A General Formula for Calculating Meridian Arc Length and its Application to Coordinate Conversion in the Gauss-Krüger Projection

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Abstract

The meridian arc length from the equator to arbitrary latitude is of utmost importance in map projection, particularly in the Gauss-Krüger projection. In Japan, the previously used formula for the meridian arc length was a power series with respect to the first eccentricity squared of the earth ellipsoid, despite the fact that a more concise expansion using the third flattening of the earth ellipsoid has been derived. One of the reasons that this more concise formula has rarely been recognized in Japan is that its derivation is difficult to understand. This paper shows an explicit derivation of a general formula in the form of a power series with respect to the third flattening of the earth ellipsoid. Since the derived formula is suitable for implementation as a computer program, it has been applied to the calculation of coordinate conversion in the Gauss-Krüger projection for trial.

1. Introduction

As is well known in geodesy, the meridian arc length $S(\varphi)$ on the earth ellipsoid from the equator to the geographic latitude φ is given by

$$S(\varphi)=\int_0^{\varphi}\frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta\,, \tag{1}$$

where a and e are the semi-major axis and the first eccentricity of the earth ellipsoid, respectively.

Since formula (1) includes an elliptic integral, it cannot be expressed explicitly using a combination of elementary functions. As a rule, we evaluate this elliptic integral as an approximate expression by first expanding the integrand in a binomial series with respect to e^2, regarding e as a small quantity, then readjusting with a trigonometric function, and finally carrying out termwise integration.

The following example is an approximation of formula (1) obtained by truncating the expansion at order e^{10}:

$$\begin{gathered}
S(\varphi)\approx a\left(1-e^2\right)\left(C_1\varphi-C_2\frac{\sin 2\varphi}{2}+C_3\frac{\sin 4\varphi}{4}-C_4\frac{\sin 6\varphi}{6}+C_5\frac{\sin 8\varphi}{8}-C_6\frac{\sin 10\varphi}{10}\right),\\
C_1=1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\frac{11025}{16384}e^8+\frac{43659}{65536}e^{10}\,,\\
C_2=\frac{3}{4}e^2+\frac{15}{16}e^4+\frac{525}{512}e^6+\frac{2205}{2048}e^8+\frac{72765}{65536}e^{10}\,,\quad C_3=\frac{15}{64}e^4+\frac{105}{256}e^6+\frac{2205}{4096}e^8+\frac{10395}{16384}e^{10}\,,\\
C_4=\frac{35}{512}e^6+\frac{315}{2048}e^8+\frac{31185}{131072}e^{10}\,,\quad C_5=\frac{315}{16384}e^8+\frac{3465}{65536}e^{10}\,,\quad C_6=\frac{693}{131072}e^{10}.
\end{gathered} \tag{2}$$

Since the above expression has fractional coefficients too complicated for most readers, except for those specialized in the given field, to memorize at a glance, users face a perpetual risk of making mistakes owing to errors in typing the fractional coefficients. Despite this fact, formula (2) has long been used generally in Japan to calculate the meridian arc length of the earth ellipsoid.

2. Other expressions

2.1 Bessel's formula

On the other hand, an alternative derivation using yet another quantity n, the third flattening of the earth ellipsoid defined as

$$n = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} \cong \frac{e^2}{4}, \tag{3}$$

has been presented by Bessel (Bessel, 1837).

Bessel rewrote formula (1) using the relation expressed in formula (3) as

$$S(\varphi) = \int_0^{\varphi} \frac{a(1-n)^2(1+n)}{\left(1+2n\cos 2\theta + n^2\right)^{3/2}} \mathrm{d}\theta, \tag{4}$$

expanded the denominator of the integrand in a series with respect to n, and then carried out termwise integration.

The following formula is a summary of the expansion result that appeared in Bessel's original paper:

$$S(\varphi) \approx a(1-n)^2(1+n) \\ \times \left\{\left(1+\frac{9}{4}n^2+\frac{225}{64}n^4\right)\varphi - \frac{3}{2}\left(n+\frac{15}{8}n^3+\frac{175}{64}n^5\right)\sin 2\varphi + \frac{15}{16}\left(n^2+\frac{7}{4}n^4\right)\sin 4\varphi - \frac{35}{48}\left(n^3+\frac{27}{16}n^5\right)\sin 6\varphi\right\}. \tag{5}$$

The main advantages of this formula over formula (2) are its good convergence, since n is about a quarter of the value of e^2, and the reduced number of terms appearing in the formula despite its almost equal precision.

2.2 Helmert's formula

About forty years after the publication of Bessel's result, Helmert derived a simpler expression by replacing the $(1-n)^2(1+n)$ factor appearing in formula (5) with an equivalent value $\left(1-n^2\right)^2/(1+n)$. Helmert then multiplied the terms in the curly brackets in formula (5) by $\left(1-n^2\right)^2$, the numerator of the substituted value, aiming to extract the factor $1/(1+n)$ and simplify the fractional coefficients appearing in formula (5) (Helmert, 1880).

More than thirty years after that, Krüger summarized Helmert's result in his paper published in 1912 (Krüger, 1912) as the following formula:

$$S(\varphi) \approx \frac{a}{1+n}\left\{\left(1+\frac{n^2}{4}+\frac{n^4}{64}\right)\varphi - \frac{3}{2}\left(n-\frac{n^3}{8}\right)\sin 2\varphi + \frac{15}{16}\left(n^2-\frac{n^4}{4}\right)\sin 4\varphi - \frac{35}{48}n^3\sin 6\varphi + \frac{315}{512}n^4\sin 8\varphi\right\}. \tag{6}$$

As formula (6) shows, the result derived by Helmert has a simpler and more concise expression than does formula (2), which was truncated at order e^{10}, and has almost the same or greater precision despite its truncation to no more than order n^4 (corresponds to e^8), as reported by Tobita et al. (Tobita et al., 2009). Nevertheless, the derivation process used by Helmert (1880) seems to be not only difficult to understand but also hard to generalize.

