e: The Master of All

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Read Euler, read Euler. He is the master of us all.
—P. S. de Laplace

Laplace's dictum may rightfully be transcribed as: "Study *e*, study *e*. It is the master of all."

Just as Gauss earned the moniker *Princeps mathematicorum* amongst his contemporaries [1], *e* may be dubbed *Princeps constantium symbolum*. Certainly, Euler would concur, or why would he have endowed it with his own initial [2]?

The historical roots of *e* have been exhaustively traced and are readily available [3, 4, 5]. Likewise, certain fundamental properties of *e* such as its limit definition, its series representation, its close association with the rectangular hyperbola, and its relation to compound interest, radioactive decay, and the trigonometric and hyperbolic functions are too well known to warrant treatment here [6, 7].

Rather, the focus of the present article is on matters related to *e* which are not so widely appreciated or at least have never been housed under one roof. When I do indulge in reviewing well-known facts concerning *e*, it is to forge a link to results of a more exotic variety.

Occurrences of *e* throughout pure and applied mathematics are considered; exhaustiveness is not the goal. Nay, the breadth and depth of our treatment of *e* has been chosen to convey the versatility of this remarkable number and to whet the appetite of the reader for further investigation.

The Cast

Unlike its elder sibling π, *e* cannot be traced back through the mists of time to some prehistoric era [8]. Rather, *e* burst into existence in the early seventeenth century in the context of commercial transactions involving compound interest [5]. Unnamed usurers observed that the profit from interest increased with increasing frequency of compounding, but with diminishing returns.

Thus, *e* was first conceived as the limit

$$e = \lim_{n\to\infty} (1 + 1/n)^n = 2.718281828459045\cdots, \qquad (1)$$

although its baptism awaited Euler in the eighteenth century [2]. One might naively expect that all that could be gleaned from equation (1) would have been mined long ago. Yet, it was only very recently that the asymptotic development

$$\tilde{e}_n = (1 + 1/n)^n = \sum_{\nu=0}^{\infty} \frac{e_\nu}{n^\nu};$$
$$e_\nu = e \sum_{k=0}^{\nu} \frac{S_1(\nu + k, k)}{(\nu + k)!} \sum_{l=0}^{\nu-k} \frac{(-1)^l}{l!}, \qquad (2)$$

with S_1 denoting the Stirling numbers of the first kind [9], was discovered [10].

Although equation (1) is traditionally taken as the definition of *e*, it is much better approximated by the limit

$$e = \lim_{n\to\infty} \left[\frac{(n+2)^{n+2}}{(n+1)^{n+1}} - \frac{(n+1)^{n+1}}{n^n} \right] \qquad (3)$$

discovered by Brothers and Knox in 1998 [11]. Figure 1 displays the sequences involved in equations (1) and (3). The superior convergence of equation (3) is apparent.

In light of the fact that both equations (1) and (3) provide rational approximations to *e*, it is interesting to note that $\frac{878}{323} = 2.71826\cdots$ provides the *best* rational approximation to *e*, with a numerator and denominator of fewer than four decimal digits [12]. (Note the palindromes!) Considering that the fundamental constants of nature (speed of light *in vacuo*, mass of the electron, Planck's constant, and so on) are known reliably to only six decimal digits, this is remarkable accuracy indeed. Mystically, if we simply delete

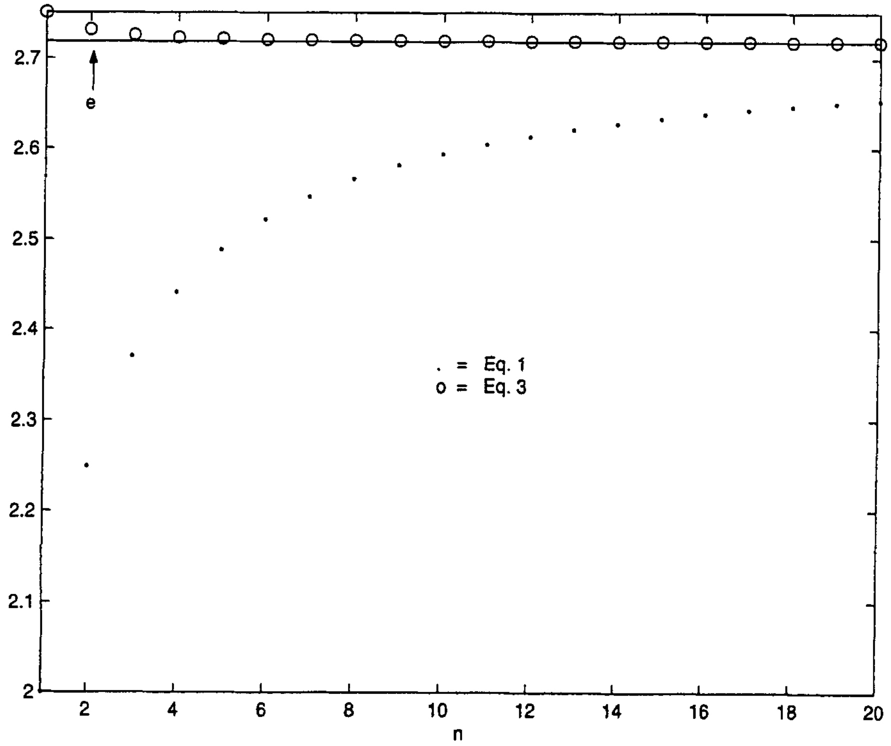


Figure 1. Exponential sequences.

the last digit of both numerator and denominator then we obtain $\frac{87}{32}$, the best rational approximation to e using fewer than three digits [13]. Is this to be regarded as a singular property of e or of base 10 numeration?

In 1669, Newton published the famous series representation for e [14],

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} =$$

$$1 + 1 + \frac{1}{2} \cdot (1 + \frac{1}{3} \cdot (1 + \frac{1}{4} \cdot (1 + \frac{1}{5} \cdot (1 + \cdots)))), \quad (4)$$

established by application of the binomial expansion to equation (1). Many more rapidly convergent series representations have been devised by Brothers [14] such as

$$e = \sum_{k=0}^{\infty} \frac{2k+2}{(2k)!}. \quad (5)$$

Figure 2 displays the partial sums of equations (4) and (5) and clearly reveals the enhanced rate of convergence. A variety of series-based approximations to e are offered in [15].

Euler discovered a number of representations of e by continued fractions. There is the simple continued fraction [16]

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cdots}}}}}}}}, \quad (6)$$

or the more visually alluring [5]

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \cfrac{6}{7 + \cfrac{7}{8 + \cdots}}}}}}}}. \quad (7)$$

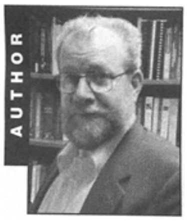
In 1655, John Wallis published the exhilarating infinite product

$$\frac{\pi}{2} = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\,\frac{8}{7}\,\frac{8}{9}\,\frac{10}{9}\,\frac{10}{11}\,\frac{12}{11}\,\frac{12}{13}\,\frac{14}{13}\,\frac{14}{15}\,\frac{16}{15}\cdots. \quad (8)$$

However, the world had to wait until 1980 for the "Pippenger product" [17]

$$\frac{e}{2} =$$

$$\left(\frac{2}{1}\right)^{1/2}\left(\frac{2}{3}\,\frac{4}{3}\right)^{1/4}\left(\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\,\frac{8}{7}\right)^{1/8}\left(\frac{8}{9}\,\frac{10}{9}\,\frac{10}{11}\,\frac{12}{11}\,\frac{12}{13}\,\frac{14}{13}\,\frac{14}{15}\,\frac{16}{15}\right)^{1/16}\cdots. \quad (9)$$



