

Research Article

Supercritical Neimark–Sacker Bifurcation and Hybrid Control in a Discrete-Time Glycolytic Oscillator Model

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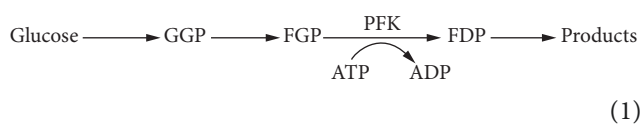
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We study the local dynamical properties, Neimark–Sacker bifurcation, and hybrid control in a glycolytic oscillator model in the interior of \mathbb{R}_+^2 . It is proved that, for all parametric values, $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is the unique positive equilibrium point of the glycolytic oscillator model. Further local dynamical properties along with different topological classifications about the equilibrium $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ have been investigated by employing the method of linearization. Existence of prime period and periodic points of the model under consideration are also investigated. It is proved that, about the fixed point $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$, the discrete-time glycolytic oscillator model undergoes no bifurcation, except Neimark–Sacker bifurcation. A further hybrid control strategy is applied to control Neimark–Sacker bifurcation in the discrete-time model. Finally, theoretical results are verified numerically.

1. Introduction

In glycolysis, glucose decomposes in the presence of various enzymes including ten steps in which five are termed the preparatory phase or phosphorylation, while the remaining steps are called the pay-off phase. Phosphofructokinase is one of the enzymes which is responsible for the occurrence of glycolytic oscillation [1–4]. This step is considered the control unit of glycolysis due to the presence of enzyme phosphofructokinase. A biochemical reaction that occurs in metabolic systems has the following sequence of steps [5]:



where the quantities involved in the above biochemical reaction are depicted in Table 1.

Generally, it is assumed that the enzyme PFK consists of two steps as given below. Among these, in one step,

adenosine diphosphate simulates the allosteric regulatory enzyme and yields a more active form. Therefore, the product formed during the reaction mediated by phosphofructokinase increases the rate of reaction. The systematic description of the kinetics is as follows:



and we have

$$\begin{aligned} \frac{dx}{dt} &= \alpha - \beta x - xy^2, \\ \frac{dy}{dt} &= \beta x + xy^2 - y, \end{aligned} \quad (3)$$

where x_n and y_n denote FGP and ADP, respectively, and α and β are positive constants. Using the nonstandard finite difference scheme, the discrete analogue of the glycolytic oscillator model, which is depicted in (3), is

TABLE 1: Quantities along their chemical interpretation.

Quantities	Corresponding chemical interpretation
GGP	Glucose-6-phosphate
FGP	Fructose-6-phosphate
FDP	Fructose-1,6-diphosphate
PFK	Phosphofructokinase
ATP	Adenosine triphosphate
ADP	Adenosine diphosphate

$$\begin{aligned} x_{n+1} &= \frac{\alpha h + x_n}{1 + h\beta + h y_n^2}, \\ y_{n+1} &= \frac{y_n + h\beta x_n + h x_n y_n^2}{1 + h}. \end{aligned} \quad (4)$$

Here, our purpose is to explore the local dynamics, N-S bifurcation, and hybrid control in a glycolytic oscillator model (4) in the interior of \mathbb{R}_+^2 . Specifically, our key findings are as follows:

- (1) About the unique +ve fixed point $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$, we explored the dynamical properties along with different topological classifications.
- (2) We explored the existence of bifurcation about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$.
- (3) Prime period and periodic points of the glycolytic oscillator model (4) are investigated.
- (4) To prove, the glycolytic oscillator model (4) undergoes an N-S bifurcation when parameters vary in a small neighborhood of $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$, and no other bifurcation occurs except the N-S bifurcation.
- (5) The N-S bifurcation is controlled by the hybrid control strategy.
- (6) Theoretical results are verified numerically.

The flow pattern of the remaining of this paper is as follows: Section 2 is about the existence of the unique +ve equilibrium point and corresponding linearized form of the glycolytic oscillator model (4). In Section 3, we study the local dynamical properties with topological classification about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the glycolytic oscillator model (4), whereas existence of prime period and periodic points of model (4) is explored in Section 4. The comprehensive N-S bifurcation analysis about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is given in Section 5. In Section 6, simulations are presented to verify theoretical results. In Section 7, a hybrid control strategy is applied to control N-S bifurcation. Concluding remarks are given in Section 8.