3. Derivation of general formula

In order to generalize formula (6), we choose another approach that does not start from formula (4) directly. First, in accordance with a well-known result regarding elliptic integrals and elliptic functions (e.g., Byrd et al., 1954), we find that the integral in formula (1) can be regarded as a special case of incomplete elliptic integrals of the third kind. Thus, it can be divided into an incomplete elliptic integral of the second kind and a term of elementary functions as*

$$S(\varphi)=a\left(\int_0^{\varphi}\sqrt{1-e^2\sin^2\theta}\,\mathrm{d}\theta-\frac{e^2\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}}\right). \quad (7)$$

Bearing the relation expressed in formula (3) in mind and letting $2\theta=\tau$, it is not hard to see that we can rewrite formula (7) as

$$S(\varphi)=\frac{a}{1+n}\left(\frac{1}{2}\int_0^{2\varphi}\sqrt{1+2n\cos\tau+n^2}\,\mathrm{d}\tau-\frac{2n\sin 2\varphi}{\sqrt{1+2n\cos 2\varphi+n^2}}\right). \quad (8)$$

At a glance, the result of formula (8) appears to be a favorable transformation, since the factor $a/(1+n)$, which also appears in formula (6), appears spontaneously. Now, we can begin to examine formula (8) in order to generalize Helmert's result.

First, since the integrand appearing in the first term in the parentheses in formula (8) (we denote this term S_1) can be regarded as a generating function of Gegenbauer polynomials $C_i^{(-1/2)}(\cos\tau)$ (e.g., Abramowitz et al., 1965), we can expand S_1 in a power series with respect to $-n$ and rearrange all the terms as

*Since we could find no references describing the derivation of this result in detail, a simple proof is given in the appendix.

$$
\begin{aligned}
S_1 &\equiv \frac{1}{2}\int_0^{2\varphi}\sqrt{1+2n\cos\tau+n^2}\,\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^{2\varphi}\sum_{i=0}^{\infty}(-n)^i C_i^{(-1/2)}(\cos\tau)\,\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^{2\varphi}\sum_{i=0}^{\infty}(-n)^i \sum_{k=0}^{i}\frac{\Gamma(k-1/2)\Gamma(i-k-1/2)}{k!(i-k)![\Gamma(-1/2)]^2}\cos(i-2k)\tau\,\mathrm{d}\tau \\
&= \int_0^{2\varphi}\sum_{j=0}^{\infty}\frac{n^{2j}}{2}\sum_{k=0}^{2j}\frac{\Gamma(k-1/2)\Gamma(2j-k-1/2)}{k!(2j-k)![\Gamma(-1/2)]^2}\cos(2j-2k)\tau\,\mathrm{d}\tau \\
&\quad -\int_0^{2\varphi}\sum_{j=0}^{\infty}\frac{n^{2j+1}}{2}\sum_{k=0}^{2j+1}\frac{\Gamma(k-1/2)\Gamma(2j+1-k-1/2)}{k!(2j+1-k)![\Gamma(-1/2)]^2}\cos(2j+1-2k)\tau\,\mathrm{d}\tau \\
&= \int_0^{2\varphi}\sum_{j=0}^{\infty}n^{2j}\left\{\frac{1}{2}\prod_{r=1}^{j}\left(1-\frac{3}{2r}\right)^2+\sum_{k=0}^{j-1}\cos(2j-2k)\tau\prod_{r=1}^{k}\left(1-\frac{3}{2r}\right)\prod_{s=1}^{2j-k}\left(1-\frac{3}{2s}\right)\right\}\mathrm{d}\tau \\
&\quad -\int_0^{2\varphi}\sum_{j=0}^{\infty}n^{2j+1}\sum_{k=0}^{j}\cos(2j+1-2k)\tau\prod_{r=1}^{k}\left(1-\frac{3}{2r}\right)\prod_{s=1}^{2j+1-k}\left(1-\frac{3}{2s}\right)\mathrm{d}\tau \\
&= \int_0^{2\varphi}\sum_{j=0}^{\infty}\left\{\frac{1}{2}\prod_{r=1}^{j}\left(\frac{3n}{2r}-n\right)^2+\sum_{k=1}^{j}\cos 2k\tau\prod_{r=1}^{j-k}\left(\frac{3n}{2r}-n\right)\prod_{s=1}^{j+k}\left(\frac{3n}{2s}-n\right)\right\}\mathrm{d}\tau \\
&\quad +\int_0^{2\varphi}\sum_{j=0}^{\infty}\sum_{k=1}^{j+1}\cos(2k-1)\tau\prod_{r=1}^{j-k+1}\left(\frac{3n}{2r}-n\right)\prod_{s=1}^{j+k}\left(\frac{3n}{2s}-n\right)\mathrm{d}\tau \\
&= \sum_{j=0}^{\infty}\left\{\varphi\prod_{r=1}^{j}\varepsilon_r^2+\sum_{k=1}^{j}\frac{\sin 4k\varphi}{2k}\prod_{r=1}^{j-k}\varepsilon_r\prod_{s=1}^{j+k}\varepsilon_s+\sum_{k=1}^{j+1}\frac{\sin(4k-2)\varphi}{2k-1}\prod_{r=1}^{j-k+1}\varepsilon_r\prod_{s=1}^{j+k}\varepsilon_s\right\} \\
&= \sum_{j=0}^{\infty}\left\{\varphi\prod_{k=1}^{j}\varepsilon_k^2+\sum_{l=1}^{2j}\frac{\sin 2l\varphi}{l}\prod_{r=1}^{j-\lfloor (l+1)/2\rfloor}\varepsilon_r\prod_{s=1}^{j+\lfloor l/2\rfloor}\varepsilon_s\right\} \\
&= \sum_{j=0}^{\infty}\left(\prod_{k=1}^{j}\varepsilon_k\right)^2\left\{\varphi+\sum_{l=1}^{2j}\frac{\sin 2l\varphi}{l}\prod_{m=1}^{l}\varepsilon_{j+(-1)^m\lfloor m/2\rfloor}^{(-1)^m}\right\}.
\end{aligned}
\tag{9}
$$