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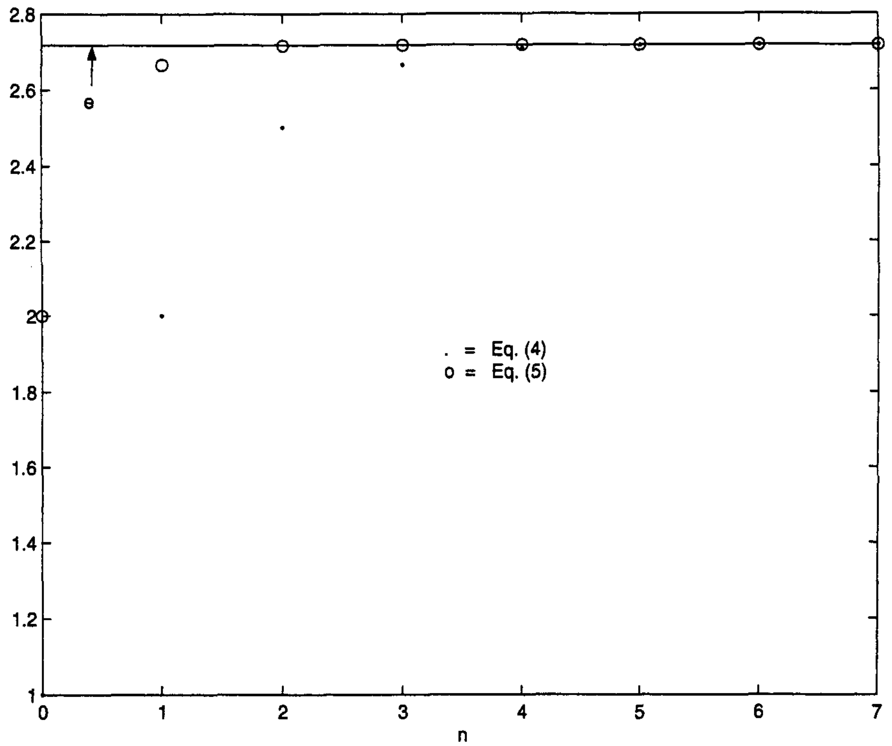

Figure 2. Exponential series.

In spite of their beauty, equations (8) and (9) converge very slowly. A product representation for e which converges at the same rate as equation (4) is given by [18]

$$u_1 = 1;\ u_{n+1} = (n+1)(u_n + 1) \Rightarrow$$

$$e = \prod_{n=1}^{\infty} \frac{u_n + 1}{u_n} = \frac{2}{1}\,\frac{5}{4}\,\frac{16}{15}\,\frac{65}{64}\,\frac{326}{325} \cdots \qquad (10)$$

An Interesting Inequalit-*e*

In 1744, Euler showed that e is irrational by considering the simple infinite continued fraction (6) [19]. In 1840, Liouville showed that e was not a quadratic irrational. Finally, in 1873, Hermite showed that e is in fact transcendental.

Since then, Gelfond has shown that e^{π} is also transcendental. Although now known as Gelfond's constant, this number had previously attracted the attention of the influential nineteenth-century American mathematician Benjamin Peirce, who was wont to write on the blackboard the following alteration of Euler's identity [2, 5]:

$$\imath^{-\imath} = \sqrt{e^{\pi}}, \qquad (11)$$

then turn to the class and cryptically remark, "Gentlemen, we have not the slightest idea what this equation means, but we may be sure that it means something very important" [4].

But what of π^e? Well, it is not even known whether it is rational! Niven [20] playfully posed the question "Which is larger, e^{π} or π^e?". Not only did he provide the answer

$$e^{\pi} > \pi^e, \qquad (12)$$

he also established the more general inequalities

$$\beta > \alpha \geq e \Rightarrow \alpha^{\beta} > \beta^{\alpha};\ e \geq \beta > \alpha > 0 \Rightarrow \beta^{\alpha} > \alpha^{\beta}, \qquad (13)$$

where e plays a pivotal role. This result is displayed graphically in Figure 3.

Barel-*e* Transcendental

As noted above, e is transcendental, but just barely so [21, 22]. In 1844, Liouville proved that the degree to which an algebraic irrational number can be approximated by rationals is limited [23], so that, from the point of view of rational approximation, *the simplest numbers are the worst* [24]. Liouville used this property—that if an irrational number is rapidly approximated by rationals then it must be transcendental—to construct the first known transcendental number, the so-called Liouville number $L = 10^{-1!} + 10^{-2!} + 10^{-3!} + \cdots 10^{-m!} + \cdots$. In 1955, Klaus Roth provided the definitive refinement of Liouville's theorem by finding the ultimate limit to which algebraic irrationals may be approximated by rationals [19]. For this work, he was awarded the Fields Medal in 1958.

To phrase these results quantitatively, for any real number $0 < \xi < 1$ and any fraction p/q in lowest terms, let R denote the set of all positive real numbers r for which the inequality

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^r} \qquad (14)$$

possesses at most finitely many solutions. Then, define the Liouville-Roth constant (irrationality measure) as

$$r(\xi) \equiv \inf_{r \in R} r, \qquad (15)$$

that is, the critical rate threshold above which ξ is not approximable by rationals.

Then, the above may be summarized as follows [22]:

$$r(\xi) = \begin{cases} = 1 \text{ when } \xi \text{ is rational,} \\ = 2 \text{ when } \xi \text{ is algebraic of degree} > 1, \\ \geq 2 \text{ when } \xi \text{ is transcendental.} \end{cases} \qquad (16)$$

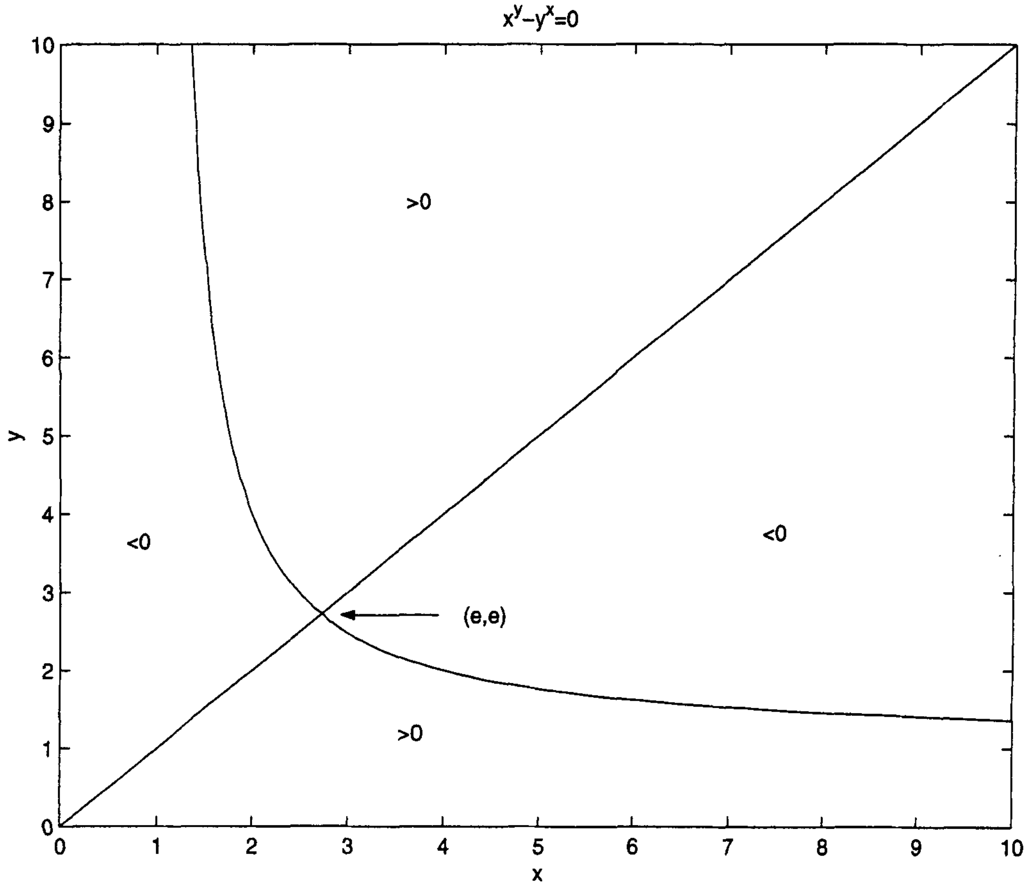

Figure 3. $e^{\pi} > \pi^{e}$.

