors. We find the mirrors in 21c. Note there are more diagonal mirrors, but we have already found enough mirrors to say that all centres of rotation lie on mirror lines, so we can say the wallpaper group acting on the pattern is p4mm.

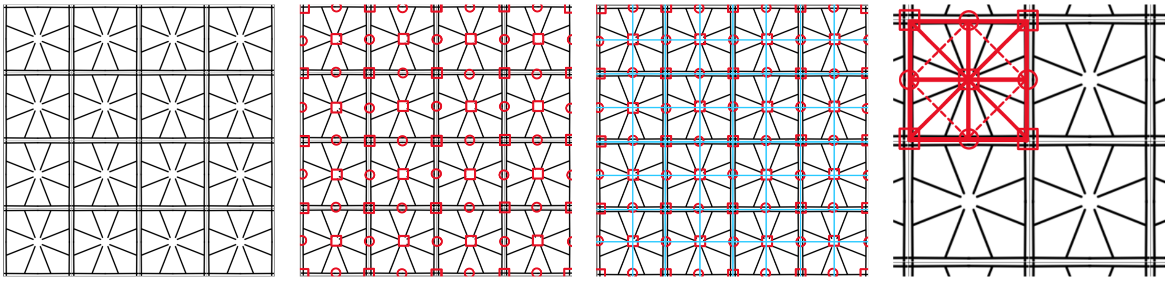
We follow the exact same steps to see that p4mm is also the wallpaper group acting on the pattern in Figure 22a. See Figure 22.

So p4mm is the wallpaper group acting on both of these patterns, so mathematically these patterns are the same.



(a) Wallpaper pattern a (b) Centres of rotation (c) Mirror lines (d) p4mm unit cell

Figure 21: Analysing wallpaper pattern (a)



(a) Wallpaper pattern (b) Centres of rotation (c) Mirror lines (d) p4mm unit cell

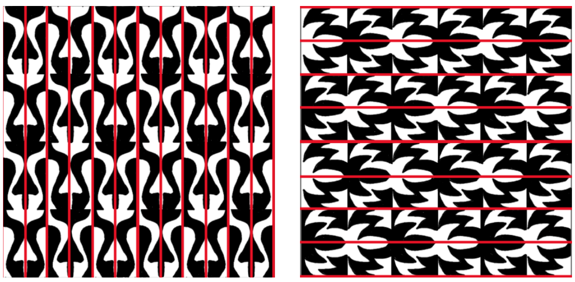
Figure 22: Analysing wallpaper pattern (b)

9.2 Two-Colour Groups

We want to show that the same two-colour group acts on both patterns in Figure 20.

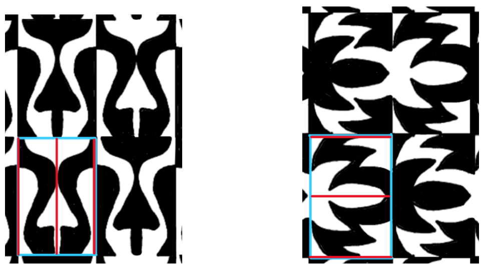
Recall the first step is to classify the wallpaper pattern when we disregard colours. Lets start with the pattern in Figure 20a. Look at this pattern and see there are no centres of rotation. The next question in Table 4 is whether there are any mirrors. There are, these are shown in red in Figure 23a. The next question is whether there are any glide lines which are not mirror lines which there are not, so the wallpaper group acting on this pattern ignoring colours is pm.

Similarly, we see for pattern b in Figure 20b this is also wallpaper group pm when we ignore colours. See Figure 23b for the mirrors drawn on this pattern.



(a) (b)

Figure 23: Mirrors for patterns ignoring colours



(a) (b)

Figure 24: Unit cells of pm

Now we need to look at the unit cells, drawn in blue, in Figure 24. Looking at the pattern in Figure 24a, translation by both **a** and **b** is type (ii), so the prefix is p'_c. Reflection in some, but not all mirrors reverses colours, so the two-colour group acting on this pattern is $p'_c m$. We follow the same steps for the pattern in Figure 24b to see it is also $p'_c m$ acting on this pattern. So the same two-colour group acts on both of these patterns, so we can say that the patterns are mathematically the same.

9.3 Discussion

In conclusion, we have investigated isometries to establish that there are seventeen wallpaper groups, and then moved onto two-colour symmetries to define and investigate the two-colour groups based on two of the wallpaper groups. We have also laid out a method, so that given any wallpaper or two-colour pattern, we can find the wallpaper or two-colour group acting on it,

References

[1] M.A. Armstrong. *Groups and symmetry.* New York: Springer, 1988.

[2] R.P Burn. *Groups: a path to geometry.* Cambridge: Cambridge University Press, 1985.

[3] Brock C.P. "Symmetry diagrams for the 17 plane groups now freely available". In: *IUCr Newsletter* 30.4 (2022).

[4] David Eck. *Wallpaper Symmetry- Hobart and William Smith Colleges.* URL: https://math.hws.edu/eck/js/symmetry/wallpaper.html. accessed 14/03/2023.

[5] C. H. MacGillavry. *Symmetry aspects of MC Escher's periodic drawings.* International Union of Crystallography Utrect, 1965.

[6] Math & The Art of MC Escher. *Wallpaper Patterns.* URL: https://mathstat.slu.edu/escher/index.php/Wallpaper_Patterns. accessed 21/03/2023.

[7] World Intellectual Property Organization. *What is Intellectual Property?* URL: https://www.wipo.int/about-ip/en/. accessed 16/03/2023.

[8] D. Schattschneider. "In black and white: How to create perfectly colored symmetric patterns". In: *Computers & Mathematics with Applications* 12.3-4 (1986).

[9] Holser W.T. Shubnikov A.V. Belov N.V. *Colored symmetry.* Oxford: Pergamon, 1964.

[10] B. G. Thomas. "Colour symmetry: the systematic coloration of patterns and tilings". In: *Colour Design* (2012), pp. 381–423.

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