In formula (9), $\varepsilon_i = 3n/2i - n$, $\Gamma(x)$ denotes the Gamma function, and $\lfloor x\rfloor$ denotes the floor function. From the above result, for the time being we rewrite formula (8) as

$$
S(\varphi)=\frac{a}{1+n}\left[\sum_{j=0}^{\infty}\left(\prod_{k=1}^{j}\varepsilon_k\right)^2\left\{\varphi+\sum_{l=1}^{2j}\frac{\sin 2l\varphi}{l}\prod_{m=1}^{l}\varepsilon_{j+(-1)^m\lfloor m/2\rfloor}^{(-1)^m}\right\}-\frac{2n\sin 2\varphi}{\sqrt{1+2n\cos 2\varphi+n^2}}\right].
\tag{10}
$$

Next, we turn to the second term in the parentheses (we denote this term S_2) in formula (8). First, we take the following favorable relation between S_1 and S_2 into account:

$$
\begin{aligned}
\frac{\mathrm{d}^2 S_1}{\mathrm{d}\varphi^2} &= \frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\left(\frac{1}{2}\int_0^{2\varphi}\sqrt{1+2n\cos\tau+n^2}\,\mathrm{d}\tau\right) \\
&= -\frac{2n\sin 2\varphi}{\sqrt{1+2n\cos 2\varphi+n^2}} = S_2\,.
\end{aligned}
\tag{11}
$$

On the other hand, as we have already seen the final result of formula (9), it is easy to understand that S_1 can be represented as a trigonometric expansion that has the form

$$S_1 = \alpha_0 \varphi + \sum_{l=1}^{\infty} \alpha_l \sin 2l\varphi. \tag{12}$$

Bering the relation of formula (11) in mind, we find that S_2 can be represented with a very similar form to that of S_1, i.e.,

$$\begin{aligned} S_2 = \frac{\mathrm{d}^2 S_1}{\mathrm{d}\varphi^2} &= \frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\left(\alpha_0\varphi + \sum_{l=1}^{\infty}\alpha_l \sin 2l\varphi\right) \\ &= \sum_{l=1}^{\infty} -4l^2 \alpha_l \sin 2l\varphi. \end{aligned} \tag{13}$$

Combining S_1 and S_2, we finally arrive at a general formula for calculating the meridian arc length from the equator to an arbitrary geographic latitude,

$$\begin{aligned} S(\varphi) &= \frac{a}{1+n}(S_1 + S_2) = \frac{a}{1+n}\sum_{j=0}^{\infty}\left(\prod_{k=1}^{j}\varepsilon_k\right)^2 \left\{\varphi + \sum_{l=1}^{2j}\left(\frac{1}{l} - 4l\right)\sin 2l\varphi \prod_{m=1}^{l} \varepsilon_{j+(-1)^m \lfloor m/2 \rfloor}^{(-1)^m}\right\} \\ &= \frac{a}{1+n}\sum_{j=0}^{\infty}\left\{\prod_{k=1}^{j}\left(\frac{3n}{2k} - n\right)\right\}^2 \left[\varphi + \sum_{l=1}^{2j}\left(\frac{1}{l} - 4l\right)\sin 2l\varphi \prod_{m=1}^{l}\left\{\frac{3n}{2j + 2\cdot(-1)^m \lfloor m/2 \rfloor} - n\right\}^{(-1)^m}\right]. \end{aligned} \tag{14}$$

Truncating the summation with respect to index j in formula (14) at $j = 2$, we can confirm that this formula yields Helmert's result summarized in formula (6).

Note that we can derive another general formula for $S(\varphi)$ without dividing terms as in formula (8) using the fact that the denominator of the integrand in formula (4) can be regarded as a generating function of Gegenbauer polynomials $C_i^{(3/2)}(\cos 2\theta)$. In this case, just replacing $\varepsilon_i = 3n/2i - n$ in formula (9) with $\varepsilon_i' = -n/2i - n$ by analogy with the expansion with $C_i^{(-1/2)}(\cos\tau) = C_i^{(-1/2)}(\cos 2\theta)$, we have

$$S(\varphi) = a(1-n)^2(1+n)\sum_{j=0}^{\infty}\left\{\prod_{k=1}^{j}\left(\frac{-n}{2k} - n\right)\right\}^2 \left[\varphi + \sum_{l=1}^{2j}\frac{\sin 2l\varphi}{l}\prod_{m=1}^{l}\left\{\frac{-n}{2j + 2\cdot(-1)^m \lfloor m/2 \rfloor} - n\right\}^{(-1)^m}\right]. \tag{15}$$

Although formula (15) is a little less effective with respect to convergence at obtaining the meridian arc length itself, it might be better than formula (14) when it comes to calculating the rectifying latitude, because of cancelation of the redundant factor in front of the summation sign during the calculation. That is, the rectifying latitude μ is given by

$$\mu \equiv \frac{\pi S(\varphi)}{2S(\pi/2)} = \varphi + \frac{\displaystyle\sum_{j=0}^{\infty}\left\{\prod_{k=1}^{j}\left(\frac{n}{2k} + n\right)\right\}^2 \sum_{l=1}^{2j}\frac{\sin 2l\varphi}{l}\prod_{m=1}^{l}\left\{\frac{-n}{2j + 2\cdot(-1)^m \lfloor m/2 \rfloor} - n\right\}^{(-1)^m}}{\displaystyle\sum_{j=0}^{\infty}\left\{\prod_{k=1}^{j}\left(\frac{n}{2k} + n\right)\right\}^2} \equiv \varphi + s(\varphi). \tag{16}$$

For practical purposes in the next section, we have defined the function $x+s(x)$ at the end of formula (16).

4. Application to coordinate conversion in the Gauss-Krüger projection

4.1 Current status of coordinate conversion in the Gauss-Krüger projection

Recently, formulae for direct projection to the Gauss-Krüger coordinate system using the third flattening of the earth ellipsoid were implemented by Karney (Karney, 2011). In this paper, almost all formulae for coordinate conversion between geographic and plane rectangular coordinates in the Gauss-Krüger projection are described as series expansions with respect to the third flattening of the earth ellipsoid.