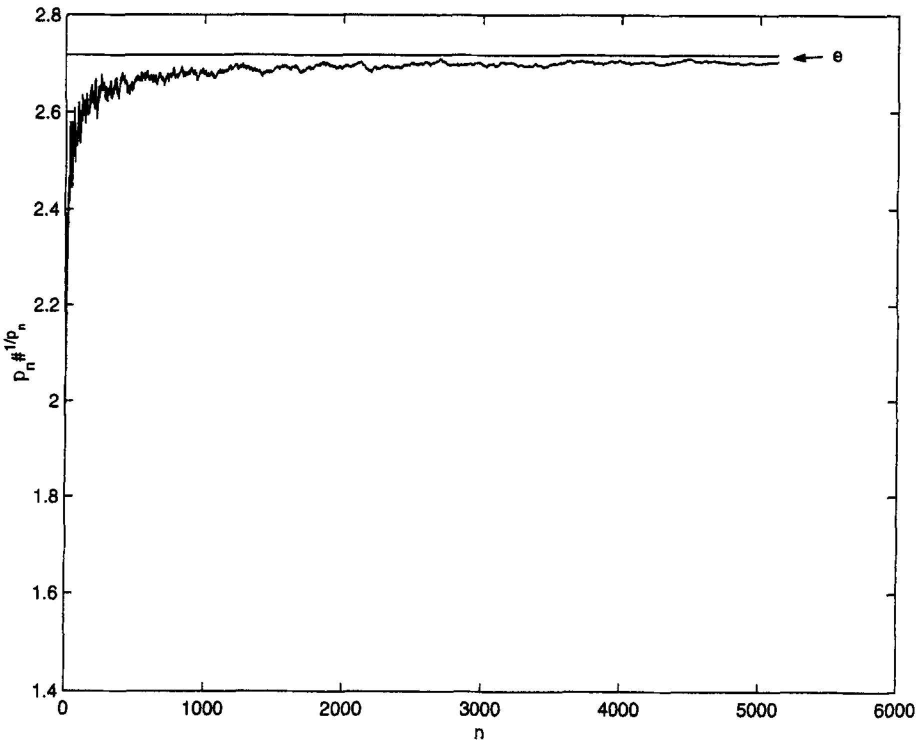

Figure 4. Primorial limit.

More germane to our present deliberations, it is known that [21]

$$r(e) = 2. \tag{17}$$

Thus, e dwells in the twilight zone of overlap between the algebraic irrationals and the transcendental numbers.

The Primor-*e*-al

Letting p_n denote the nth prime number, we define the primorial of p_n as follows [25]:

$$p_n^{\#} \equiv \prod_{k=1}^{n} p_k. \tag{18}$$

Thus, the primorial sequence proceeds as follows: 2, 6, 30, 210, . . . An integer is called a *primorial prime* if it is a prime of the form $p_n^{\#} \pm 1$. For example, $211 = 2 \cdot 3 \cdot 5 \cdot 7 + 1$ is a primorial prime. It is not known whether there are infinitely many primes p for which $p^{\#} \pm 1$ is prime or whether there are infinitely many primes p for which $p^{\#} \pm 1$ is composite.

However, Ruiz [26] has established the following remarkable limit relating e to the primes:

$$\lim_{n\to\infty} (p_n^{\#})^{1/p_n} = e. \tag{19}$$

As Figure 4 shows, the approach to the limit is not nearly as tame or as speedy as that in Figures 1 and 2.

A related result certainly worthy of note is the following: Let A_n and G_n denote the arithmetic and geometric means, respectively, of the integers 1, 2, 3, . . . , n. Then [18]

$$\lim_{n\to\infty} \frac{A_n}{G_n} = \frac{e}{2}. \tag{20}$$

e-quiangular Spiral

Figure 5 displays the equiangular (logarithmic) spiral defined by

$$r = e^{\cot \alpha \cdot \theta} \tag{21}$$

and possessing the distinguishing characteristic of intersecting any radial line drawn from its center at a constant angle α. This spiral has many remarkable self-reproducing properties, such as that of being its own evolute [5]. Jakob Bernoulli (brother and sometimes-adversary of Euler's teacher, Johann [27]) was so taken by it that he had it placed on his gravestone with the inscription "*Eadum mutata resurgo* (Though changed, I shall arise the same)."

While e has the straightforward geometrical interpretation as delimiting a unit area under the hyperbola $y = 1/x$ starting from $x = 1$, π is defined as the ratio of two *lengths* (circumference to diameter of a circle). However, with reference to Figure 5, e may also be defined as the ratio of two lengths [28]:

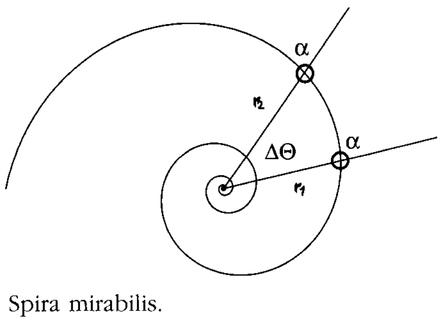

Figure 5. Spira mirabilis.

$$\Delta\theta = \tan \alpha \Rightarrow \frac{r_2}{r_1} = e. \tag{22}$$

(For example, if $\alpha = \pi/4$, then $\Delta\theta$ should be chosen to be one radian.) But since, according to equation (21), the spiral requires e as part of its very definition, is this not a circular argument?

No, it is a spiral argument, for an equiangular spiral may be drawn without reference to equation (21) [7]! Figure 6 displays a spiral compass designed for this purpose. The compass point is located at O and is free to slide smoothly in the slotted rod BC. The wheel F, which lies in a plane perpendicular to the page, may be locked in place at the desired angle relative to this rod. A small handle protrudes at G which permits grasping the apparatus between the thumb and two first fingers. Its sharp edge keeps the wheel from slipping sideways. Hence, as the wheel revolves it maintains a constant angle with the slotted rod and thereby traces out an equiangular spiral of any desired eccentricity.

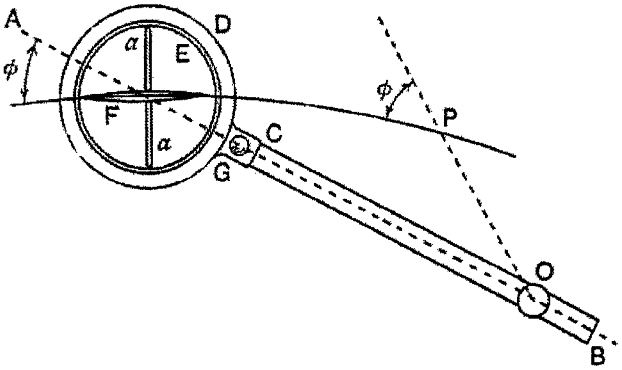

Figure 6. Spiral compass.

The equiangular spiral arises in many natural settings (and some unnatural ones as well: the beginning of the Yellow Brick Road is reputed to be a golden spiral). One of these is shown in Figure 7. The cauliflower (Broccoli Romanesco) there displayed is resplendent with equiangular spirals. Deliciously enough, it is also a rich source of vitamin E!