However, for simplicity of evaluation and inversion, Newton's method, which belongs to a class of iteration methods, was adopted instead of series expansions such as those given by Engsager and Poder (Engsager et al., 2007) for the transformation between geographic latitude and conformal latitude. Although the details of the Engsager and Poder implementations are alleged to be available on the Internet (as mentioned in the original paper), they are still difficult (or impossible) to access and follow. On the other hand, it is also true that adopting the simpler iteration method and sacrificing calculation efficiency might lead to inconsistent uniformity of the evaluation because of intermixing of iteration and truncation round-off errors.

4.2 Applying the general formula to the Gauss-Krüger projection

Bearing the above discussion of the current status in mind, we now consider applying the general formula for meridian arc length derived in the previous section to coordinate conversion in the Gauss-Krüger projection. The goal of this paper is an explicit and self-contained presentation of series expansion coefficients to the 10th order of the quantity n using no iteration methods. As we shall see below, all we have to do is to boot wxMaxima, which is a document-based interface for the computer algebra system Maxima (Maxima.sourceforge.net, 2011), and input six command lines.

We start from the following formula describing the relation between geographic latitude φ and conformal latitude χ:

$$\mathrm{gd}^{-1}\chi = \mathrm{gd}^{-1}\varphi - \frac{2\sqrt{n}}{1+n}\tanh^{-1}\left(\frac{2\sqrt{n}}{1+n}\sin\varphi\right). \tag{17}$$

Here, $\mathrm{gd}^{-1}x$ denotes the inverse function of the Gudermannian function $\mathrm{gd}\,x$ defined as

$$\mathrm{gd}\,x = \int_0^x \frac{\mathrm{d}t}{\cosh t}, \quad \mathrm{gd}^{-1}x = \int_0^x \frac{\mathrm{d}t}{\cos t}. \tag{18}$$

Now, we introduce a function g and variables u and v, which temporarily replace the function and variables in formula (17) as

$$\begin{aligned} &u = \mathrm{gd}^{-1}\varphi, \quad v = \mathrm{gd}^{-1}\chi, \\ &g(u) = g\left(\mathrm{gd}^{-1}\varphi\right) = \frac{2\sqrt{n}}{1+n}\tanh^{-1}\left(\frac{2\sqrt{n}}{1+n}\sin\varphi\right). \end{aligned} \tag{19}$$

It follows from formula (19) that we can rewrite formula (17) as $u = v + g(u)$. From this equation, by applying the Lagrange inversion theorem (Lagrange, 1770; Weisstein, 2011) with respect to the Gudermannian function, we can obtain the expression

$$\mathrm{gd}\,u = \mathrm{gd}\,v + \sum_{k=1}^{\infty}\frac{1}{k!}\frac{\partial^{k-1}}{\partial v^{k-1}}\left\{g(v)^k\,\mathrm{gd}'(v)\right\}. \tag{20}$$

Using the well-known characteristic of the Gudermannian function shown in formula (18), it is not hard to see that

$$\mathrm{gd}'(v) = \frac{\mathrm{d}\chi}{\mathrm{d}v} = \frac{1}{\frac{\mathrm{d}v}{\mathrm{d}\chi}} = \frac{1}{\frac{\mathrm{d}\,\mathrm{gd}^{-1}\chi}{\mathrm{d}\chi}} = \cos\chi\,,$$

and

$$\frac{\partial}{\partial v} = \frac{\mathrm{d}\chi}{\mathrm{d}v}\cdot\frac{\partial}{\partial\chi} = \cos\chi\frac{\partial}{\partial\chi}\,.$$

It follows from the above relations and formula (19) that we can rewrite formula (20) with the original variables as

$$\varphi = \chi + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\cos\chi\frac{\partial}{\partial\chi}\right)^{k-1}\left[\left\{\frac{2\sqrt{n}}{1+n}\tanh^{-1}\left(\frac{2\sqrt{n}}{1+n}\sin\chi\right)\right\}^{k}\cos\chi\right]. \tag{21}$$

With the help of powerful commands included in wxMaxima, we can obtain a readjusted trigonometric expansion of formula (21) that has the form

$$\varphi = \chi + \sum_{k=1}^{\infty}\delta_k \sin 2k\chi\,. \tag{22}$$

In addition, it is not hard to obtain the inverse equation in the form

$$\chi = \varphi + \sum_{k=1}^{\infty}\delta'_k \sin 2k\varphi = \varphi - h(\varphi). \tag{23}$$

From the equation $\varphi = \chi + h(\varphi)$ derived in formula (23), by applying the Lagrange inversion theorem with respect to the function $x + s(x)$ defined in formula (16), we have

$$\varphi + s(\varphi) = \chi + s(\chi) + \sum_{k=1}^{\infty}\frac{1}{k!}\frac{\partial^{k-1}}{\partial\chi^{k-1}}\left[h(\chi)^k\{1+s'(\chi)\}\right]. \tag{24}$$

According to formula (16), since the left-hand side of formula (24) corresponds to the rectifying latitude μ, we can confirm that the equations that represent the relation between μ and the conformal latitude χ are expressed in the form

$$\mu = \chi + \sum_{k=1}^{\infty}\alpha_k \sin 2k\chi\,, \tag{25}$$

and, as its inverse equation,

$$\chi = \mu - \sum_{k=1}^{\infty}\beta_k \sin 2k\mu\,. \tag{26}$$

Beginning on the next page, we show a series of screen captures (from Fig. 1 to Fig. 3) describing the calculation of the expansion coefficients δ_k, δ'_k, α_k, and β_k appearing in formulas (22), (23), (25), and (26), respectively, to the 10th order of n. The coefficients α_k and β_k use the same notation as those that appeared in Karney (2011). We note that it is sufficient to truncate the summation with respect to index j in $s(p)$ (corresponds to (%i2) in Fig. 1) at $j = 5$ in order to cover the coefficients to the 10th order.

```
(%i1) g(p):=taylor(2*sqrt(n)/(1+n)*atanh(2*sqrt(n)/(1+n)*sin(p)), n, 0, 10);
```

$$\text{(\%o1)}\quad g(p):=\mathrm{taylor}\left(\frac{2\sqrt{n}}{1+n}\,\mathrm{atanh}\left(\frac{2\sqrt{n}}{1+n}\sin(p)\right),n,0,10\right)$$

```
(%i2) s(p):=sum((product(n/(2*k)+n, k, 1, j))^2*sum(sin(2*l*p)/l*product((-n/(2*(j+(-1)^m*floor(m/2)))-n)^((-1)^m), m, 1, l), l, 1, 2*j), j, 0, 5)
      /sum((product(n/(2*k)+n, k, 1, j))^2, j, 0, 5);
```

$$\text{(\%o2)}\quad s(p):=\frac{\displaystyle\sum_{j=0}^{5}\left(\prod_{k=1}^{j}\left(\frac{n}{2\,k}+n\right)\right)^{2}\sum_{l=1}^{2\,j}\frac{\sin(2\,l\,p)}{l}\prod_{m=1}^{l}\left(\frac{-n}{2\left(j+(-1)^{m}\,\mathrm{floor}\left(\frac{m}{2}\right)\right)}-n\right)^{(-1)^{m}}}{\displaystyle\sum_{j=0}^{5}\left(\prod_{k=1}^{j}\left(\frac{n}{2\,k}+n\right)\right)^{2}}$$

```
(%i3) expand(trigrat(taylor( g(p)*cos(p) + cos(p)*diff(g(p)^2*cos(p), p)/2! + cos(p)*diff(cos(p)*diff(g(p)^3*cos(p), p), p)/3!
        + cos(p)*diff(cos(p)*diff(cos(p)*diff(g(p)^4*cos(p), p), p), p)/4!
        + cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(g(p)^5*cos(p), p), p), p), p)/5!
        + cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(g(p)^6*cos(p), p), p), p), p), p)/6!
        + cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(g(p)^7*cos(p), p), p), p), p), p), p)/7!
        + cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(g(p)^8*cos(p), p), p), p), p), p), p), p)/8!
        + cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(
          cos(p)*diff(g(p)^9*cos(p), p), p), p), p), p), p), p), p), p)/9!
        + cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(cos(p)*diff(
          cos(p)*diff(cos(p)*diff(g(p)^10*cos(p), p), p), p), p), p), p), p), p), p), p)/10!
      , n, 0, 10)));
```

$$\begin{aligned}\text{(\%o3)}\quad &\frac{175201343549\,n^{10}\sin(20\,p)}{297604125}-\frac{24899113566814\,n^{10}\sin(18\,p)}{29462808375}+\frac{106974149462\,n^{9}\sin(18\,p)}{402026625}-\\
&\frac{1740830660174\,n^{10}\sin(16\,p)}{2170943775}-\frac{138163416988\,n^{9}\sin(16\,p)}{402026625}+\frac{1383243703\,n^{8}\sin(16\,p)}{11351340}+\\
&\frac{493459023622\,n^{10}\sin(14\,p)}{310134825}-\frac{1182085822\,n^{9}\sin(14\,p)}{3378375}-\frac{170079376\,n^{8}\sin(14\,p)}{1216215}+\frac{38341552\,n^{7}\sin(14\,p)}{675675}+\\
&\frac{625821359\,n^{10}\sin(12\,p)}{638512875}+\frac{41561762048\,n^{9}\sin(12\,p)}{70945875}-\frac{2155215124\,n^{8}\sin(12\,p)}{14189175}-\frac{115444544\,n^{7}\sin(12\,p)}{2027025}+\\
&\frac{601676\,n^{6}\sin(12\,p)}{22275}-\frac{76998787574\,n^{10}\sin(10\,p)}{91216125}+\frac{67926842\,n^{9}\sin(10\,p)}{2837835}+\frac{258316372\,n^{8}\sin(10\,p)}{1216215}-\\
&\frac{2046082\,n^{7}\sin(10\,p)}{31185}-\frac{144838\,n^{6}\sin(10\,p)}{6237}+\frac{4174\,n^{5}\sin(10\,p)}{315}+\frac{23834033824\,n^{10}\sin(8\,p)}{91216125}-\\
&\frac{2647902052\,n^{9}\sin(8\,p)}{10135125}+\frac{14416399\,n^{8}\sin(8\,p)}{935550}+\frac{11763988\,n^{7}\sin(8\,p)}{155925}-\frac{399572\,n^{6}\sin(8\,p)}{14175}-\\
&\frac{332\,n^{5}\sin(8\,p)}{35}+\frac{4279\,n^{4}\sin(8\,p)}{630}+\frac{1674405706\,n^{10}\sin(6\,p)}{18243225}+\frac{53146406\,n^{9}\sin(6\,p)}{779625}-\frac{2363828\,n^{8}\sin(6\,p)}{31185}\\
&+\frac{98738\,n^{7}\sin(6\,p)}{14175}+\frac{73814\,n^{6}\sin(6\,p)}{2835}-\frac{1262\,n^{5}\sin(6\,p)}{105}-\frac{136\,n^{4}\sin(6\,p)}{35}+\frac{56\,n^{3}\sin(6\,p)}{15}-\\
&\frac{66355687\,n^{10}\sin(4\,p)}{1403325}+\frac{10453448\,n^{9}\sin(4\,p)}{606375}+\frac{141514\,n^{8}\sin(4\,p)}{8505}-\frac{31256\,n^{7}\sin(4\,p)}{1575}+\frac{2323\,n^{6}\sin(4\,p)}{945}+\\
&\frac{2704\,n^{5}\sin(4\,p)}{315}-\frac{227\,n^{4}\sin(4\,p)}{45}-\frac{8\,n^{3}\sin(4\,p)}{5}+\frac{7\,n^{2}\sin(4\,p)}{3}+\frac{22150106\,n^{10}\sin(2\,p)}{4465125}-\\
&\frac{1113026\,n^{9}\sin(2\,p)}{165375}+\frac{189416\,n^{8}\sin(2\,p)}{99225}+\frac{16822\,n^{7}\sin(2\,p)}{4725}-\frac{2854\,n^{6}\sin(2\,p)}{675}+\frac{26\,n^{5}\sin(2\,p)}{45}+\\
&\frac{116\,n^{4}\sin(2\,p)}{45}-2\,n^{3}\sin(2\,p)-\frac{2\,n^{2}\sin(2\,p)}{3}+2\,n\sin(2\,p)\end{aligned}$$