Stirl-*e*-ng's Formula

In 1730, Stirling published the asymptotic formula [29]

$$n! \sim e^{-n} \cdot n^n \cdot \sqrt{2\pi n}. \tag{23}$$

In addition to providing an intriguing connection between π and e, it also affords the limit

$$e = \lim_{n\to\infty} \frac{n}{(n!)^{1/n}}. \tag{24}$$

However, it should be pointed out that equation (24) can be obtained directly from equation (1) by elementary means and, furthermore, that equation (20) is now an immediate consequence of equation (24). Formula (23) bears so many

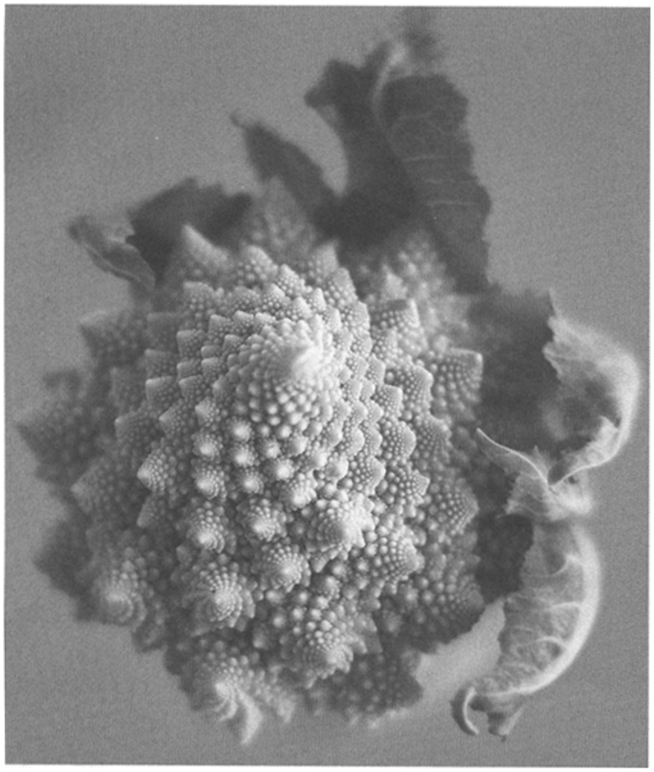

Figure 7. Vitamin *e*.

remarkable relationships to the foregoing as to make it positively *e*clectic. To begin with, Wallis's formula (8) was used by Stirling to determine the constant factor in his asymptotic formula. Furthermore, Pippenger used Stirling's formula in deriving his product (9). Moreover, Stirling's formula is an essential ingredient of Borwein's derivation of equation (17) for the Liouville-Roth constant for *e*. Lastly, both Ruiz's evaluation of the primorial limit (19) and the determination of the arithmetic/geometric mean limit (20) hinge upon the invocation of Stirling's formula.

Let us try to gain an intuitive feel for the accuracy of Stirling's formula for moderate values of n. Specifically, $100! = 9.3326 \cdots \times 10^{157}$, while Stirling's formula produces $100! \approx 9.3248 \cdots \times 10^{157}$ with an error of less than $\frac{1}{10}\%$. This also allows us to appreciate the immensity of $n!$ for even moderate values of n. For example, the age of the Universe is believed to be $O(10^{17})$ seconds, and the number of atoms in the visible Universe has been estimated to be $O(10^{79})$. Yet, we must temper our awe of the magnitude of $n!$ with the realization that the number of possible chess games has been estimated to be $O(10^{10^{50}})$ [30].

*e*igenfunction

The functional equation

$$f(x+y) \equiv f(x)f(y) \quad (25)$$

is known as the exponential equation of Cauchy [31]. If $f(x)$ is bounded on a set of positive measure then the only solutions of (25) are $f(x) = 0$ and $f(x) = e^{cx}$, where c is an arbitrary constant. This is easy to establish if we make the more restrictive assumption that $f(x)$ is differentiable.

In that case,

$$f'(x+y) = f'(x)f(y) = f(x)f'(y), \quad (26)$$

which upon rearrangement becomes

$$\frac{f'(x)}{f(x)} = c = \frac{f'(y)}{f(y)} \Rightarrow f(x) = Ae^{cx}. \quad (27)$$

Since equation (25) clearly implies that $f(0) = [f(0)]^2$, we must have either $f(0) = 0 = A$ or $f(0) = 1 = A$, thereby producing the two solutions as required.

Observe that, from equation (27), the emergence of the exponential solution was a direct consequence of its being the eigenfunction of the differentiation operator:

$$(D-c)y(x) = 0 \Rightarrow y(x) = Ae^{cx}; \quad D \equiv \frac{d}{dx}. \quad (28)$$

This same fundamental property provides the exponential function with a central role throughout the vast field of differential equations and their applications.

For example, consider the first-order, constant coefficient, scalar equation [32]

$$y'(x) - ry(x) = f(x). \quad (29)$$

Multiplication by the integrating factor $\mu \equiv e^{-rx}$ and invocation of the eigenfunction property produces the self-adjoint form

$$[e^{-rx}y(x)]' = e^{-rx}f(x), \quad (30)$$

which upon integration leads directly to the general solution

$$y(x) = e^{rx}\int e^{-rx}f(x)\,dx, \quad (31)$$

where we have utilized the shorthand $\int \phi(x)dx \equiv \int^x \phi(\xi)d\xi$.

Let us recast this result in operator form. Equation (29) becomes

$$(D-r)y(x) = f(x), \quad (32)$$

while its solution, equation (31), becomes

$$y(x) = \frac{1}{D-r}\cdot f(x) = e^{rx}\int e^{-rx}f(x)\,dx, \quad (33)$$

providing an explicit representation of the inverse of the differential operator $D - r$.

We may exploit this observation in order to bootstrap to a solution procedure for second-order equations with constant coefficients. Consider, for example,

$$y''(x) - 3y'(x) + 2y(x) = xe^x, \quad (34)$$

or, in operator form,

$$(D^2 - 3D + 2)y(x) = xe^x. \quad (35)$$

Factorization followed by inversion then leads to

$$y(x) = \frac{1}{D-1}\cdot\frac{1}{D-2}\cdot xe^x. \quad (36)$$

An initial application of the inversion formula (33) produces the intermediate result

$$\frac{1}{D-2}\cdot xe^x = e^{2x}\int e^{-2x}xe^x\,dx = -(1+x)e^x; \quad (37)$$

then a second application provides a particular solution to equation (34):

$$y(x) = \frac{1}{D-1} \cdot [-(1+x)e^x]$$
$$= -e^x \int e^{-x}(1+x)e^x\, dx = -\frac{1}{2}(1+x)^2 e^x. \quad (38)$$

Clearly, this symbolic process generalizes to nth-order constant coefficient operators.

However, it is frequently more natural to recast a higher-order equation as a system of first-order differential equations [33]:

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t);\ \vec{x}(0) = \vec{x}_0. \quad (39)$$

Defining the matrix exponential as

$$e^{tA} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \quad (40)$$

we arrive at a solution analogous to equation (31),

$$\vec{x}(t) = e^{tA}\vec{x}_0 + \int_0^t e^{(t-\tau)A} \vec{f}(\tau)\, d\tau. \quad (41)$$

This useful result is called the variation of parameters formula. Part of its allure is that it expresses the solution in a form whereby the influence of the initial conditions, $\vec{x}_0$, and the external influences, $\vec{f}(t)$, have been isolated and hence may be studied separately.