Fig. 1 Screen capture of calculation using wxMaxima (1/3)

(%i4) `expand(trigrat(taylor(sum(diff((-%)^k, p, k-1)/k!, k, 1, 10), n, 0, 10)));`

(%o4)

$$\begin{aligned}
&\frac{47279538091\,n^{10}\sin(20p)}{6249686625}+\frac{26761121546\,n^{10}\sin(18p)}{1733106375}-\frac{779685094\,n^{9}\sin(18p)}{134008875}-\\
&\frac{91990762\,n^{10}\sin(16p)}{638512875}-\frac{38717707988\,n^{9}\sin(16p)}{3618239625}+\frac{256663081\,n^{8}\sin(16p)}{56756700}-\frac{5343686906\,n^{10}\sin(14p)}{221524875}+\\
&\frac{38853428\,n^{9}\sin(14p)}{30405375}+\frac{3463678\,n^{8}\sin(14p)}{467775}-\frac{2405834\,n^{7}\sin(14p)}{675675}-\frac{11976321452\,n^{10}\sin(12p)}{638512875}+\\
&\frac{1138618072\,n^{9}\sin(12p)}{70945875}-\frac{126463\,n^{8}\sin(12p)}{72765}-\frac{941912\,n^{7}\sin(12p)}{184275}+\frac{444337\,n^{6}\sin(12p)}{155925}+\\
&\frac{477299954\,n^{10}\sin(10p)}{91216125}+\frac{90324188\,n^{9}\sin(10p)}{8513505}-\frac{12870194\,n^{8}\sin(10p)}{1216215}+\frac{1040\,n^{7}\sin(10p)}{567}+\\
&\frac{109598\,n^{6}\sin(10p)}{31185}-\frac{734\,n^{5}\sin(10p)}{315}+\frac{8134004876\,n^{10}\sin(8p)}{638512875}-\frac{109153684\,n^{9}\sin(8p)}{30405375}-\\
&\frac{1097407\,n^{8}\sin(8p)}{187110}+\frac{1077964\,n^{7}\sin(8p)}{155925}-\frac{24832\,n^{6}\sin(8p)}{14175}-\frac{12\,n^{5}\sin(8p)}{5}+\frac{1237\,n^{4}\sin(8p)}{630}+\\
&\frac{2893348606\,n^{10}\sin(6p)}{638512875}-\frac{5134016\,n^{9}\sin(6p)}{779625}+\frac{120202\,n^{8}\sin(6p)}{51975}+\frac{44644\,n^{7}\sin(6p)}{14175}-\frac{12686\,n^{6}\sin(6p)}{2835}\\
&+\frac{8\,n^{5}\sin(6p)}{5}+\frac{34\,n^{4}\sin(6p)}{21}-\frac{26\,n^{3}\sin(6p)}{15}-\frac{6423064\,n^{10}\sin(4p)}{7016625}-\frac{35853856\,n^{9}\sin(4p)}{16372125}+\\
&\frac{142607\,n^{8}\sin(4p)}{42525}-\frac{2288\,n^{7}\sin(4p)}{1575}-\frac{1522\,n^{6}\sin(4p)}{945}+\frac{904\,n^{5}\sin(4p)}{315}-\frac{13\,n^{4}\sin(4p)}{9}-\frac{16\,n^{3}\sin(4p)}{15}+\\
&\frac{5\,n^{2}\sin(4p)}{3}-\frac{4266638\,n^{10}\sin(2p)}{4465125}+\frac{263824\,n^{9}\sin(2p)}{1488375}+\frac{1514\,n^{8}\sin(2p)}{1323}-\frac{8384\,n^{7}\sin(2p)}{4725}+\\
&\frac{4642\,n^{6}\sin(2p)}{4725}+\frac{32\,n^{5}\sin(2p)}{45}-\frac{82\,n^{4}\sin(2p)}{45}+\frac{4\,n^{3}\sin(2p)}{3}+\frac{2\,n^{2}\sin(2p)}{3}-2\,n\sin(2p)
\end{aligned}$$

(%i5) `expand(trigrat(taylor(s(p)+sum(diff((-%)^k*(1+diff(s(p), p)), p, k-1)/k!, k, 1, 10), n, 0, 10)));`

(%o5)