From Continuous to Discr-*e*-te

The preceding section detailed an operational approach to differential equations (continuous models) founded on the eigenfunction property of the exponential function. An analogous symbolic calculus may be developed for the study of difference equations (discrete models). Appropriately enough, the bridge between these parallel universes (the continuous and the discrete) is afforded by e [34].

First, define the shift operator E,

$$Ef(x) \equiv f(x+h), \quad (42)$$

and the forward difference operator Δ,

$$\Delta f(x) \equiv f(x+h) - f(x). \quad (43)$$

Then by Taylor's theorem, E is expressible in terms of D of the previous section as

$$1 + \Delta = E = e^{hD}. \quad (44)$$

Equation (44) may now be used to transform back and forth between the continuous world of D and the discrete world of Δ:

$$Df(x) = \frac{1}{h}\ln(1+\Delta)f(x)$$
$$= \frac{1}{h}[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots]f(x), \quad (45)$$

$$\Delta f(x) = (e^{hD} - 1)f(x) = [hD + \frac{h^2}{2!}D^2 + \cdots]f(x). \quad (46)$$

Corresponding relations may be derived involving both backward and central difference operators, but the above should be sufficient to convey the essential role of e as intermediary between these two distant lands.

*e*xponential Generating Functions

The sequence $\{a_n\}_{n=0}^{\infty}$ is said to be *generated* by the function $g(x)$ if they are related by [35]

$$g(x) = \sum_{n=0}^{\infty} a_n x^n \quad (47)$$

For example, $(1+x)^m$ generates the binomial coefficients $\binom{m}{n}$.

Likewise, $g(x)$ is called the *exponential generating function* for the sequence $\{a_n\}_{n=0}^{\infty}$ if

$$g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \quad (48)$$

For example, the Bernoulli function $B(x)$ is the exponential generating function for the Bernoulli numbers B_n:

$$B(x) \equiv \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \cdot \frac{x^n}{n!}. \quad (49)$$

The Bernoulli numbers arise in a dizzying variety of analytical and combinatorial contexts [9], like Faulhaber's formula for the sums of powers of integers:

$$\sum_{k=1}^{n} k^p = \sum_{k=1}^{p+1} \frac{(-1)^{p-k+1} B_{p-k+1}\, p!}{k!\,(p-k+1)!} n^k. \quad (50)$$

In a later section, I will explore the use of exponential generating functions in combinatorics and graph theory. But first let us explore an important occurrence of the Bernoulli function in its own right [36].

Bernoull-*e* Function and Singular Perturbations

Many natural phenomena are governed by differential equations whose highest derivative is multiplied by a small parameter ϵ. These include [37] pollutant dispersal in a river estuary, vorticity transport in incompressible flow, atmospheric pollution, kinetic theory of gases, semiconductor devices, groundwater transport, financial modelling, melting phenomena, fluid flow over an airplane, and turbulence transport.

This field of applied mathematics is called *singular perturbation theory* because the correct physical solution for small ϵ bears little resemblance to the solution obtained by simply setting ϵ to zero. A prototypical problem is provided by [38]

$$\epsilon \cdot u'(x) + u(x) = 0 \Rightarrow u_i = e^{-\Delta x/\epsilon} \cdot u_{i-1}, \quad (51)$$

where the subscript notation indicates that the problem has been discretized on a mesh of width Δx. The exact solution afforded by equation (51) is shown in Figure 8 with $u_0 = u(0) = 1$. The steep front that is on prominent display there is called an initial layer.

If we approximate the differential equation by the explicit (forward) Euler scheme [39], we obtain

$$\epsilon \cdot \frac{u_i - u_{i-1}}{\Delta x} + u_{i-1} = 0 \Rightarrow u_i = (1 - \Delta x/\epsilon) \cdot u_{i-1} \quad (52)$$

Comparison of the solutions in equations (51) and (52) reveals that this is equivalent to the use of the first-order Taylor series approximation to the exponential function, $e^z \approx 1 + z$. This approximation is shown in Figure 8 for $\Delta x/\epsilon = 1.9608$ and is clearly seen to be inadequate.

If we instead insert a strategically placed Bernoulli function, $B(-\Delta x/\epsilon)$, into our approximation

$$\epsilon \cdot B(-\Delta x/\epsilon) \cdot \frac{u_i - u_{i-1}}{\Delta x} + u_{i-1} = 0 \Rightarrow u_i = e^{-\Delta x/\epsilon} \cdot u_{i-1}, \quad (53)$$

we obtain the exact solution! This spectacularly successful trick is known as exponential fitting [40].

Let us now employ the implicit (backward) Euler scheme to provide our approximation to the differential equation

$$\epsilon \cdot \frac{u_i - u_{i-1}}{\Delta x} + u_i = 0 \Rightarrow u_i = \frac{1}{1 + \Delta x/\epsilon} \cdot u_{i-1}. \quad (54)$$

Comparison of the solutions in equations (51) and (54) now reveals that this is equivalent to the use of the (0,1)-Padé approximation to the exponential function [41], $e^z \approx 1/(1 - z)$. Revisiting Figure 8, we find that, while the implicit scheme vastly improves over the explicit scheme, it is still inadequate for many applications.

Introduction of a strategically placed Bernoulli function, $B(\Delta x/\epsilon)$, into our approximation

$$\epsilon \cdot B(\Delta x/\epsilon) \cdot \frac{u_i - u_{i-1}}{\Delta x} + u_i = 0 \Rightarrow u_i = e^{-\Delta x/\epsilon} \cdot u_{i-1} \quad (55)$$

again produces the exact solution. The crucial role played by the Bernoulli function in the approximation of singularly perturbed differential equations is thereby revealed.

*e*xponential Asymptotics

Continuing with the theme of the previous section—just because a term is small does not imply that it can be neglected in a respectable mathematical analysis of a physical problem. Consider Figure 9, where we observe a duck moving along the surface of a pond, leaving in its wake a "caustic" enclosing an angle of approximately 40° [42]. Outside the wake, the effect of the duck's motion is "exponentially small"; inside and on the caustic, its effect may be obtained through asymptotic approximations [43]. However, as first observed by Stokes in 1847, these exponentially small terms must be accounted for in order to produce refined asymptotic approximations.

Quantitative accuracy is not the only concern of such so-called exponential asymptotics. The very existence of a solution may hinge on properly accounting for exponentially small terms. For example, in crystal growth the formation of dendritic fingers is governed by an exponentially small term in the surface energy between solid and liquid [43]. This is an area of intense current interest in applied mathematics [42].

*e*xponential Transform

Let us now return to the subject of exponential generating functions. In combinatorial analysis [44], generating functions are used for problems involving combinations while exponential generating functions find application in problems involving permutations. However, exponential generating functions find wider application to graphical enumeration problems [45].

Let $A(x) = \Sigma_{n=1}^{\infty} a_n x^n/n!$ be the exponential generating function of the sequence $\{a_n\}_{n=1}^{\infty}$ and $B(x) = \Sigma_{n=1}^{\infty} b_n x^n/n!$ be the exponential generating function of the sequence $\{b_n\}_{n=1}^{\infty}$. If $1 + B(x) = e^{A(x)}$ then $\{b_n\}_{n=1}^{\infty}$ is called the exponential transform of $\{a_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ is called the logarithmic transform of $\{b_n\}_{n=1}^{\infty}$ [46].

Riddell's formula relates the number of graphs with n vertices to the number of such graphs which are connected [47]. It is stated in [48] that "if a_n is the number of connected labeled graphs with a certain property, then b_n is the total num-

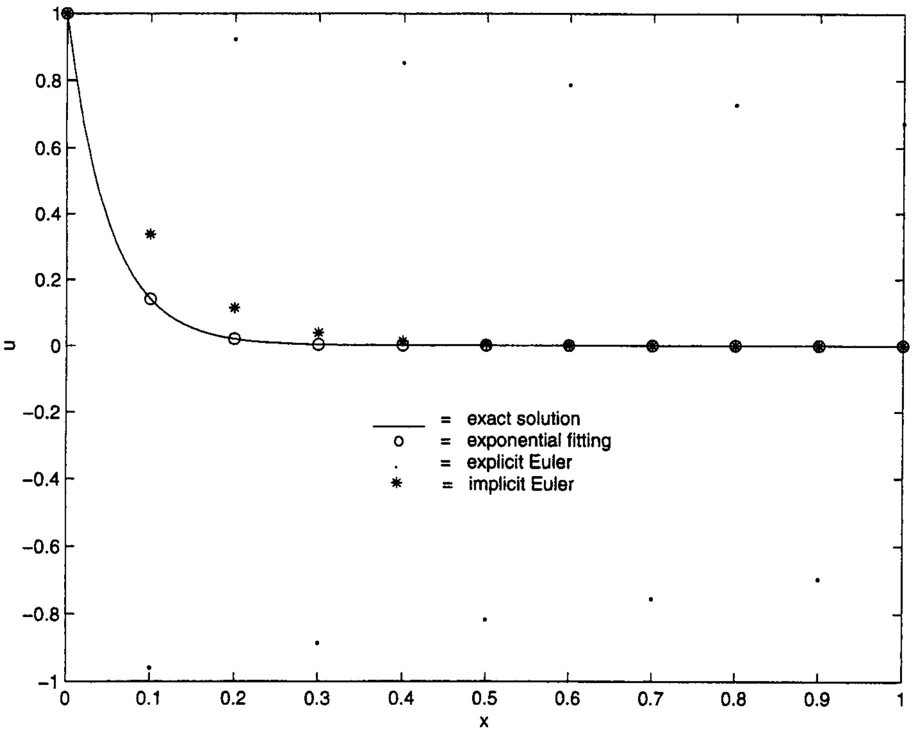

Figure 8. Another Bernoulli miracle!

Figure 9. Duck-e with attendant wake.

ber of labeled graphs with that property." Letting the property in question be "having exactly two connected components," we see that as stated this is clearly false. ($a_n = 0$ for all n so that Riddell's formula implies that $b_n = 0$ for all n, but $b_2 = 1$.) Unfortunately, this error is repeated by Weisstein [46].

Sloane and Plouffe cite Harary and Palmer [45] who in turn cite Riddell [49]. However, inspection of Riddell's original work [49] reveals that he never applied the exponential transform to any property other than connectedness. The correct generalization of Riddell's formula requires amendment of the above: "then b_n is the total number of labeled graphs with n vertices whose connected components possess the same property." Thus, if one restricts oneself to properties inherited by the connected components of a graph, then Riddell's formula may be confidently used as an enumerative tool. For example, if we let the property be "even-ness" then Riddell's formula correctly relates the number of Eulerian graphs to the number of even graphs (see [47] for the definitions of these graph-theoretic concepts). Otherwise, employment of Riddell's formula produces erroneous results.

Pron-*e*'s Method

It is sometimes necessary to approximate a function by a sum of exponential functions. Essentially a nonlinear problem, this may be accomplished by transforming it to the problem of finding the zeros of a certain polynomial. This technique is known as Prony's method [39].

Given $f(x)$ at $x = 0, 1, \ldots, N-1$, we seek an exponential approximation

$$f(x) \approx c_1 e^{\alpha_1 x} + \cdots + c_n e^{\alpha_n x}, \qquad (56)$$

where $N \geq 2n$. Defining $f_j = f(j)$, $\mu_k := e^{\alpha_k}$, Prony's method proceeds as follows:

1. Solve (by least squares if necessary) for $\beta_1, \ldots, \beta_n$ in the $N - n$ linear equations

$$\begin{aligned} f_n + f_{n-1}\beta_1 + \cdots + f_0\beta_n &= 0 \\ &\vdots \\ f_{N-1} + f_{N-2}\beta_1 + \cdots + f_{N-n-1}\beta_n &= 0. \end{aligned} \qquad (57)$$

2. Find the roots $\mu_1, \ldots, \mu_n$ of

$$\mu^n + \beta_1\mu^{n-1} + \cdots + \beta_{n-1}\mu + \beta_n = 0, \qquad (58)$$

and define $\alpha_k = \ln\mu_k$ $(k = 1, \ldots n)$.

3. Solve by least squares for $c_1, \ldots, c_n$ in the N linear equations

$$\begin{aligned} c_1 + c_2 + \cdots + c_n &= f_0 \\ \mu_1 c_1 + \mu_2 c_2 + \cdots + \mu_n c_n &= f_1 \\ &\vdots \\ \mu_1^{N-1} c_1 + \mu_2^{N-1} c_2 + \cdots + \mu_n^{N-1} c_n &= f_{N-1}. \end{aligned} \qquad (59)$$

Depending on the size of n, we may either solve equation (58) by directly applying a polynomial root-finding algorithm or by recasting it as an eigenvalue problem for the companion matrix [50]

$$\begin{bmatrix} -\beta_1 & 1 & 0 & \cdots & 0 \\ -\beta_2 & 0 & 1 & \cdots & 0 \\ -\beta_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_n & 0 & 0 & \cdots & 0 \end{bmatrix}. \qquad (60)$$

Modern-day applications of Prony's method include digital-filter design, radar and sonar signal processing, determination of atmospheric transfer functions, and even radiation therapy planning for cancer patients [51]. The original algorithm of Baron de Prony [52] dates back to 1795 and was concerned only with the case $N = 2n$. Unlike his contemporaries Monge and Fourier, de Prony refused to join Napoleon's invasion of Egypt. Only his wife's close friendship with Josephine averted the anticipated dire consequences of such lack of military zeal [53].

Probabl-*e*

e has the curious habit of asserting itself in strange probabilistic contexts [54]. For example, suppose that numbers are selected at random from the interval [0,1]. What is the expected number of draws necessary for the sum of these numbers to exceed 1? There are several elementary proofs [54, 55, 56] that the answer is emphatically e.

Then there is the old chestnut of the "secretary problem" [22, 54]. Here, it is required to select a new secretary from a pool of n applicants. Candidates are interviewed sequentially and the decision whether or not to hire an individual must be made at the conclusion of the interview. If the candidate is not selected then he/she is excluded from further consideration. If this process ever reaches the last applicant then that individual is automatically hired. The goal is now to devise a strategy that maximizes the probability of hiring the most qualified applicant. There turns out to be a simple optimal strategy: for some integer k (to be determined as part of the optimization problem) with $1 \le k < n$, interview and reject the first k applicants. Then, from the remaining $n - k$ applicants, choose the first one that is the best seen to date. In the limiting case of large n, the optimal value is $k \sim n/e \approx 0.368n$ and the probability of finding the best applicant is asymptotically $1/e$ (about 36.8%). This is but one of a variety of optimal stopping problems [22]; remarkably, this very result has been applied to the allocation of cadaveric kidneys for transplantation [57].

Well, our newly hired secretary is working out fine until the Christmas luncheon, at which one too many spiked eggnogs is imbibed. Upon returning to the office, we are confronted with the "drunken secretary problem." Waiting on the desk are n different letters, each with a corresponding addressed envelope. What is the probability that he/she will produce a "derangement" whereby no letter is placed in its correct envelope? Euler first posed this problem, answered it, and showed that the required probability asymptotically approaches $1/e$ [2].

Of course, the flip side of this problem is: What is the probability, p, that a permutation of n distinct objects has at least one fixed point? According to the above, $p \sim 1 - 1/e \approx 0.632$. We have hardly exhausted all of the appearances of e in probability [54], but it is time we moved on to her sister discipline of statistics.