$$\begin{aligned}
&\frac{7791151562323282\,n^{10}\sin(20p)}{12014330904576000}-\frac{6719618213835585\,n^{10}\sin(18p)}{3379030566912000}+\\
&\frac{2109164619535\,n^{9}\sin(18p)}{6080126976000}+\frac{2468749292989891\,n^{10}\sin(16p)}{203249958912000}-\frac{256783708069\,n^{9}\sin(16p)}{25204608000}+\\
&\frac{1424729850961\,n^{8}\sin(16p)}{743921418240}+\frac{7180998783745\,n^{10}\sin(14p)}{3629463552000}+\frac{1315149374443\,n^{9}\sin(14p)}{221405184000}-\\
&\frac{16759934899\,n^{8}\sin(14p)}{3113510400}+\frac{1522256789\,n^{7}\sin(14p)}{1383782400}-\frac{1328004581729009\,n^{10}\sin(12p)}{47823519744000}+\\
&\frac{870492877\,n^{9}\sin(12p)}{96096000}+\frac{175214326799\,n^{8}\sin(12p)}{58118860800}-\frac{30705481\,n^{7}\sin(12p)}{10378368}+\frac{212378941\,n^{6}\sin(12p)}{319334400}+\\
&\frac{5820486440369\,n^{10}\sin(10p)}{1307674368000}-\frac{31015475399\,n^{9}\sin(10p)}{2583060480}+\frac{2605413599\,n^{8}\sin(10p)}{622702080}+\\
&\frac{14644087\,n^{7}\sin(10p)}{9123840}-\frac{3418889\,n^{6}\sin(10p)}{1995840}+\frac{34729\,n^{5}\sin(10p)}{80640}+\frac{48087451385201\,n^{10}\sin(8p)}{5230697472000}+\\
&\frac{138471097\,n^{9}\sin(8p)}{66528000}-\frac{40176129013\,n^{8}\sin(8p)}{7664025600}+\frac{97445\,n^{7}\sin(8p)}{49896}+\frac{6601661\,n^{6}\sin(8p)}{7257600}-\\
&\frac{179\,n^{5}\sin(8p)}{168}+\frac{49561\,n^{4}\sin(8p)}{161280}-\frac{6601904925257\,n^{10}\sin(6p)}{1307674368000}+\frac{6304945039\,n^{9}\sin(6p)}{2128896000}+\\
&\frac{79682431\,n^{8}\sin(6p)}{79833600}-\frac{67102379\,n^{7}\sin(6p)}{29030400}+\frac{167603\,n^{6}\sin(6p)}{181440}+\frac{15061\,n^{5}\sin(6p)}{26880}-\frac{103\,n^{4}\sin(6p)}{140}+\\
&\frac{61\,n^{3}\sin(6p)}{240}+\frac{1703267974087\,n^{10}\sin(4p)}{3218890752000}-\frac{705286231\,n^{9}\sin(4p)}{465696000}+\frac{148003883\,n^{8}\sin(4p)}{174182400}+\\
&\frac{13769\,n^{7}\sin(4p)}{28800}-\frac{1983433\,n^{6}\sin(4p)}{1935360}+\frac{281\,n^{5}\sin(4p)}{630}+\frac{557\,n^{4}\sin(4p)}{1440}-\frac{3\,n^{3}\sin(4p)}{5}+\frac{13\,n^{2}\sin(4p)}{48}+\\
&\frac{134592031\,n^{10}\sin(2p)}{1026432000}+\frac{60193001\,n^{9}\sin(2p)}{290304000}-\frac{18975107\,n^{8}\sin(2p)}{50803200}+\frac{72161\,n^{7}\sin(2p)}{387072}+\\
&\frac{7891\,n^{6}\sin(2p)}{37800}-\frac{127\,n^{5}\sin(2p)}{288}+\frac{41\,n^{4}\sin(2p)}{180}+\frac{5\,n^{3}\sin(2p)}{16}-\frac{2\,n^{2}\sin(2p)}{3}+\frac{n\sin(2p)}{2}
\end{aligned}$$

Fig. 2 Screen capture of calculation using wxMaxima (2/3)

(%i6) `expand(trigrat(taylor(-sum(diff((-%)^k, p, k-1)/k!, k, 1, 10), n, 0, 10)));`

(%o6)
$$
\begin{aligned}
&\frac{7028504530429621\, n^{10} \sin(20\, p)}{72085985427456000}-\frac{4922931584446 91\, n^{10} \sin(18\, p)}{6758061133824000}+\\
&\frac{11025641854267\, n^{9} \sin(18\, p)}{158083301376000}-\frac{497155444501631\, n^{10} \sin(16\, p)}{1422749712384000}-\frac{17822319343\, n^{9} \sin(16\, p)}{336825216000}+\\
&\frac{191773887257\, n^{8} \sin(16\, p)}{3719607091200}+\frac{4346429528407\, n^{10} \sin(14\, p)}{12703122432000}-\frac{79431132943\, n^{9} \sin(14\, p)}{332107776000}-\\
&\frac{497323811\, n^{8} \sin(14\, p)}{12454041600}+\frac{219941297\, n^{7} \sin(14\, p)}{5535129600}+\frac{5489489029 8749\, n^{10} \sin(12\, p)}{167382319104000}+\\
&\frac{4543317553\, n^{9} \sin(12\, p)}{18162144000}-\frac{2204645983\, n^{8} \sin(12\, p)}{12915302400}-\frac{16363163\, n^{7} \sin(12\, p)}{518918400}+\frac{20648693\, n^{6} \sin(12\, p)}{638668800}-\\
&\frac{5391039814733\, n^{10} \sin(10\, p)}{10461394944000}+\frac{112731569449\, n^{9} \sin(10\, p)}{557941063680}+\frac{22894433\, n^{8} \sin(10\, p)}{124540416}-\frac{8005831\, n^{7} \sin(10\, p)}{63866880}\\
&-\frac{108847\, n^{6} \sin(10\, p)}{3991680}+\frac{4583\, n^{5} \sin(10\, p)}{161280}-\frac{89112264211\, n^{10} \sin(8\, p)}{5230697472000}-\frac{937932223\, n^{9} \sin(8\, p)}{3891888000}+\\
&\frac{324154477\, n^{8} \sin(8\, p)}{7664025600}+\frac{466511\, n^{7} \sin(8\, p)}{2494800}-\frac{830251\, n^{6} \sin(8\, p)}{7257600}-\frac{11\, n^{5} \sin(8\, p)}{504}+\frac{4397\, n^{4} \sin(8\, p)}{161280}-\\
&\frac{852549456029\, n^{10} \sin(6\, p)}{20922789888000}+\frac{2473691167\, n^{9} \sin(6\, p)}{9289728000}-\frac{6457463\, n^{8} \sin(6\, p)}{17740800}+\frac{9261899\, n^{7} \sin(6\, p)}{58060800}+\\
&\frac{5569\, n^{6} \sin(6\, p)}{90720}-\frac{209\, n^{5} \sin(6\, p)}{4480}-\frac{37\, n^{4} \sin(6\, p)}{840}+\frac{17\, n^{3} \sin(6\, p)}{480}+\frac{5487737251099\, n^{10} \sin(4\, p)}{51502252032000}-\\
&\frac{115295683\, n^{9} \sin(4\, p)}{1397088000}+\frac{24749483\, n^{8} \sin(4\, p)}{348364800}+\frac{51841\, n^{7} \sin(4\, p)}{1209600}-\frac{1118711\, n^{6} \sin(4\, p)}{3870720}+\frac{46\, n^{5} \sin(4\, p)}{105}-\\
&\frac{437\, n^{4} \sin(4\, p)}{1440}+\frac{n^{3} \sin(4\, p)}{15}+\frac{n^{2} \sin(4\, p)}{48}+\frac{25123531261\, n^{10} \sin(2\, p)}{804722688000}-\frac{7378753979\, n^{9} \sin(2\, p)}{97542144000}+\\
&\frac{7944359\, n^{8} \sin(2\, p)}{67737600}-\frac{5406467\, n^{7} \sin(2\, p)}{38707200}+\frac{96199\, n^{6} \sin(2\, p)}{604800}-\frac{81\, n^{5} \sin(2\, p)}{512}-\frac{n^{4} \sin(2\, p)}{360}+\frac{37\, n^{3} \sin(2\, p)}{96}\\
&-\frac{2\, n^{2} \sin(2\, p)}{3}+\frac{n \sin(2\, p)}{2}
\end{aligned}
$$