Statisticall-*e*

While occurrences of e throughout probability theory might be characterized as sporadic, its influence on statistical theory has been pervasive. First introduced by DeMoivre in 1733 as the limit of the binomial distribution [58], the normal distribution, $N(x;\mu,\sigma) = (\sigma\sqrt{2\pi})^{-1}e^{-(x-\mu)^2/(2\sigma^2)}$, has come to dominate the statistical landscape. Spurred on by Gauss's 1809 theory of errors of observation and Laplace's publication in 1812 of the Central Limit Theorem [59], study of the normal distribution became virtually synonymous with statistical inquiry. This fixation on the normal distribution reached an abnormal pitch in the late nineteenth century at the hands of Adolphe Quetelet and Sir Francis Galton [59]. This frenzy was tempered somewhat in the twentieth century by Galton's protégé Karl Pearson and his nemesis Sir Ronald Fisher, but to this day the normal distribution holds a special place in the hearts and minds of statisticians.

e is intimately involved with at least two other statistical distributions [60], one discrete, the Poisson distribution with parameter a (mean)

$$Prob\{X = k\} = e^{-a} \cdot \frac{a^k}{k!}, \tag{61}$$

and the other continuous, the exponential distribution with parameter c (1/mean)

$$f(x) = ce^{-cx} \cdot U(x), \tag{62}$$

where $U(x)$ denotes the Heaviside unit step function.

These two distributions are related by the theory of stochastic processes [60], where the Poisson distribution is used to model the number of arrivals for a queue in a given interval of time and, correspondingly, the exponential distribution is used to model the actual arrival times. Thus, they give rise to mathematical models of supermarket waiting lines and car accidents. If the independent variable is not time but space, then the exponential distribution can be used to model distance between roadkill or between mutations on a strand of DNA.

The exponential distribution possesses a special property that endows it with great utility in reliability theory [60, pp. 82–83]. This can be seen by computing the probability of failure of some physical device during the interval $(x, x + dx)$, assuming that it did not fail prior to time t:

$$f(x|X \ge t)\ dx = f(x - t)\ dx; \quad x \ge t. \tag{63}$$

where X is the random failure time of the device.

Thus, the conditional failure rate, $\beta(t)$, defined as

$$\beta(t) \equiv f(t|X \ge t) = f(0) = c, \tag{64}$$

is independent of t. This is called the memoryless property, and, significantly, only the exponential distribution possesses it. This implies that, in any application where the failure rate is approximately constant, the exponential distribution must arise. This is evidenced by its frequent appearance in modeling the "middle years" of the lifetime of diverse systems ranging from industrial machines to human beings.

The exponential function is also used to define the moment generating function of the statistical distribution with density function $f(x)$ [60, p. 116]:

$$\Phi(s) \equiv \int_{-\infty}^{\infty} f(x) \cdot e^{sx}\ dx \Rightarrow m_n \equiv E\{X^n\} = \Phi^{(n)}(0), \tag{65}$$

where E is the expected value. Consequently, $\Phi(s)$ is the exponential generating function of the moment sequence $\{m_n\}_{n=1}^{\infty}$. In point of fact, generating functions were first exploited by Lagrange and Laplace in their probabilistic investigations [59]. Providing yet another unifying thread, I note that Stirling's formula is an essential ingredient in one of the proofs of the DeMoivre theorem alluded to above [60]!

Family Tr-*e*

Over the course of four centuries, e has dutifully obeyed the biblical directive "be fruitful and multiply" and spawned an illustrious lineage. Among its distinguished progeny [35, 61]:

- hyperbolic functions: $\sinh(z) = (e^z - e^{-z})/2$, etc.
- trigonometric functions: $\cos(z) = (e^{\imath z} + e^{-\imath z})/2$, etc.
- Bernoulli function: $B(x) = x/(e^x - 1)$
- Gaussian function: $G(x;\ \alpha,\beta) = e^{-[(x-\alpha)/\beta]^2}$
- error function: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$
- Gamma function: $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ (Re $z > 0$)
- exponential integral: $Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt$
- orthogonal polynomials:
  Hermite: weight $w(x) = e^{-x^2}$ on $(-\infty,\infty)$
  Laguerre: weight $w(x) = x^\alpha e^{-x}$ on $(0,\infty)$, $\alpha > -1$
- integral transforms:
  Fourier: $\int_{-\infty}^{\infty} e^{\imath xy} f(y)dy$
  Laplace: $\int_0^\infty e^{-xy} f(y)dy$
  Gauss: $\int_{-\infty}^{\infty} e^{-(x-y)^2} f(y)dy$

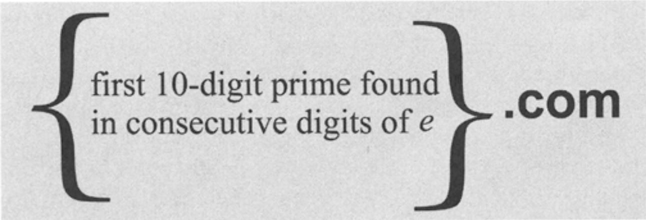

Figure 10. Google-plexed.

It should be clear from the above that one would be challenged to find a branch of pure or applied mathematics which has not felt the enriching presence of e. Readers are invited to add their own favorite branches to this already robust family tree.

Conclusion

The time has come to end our stroll through the Garden of e-den, even though we have yet to taste many of its most succulent fruits, such as exponential splines [62]. However, I would like to conclude my p-e-an to e in a lighter vein.

The initial decimal digits of e are quite easy to remember: 2.7 1828 1828 45 90 45. In spite of this, many mnemonics based on the number of characters of each word have been developed. A particular favorite is: "It enables a numskull to memorize a quantity of numerals" [63]. For the more sophisticated, I offer the pleasingly self-referential: "I'm forming a mnemonic to remember a function in analysis" [13]. Of course, anything can be taken to excess, as is evidenced by the gargantuan 40-digit mnemonic of [64].

Internet giant Google holds the distinction of having transformed a mathematical noun (the googol [65]) into a verb. As further evidence of their mathematical pedigree, when filing the registration for their 2004 initial public offering of stock (IPO) with the Securities and Exchange Commission (SEC), they chose not to state the number of shares to be offered. Instead, they declared $2,718,281,828 as their estimate of the money that would be so raised.

Later that same year, from deep within their corporate headquarters in Mountain View, California (the Googleplex), a most curious recruitment strategy was hatched. Billboards and banners, such as that in Figure 10, appeared anonymously in Silicon Valley and Cambridge, Massachusetts. After deciphering the puzzle (whose solution begins strangely enough at the 101st digit of e) and visiting 7427466391.com, one was eventually led to a solicitation of employment for Google.

Even though e finally has its own biography [5], it has traditionally had to dwell in the shadow cast by the headline-grabbing π. Witness the fact that in [12], π has 8 pages devoted to it, while e is allocated a paltry 1/2 page! However, the situation has improved to the point where e now has its own Web page [66], from which Figure 11 is borrowed.

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Top $\ln(e^{10})$ Reasons Why e Is Better than π

10) e is easier to spell than π.
9) $\pi \approx 3.14$ while $e \approx 2.718281828459045$.
8) The character for e can be found on a keyboard, but π sure can't.
7) Everybody fights for a piece of the pie.
6) $\ln(\pi)$ is a really nasty number, but $\ln(e) = 1$.
5) e is used in calculus while π is used in baby geometry.
4) 'e' is the most commonly picked vowel in Wheel of Fortune.
3) e stands for Euler's Number; π doesn't stand for squat.
2) You don't need to know Greek to be able to use e.
1) You can't confuse e with a food product.

Figure 11. e-gotism.

REFERENCES

[1] C. C. Gillispie, *Biographical Dictionary of Mathematics*, Charles Scribner's Sons, New York, 1991.
[2] W. Dunham, *Euler: The Master of Us All*, Mathematical Association of America, Washington, DC, 1999.
[3] U. G. Mitchell and M. Strain, The Number e, *Osiris* 1 (1936), 476–496.
[4] J. L. Coolidge, The Number e, *American Mathematical Monthly* 57 (1950), 591–602.
[5] E. Maor, *e: The Story of a Number*, Princeton University Press, Princeton, NJ, 1994.
[6] A. H. Bell, *The Exponential and Hyperbolic Functions and Their Applications*, Pitman & Sons, London, 1932.
[7] M. E. J. Gheury de Bray, *Exponentials Made Easy*, Macmillan & Co., London, 1921.
[8] P. Beckmann, *A History of π*, Barnes & Noble, New York, 1993.
[9] J. H. Conway and R. H. Guy, *The Book of Numbers*, Springer-Verlag, New York, 1996.
[10] M. Brede, On the Convergence of the Sequence Defining Euler's Number, *Mathematical Intelligencer* 27 (2005), 6–7.
[11] H. J. Brothers and J. A. Knox, New Closed-Form Approximations to the Logarithmic Constant e, *Mathematical Intelligencer* 20 (1998), 25–29.
[12] D. Wells, *The Penguin Dictionary of Curious and Interesting Numbers*, Penguin Books Ltd., Middlesex, England, 1986.
[13] M. Gardner, *The Unexpected Hanging and Other Mathematical Diversions*, Simon and Schuster, New York, 1969.
[14] H. J. Brothers, Improving the Convergence of Newton's Series Approximation for e, *College Mathematics Journal* 35(1) (2004), 34–39.
[15] J. A. Knox and H. J. Brothers, Novel Series-Based Approximations to e, *College Mathematics Journal* 30(4) (1999), 269–275.
[16] C. D. Olds, The Simple Continued Fraction Expression for e, *American Mathematical Monthly* 77 (1970), 968–974.
[17] N. Pippenger, An Infinite Product for e, *American Mathematical Monthly* 87(5) (1980), 391.
[18] X. Gourdon and P. Sebah, *The Constant e and Its Computation*, http://numbers.computation.free.fr/Constants/E/e.html.

[19] A. Baker, *Transcendental Number Theory*, Cambridge University Press, Cambridge, 1975.

[20] I. Niven, Which is Larger, e^π or π^e? in *A Century of Calculus, Part II: 1969–1991*, Mathematical Association of America, Washington, DC, 1992, 445–447.

[21] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Wiley-Interscience, New York, 1987.

[22] S. R. Finch, *Mathematical Constants*, Cambridge University Press, Cambridge, 2003.

[23] R. Courant and H. Robbins, *What Is Mathematics?*, Oxford University Press, New York, 1941.

[24] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Third Edition, Oxford University Press, London, 1954.

[25] P. Ribenboim, *The New Book of Prime Number Records*, Springer, New York, 1996.

[26] S. M. Ruiz, A Result on Prime Numbers, *Mathematical Gazette* 81 (1997), 269–270.

[27] E. T. Bell, *Men of Mathematics*, Simon and Schuster, New York, 1965.

[28] H. V. Baravalle, The Number *e*—The Base of the Natural Logarithms, *The Mathematics Teacher* 38 (1945), 350–355.

[29] B. Friedman, *Lectures on Applications-Oriented Mathematics*, Holden-Day, San Francisco, 1969.

[30] G. H. Hardy, *Ramanujan*, AMS Chelsea, Providence, RI, 1999.

[31] J. Aczél, *On Applications and Theory of Functional Equations*, Academic Press, New York, 1969.

[32] H. Bateman, *Differential Equations*, Chelsea, New York, 1966.

[33] R. E. O'Malley, *Thinking About Ordinary Differential Equations*, Cambridge University Press, New York, 1997.

[34] G. Boole, *Calculus of Finite Differences*, Chelsea, New York, 1970.

[35] S. Iyanaga and Y. Kawada, *Encyclopedic Dictionary of Mathematics*, MIT Press, Cambridge, MA, 1980.

[36] J. Spanier and K. B. Oldham, *An Atlas of Functions*, Hemisphere, New York, 1987.

[37] K. W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Chapman & Hall, London, 1996.

[38] E. P. Doolan, J. J. H. Miller, and W. H. A. Schilders, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980.

[39] F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.

[40] B. J. McCartin, Exponential Fitting of the Delayed Recruitment/Renewal Equation, *Journal of Computational and Applied Mathematics* 136 (2001), 343–356.

[41] G. A. Baker, Jr. and P. Graves-Morris, *Padé Approximants*, 2nd Edition, Cambridge University Press, Cambridge, 1996.

[42] D. Mortimer, *Research on Asymptotics*, http://www.maths.ox.ac.uk/~mortimer/research/davidresearch.shtml.

[43] A. C. Fowler, *Mathematical Models in the Applied Sciences*, Cambridge University Press, Cambridge, 1997.

[44] J. Riordan, *An Introduction to Combinatorial Analysis*, Princeton University Press, Princeton, NJ, 1980.

[45] F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.

[46] E. M. Weisstein, *Exponential Transform*, http://mathworld.wolfram.com/ExponentialTransform.html.

[47] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.

[48] N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic, San Diego, 1995.

[49] R. J. Riddell, Jr., *Contributions to the Theory of Condensation*, Ph.D. Dissertation, University of Michigan, UMI Dissertation Services, Ann Arbor, MI, 1951.

[50] A. S. Householder, *The Numerical Treatment of a Single Nonlinear Equation*, McGraw-Hill, New York, 1970.

[51] K. Holmström, A. Ahnesjö, and J. Petersson, Algorithms for Exponential Sum Fitting in Radiotherapy Planning, *Technical Report IMa-TOM-2001-02*, Department of Mathematics and Physics, Mälarden University, Sweden (2001).

[52] G. R. de Prony, Essai éxperimental et analytique, *Journal de l'École Polytechnique*, 1(22) (1795), 24–76.

[53] J. J. O'Connor and E. F. Robertson, Gaspard Clair François Marie Riche de Prony, *http://www.groups.dcs.st-and.ac.uk/history/Mathematicians/De_Prony.html.*

[54] H. S. Schultz and B. Leonard, Unexpected Occurrences of the Number *e*, *Mathematics Magazine* 62(4) (1989), 269–271.

[55] N. MacKinnon, Another Surprising Appearance of *e*, *Mathematical Gazette* 74(468) (1990), 167–169.

[56] D. J. Colwell, J. R. Gillett and B. C. Jones, That Number *e* Again, *Mathematical Gazette* 75(473) (1991), 329–330.

[57] J. Hornberger and J-H. Ahn, Deciding eligibility for transplantation when a kidney donor is available, *Medical Decision Making* 17 (1997), 160–170.

[58] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, NJ, 1946.

[59] S. M. Stigler, *The History of Statistics*, Harvard University Belknap Press, Cambridge, MA, 1986.

[60] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, Second Edition, McGraw-Hill, New York, 1984.

[61] H. Bateman, *Higher Transcendental Functions*, Krieger, Malabar, FL, 1981.

[62] B. J. McCartin, Theory of Exponential Splines, *Journal of Approximation Theory* 66(1) (1991), 1–23.

[63] M. Gardner, *The Scientific American Book of Mathematical Puzzles and Diversions*, Simon and Schuster, New York, 1959.

[64] Z. Barel, A Mnemonic for *e*, *Mathematics Magazine* 68 (1995), 253.

[65] E. Kasner and J. Newman, *Mathematics and the Imagination*, Simon and Schuster, New York, 1967.

[66] D. Sapp, *The e Home Page*, http://www.mu.org/~doug/exp/etop10.html.