Created with wxMaxima.

Fig. 3 Screen capture of calculation using wxMaxima (3/3)

It is easy to see from these figures that the coefficients δ_k (corresponds to (%o3) in Fig. 1) and δ'_k (corresponds to (%o4) in Fig. 2) coincide with G_{2k} and e_{2k} appearing in Engsager et al. (2007) to the 7th order, respectively. Likewise, we can also confirm that the coefficients α_k (corresponds to (%o5) in Fig. 2) and β_k (corresponds to (%o6) in Fig. 3) completely coincide with those appearing on the Web site (http://geographiclib.sourceforge.net/html/transversemercator.html#tmseries) presented by Karney.

Although the command (%i3) has redundant expressions (to a large extent due to the author's poor knowledge of wxMaxima) and there is still plenty of room for improvement, it does not take much time (less than 20 seconds on a 2.93 GHz Intel® processor) to carry out all the calculations. This implies that Maxima is a powerful tool even for amateur users such as the author.

5. Concluding remarks

A general formula for the calculation of the meridian arc length has been presented. The derived formula is very concise and suitable for implementation as a computer program, as well, due to its simple expression and easy handling.

The derived formula has also been applied to a core part of the calculation for coordinate conversion in the Gauss-Krüger projection. For the readers' immediate use, a practical example of the calculation using wxMaxima has also been displayed. This confirmed that an advantage of the explicit general formula is its easy implementation on the computer algebra system Maxima.

Acknowledgement

The author would like to thank Dr. Hiroshi Masaharu and Dr. Mikio Tobita of the Geospatial Information Authority of Japan for helpful suggestions and discussion.

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APPENDIX: A simple proof of the relation between formulas (1) and (7)

First, after performing integration by parts and then inserting terms that cancel each other, we have

$$\begin{aligned}\int_0^{\varphi}\frac{\cos 2\theta}{\sqrt{1-e^2\sin^2\theta}}\mathrm{d}\theta &= \left[\frac{\sin 2\theta}{2\sqrt{1-e^2\sin^2\theta}}\right]_0^{\varphi} - \int_0^{\varphi}\frac{e^2\sin^2\theta\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta \\ &= \frac{\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}} + \int_0^{\varphi}\frac{\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta - \int_0^{\varphi}\frac{e^2\sin^2\theta\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta - \int_0^{\varphi}\frac{\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta \\ &= \frac{\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}} + \int_0^{\varphi}\frac{\cos^2\theta}{\sqrt{1-e^2\sin^2\theta}}\mathrm{d}\theta - \int_0^{\varphi}\frac{\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta.\end{aligned}$$

Bearing the relation $\cos 2\theta = \cos^2\theta - \sin^2\theta$ in mind, we obtain

$$\begin{aligned}\int_0^{\varphi}\frac{\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta &= \int_0^{\varphi}\frac{\sin^2\theta}{\sqrt{1-e^2\sin^2\theta}}\mathrm{d}\theta + \frac{\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}} \\ &= \frac{1}{e^2}\int_0^{\varphi}\frac{1-\left(1-e^2\sin^2\theta\right)}{\sqrt{1-e^2\sin^2\theta}}\mathrm{d}\theta + \frac{\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}} \\ &= \frac{1}{e^2}\left(\int_0^{\varphi}\frac{\mathrm{d}\theta}{\sqrt{1-e^2\sin^2\theta}} - \int_0^{\varphi}\sqrt{1-e^2\sin^2\theta}\,\mathrm{d}\theta\right) + \frac{\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}} \\ &= \frac{F(\varphi,e)-E(\varphi,e)}{e^2} + \frac{\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}},\end{aligned}$$

where $F(\varphi,e)$ and $E(\varphi,e)$ denote the first and the second kind of incomplete elliptic integral, respectively. On the other hand, it follows from the definition of $F(\varphi,e)$ that

$$\begin{aligned}F(\varphi,e) &\equiv \int_0^{\varphi}\frac{\mathrm{d}\theta}{\sqrt{1-e^2\sin^2\theta}} \\ &= \int_0^{\varphi}\frac{\mathrm{d}\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}} - e^2\int_0^{\varphi}\frac{\sin^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta \\ &= \int_0^{\varphi}\frac{\mathrm{d}\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}} - e^2\int_0^{\varphi}\frac{1-\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta \\ &= \left(1-e^2\right)\int_0^{\varphi}\frac{\mathrm{d}\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}} + e^2\int_0^{\varphi}\frac{\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta.\end{aligned}$$

Rearranging the above results, we arrive at the final conclusion as

$$\begin{aligned}\int_0^{\varphi}\frac{1-e^2}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta &= F(\varphi,e)-e^2\int_0^{\varphi}\frac{\cos^2\theta}{\left(1-e^2\sin^2\theta\right)^{3/2}}\mathrm{d}\theta\\ &= E(\varphi,e)-\frac{e^2\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}}\\ &= \int_0^{\varphi}\sqrt{1-e^2\sin^2\theta}\,\mathrm{d}\theta-\frac{e^2\sin 2\varphi}{2\sqrt{1-e^2\sin^2\varphi}}. \qquad \blacksquare\end{aligned}$$