2. Existence of the Unique +ve Equilibrium Point and Corresponding Linearized Form of the Glycolytic Oscillator Model (4)

The existence of the +ve fixed point in the interior of \mathbb{R}_+^2 and corresponding linearized form of the glycolytic oscillator model (4) are explored in this section. More specifically, the

existence result about the equilibrium point can be summarized as follows.

Lemma 1. \forall parametric values α and β , the discrete-time glycolytic oscillator model (4) has a unique +ve equilibrium point $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$.

Now, the linearized form of the glycolytic oscillator model (4) about $P_{xy}(x, y)$ under the map $(\Psi_1, \Psi_2) \rightarrow (x_{n+1}, y_{n+1})$ becomes

$$\Lambda_{n+1} = J|_{P_{xy}(x,y)} \Lambda_n, \quad (5)$$

where

$$J|_{P_{xy}(x,y)} = \begin{pmatrix} \frac{1}{1 + h(\beta + y^2)} & \frac{-2yh(x + \alpha h)}{(1 + h(\beta + y^2))^2} \\ \frac{h(\beta + y^2)}{1 + h} & \frac{1 + 2hxy}{1 + h} \end{pmatrix}, \quad (6)$$

$$\Psi_1 = \frac{\alpha h + x_n}{1 + h\beta + h y_n^2},$$

$$\Psi_2 = \frac{y_n + h\beta x_n + h x_n y_n^2}{1 + h}.$$

3. Local Dynamical Properties of Model

(4) about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$

The local dynamics about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the glycolytic oscillator model (4) is explored in this section. $J|_{P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)}$ about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is

$$J|_{P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)} = \begin{pmatrix} \frac{1}{1 + h(\beta + \alpha^2)} & \frac{2\alpha^2 h}{(\alpha^2 + \beta)(1 + h(\beta + \alpha^2))} \\ \frac{h(\beta + \alpha^2)}{1 + h} & \frac{\alpha^2 + \beta + 2h\alpha^2}{(\beta + \alpha^2)(1 + h)} \end{pmatrix}. \quad (7)$$

The auxiliary equation of $J|_{P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)}$ about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is

$$\lambda^2 - p\left(\frac{\alpha}{\beta + \alpha^2}, \alpha\right)\lambda + q\left(\frac{\alpha}{\beta + \alpha^2}, \alpha\right) = 0, \quad (8)$$

where

$$\begin{aligned} p\left(\frac{\alpha}{\beta + \alpha^2}, \alpha\right) &= \frac{(1 + h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1 + h(\alpha^2 + \beta))}{(1 + h(\alpha^2 + \beta))(1 + h)(\beta + \alpha^2)}, \\ q\left(\frac{\alpha}{\beta + \alpha^2}, \alpha\right) &= \frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1 + h)(\alpha^2 + \beta)(1 + h(\beta + \alpha^2))}. \end{aligned} \quad (9)$$

And eigenvalues of $J|_{P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)}$ about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ are

$$\lambda_{1,2} = \frac{p(\alpha/(\beta + \alpha^2), \alpha) \pm \sqrt{\Delta}}{2}, \quad (10)$$

where

$$\begin{aligned} \Delta &= \left(p\left(\frac{\alpha}{\beta + \alpha^2}, \alpha\right) \right)^2 - 4q\left(\frac{\alpha}{\beta + \alpha^2}, \alpha\right) \\ &= \left(\frac{(1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta))}{(1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)} \right)^2 \\ &\quad - 4\left(\frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} \right). \end{aligned} \tag{11}$$

Hereafter in the following, we will explore the necessary and sufficient condition under which $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the glycolytic oscillator model (4) is a sink, a repeller, a saddle, and nonhyperbolic.

Lemma 2. For $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the glycolytic oscillator model (4), the following holds:

(i) $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is a sink iff

$$\begin{aligned} &\left| \frac{(1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta))}{(1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)} \right| \\ &< 1 - \frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} < 2. \end{aligned} \tag{12}$$

(ii) $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is a repeller iff

$$\begin{aligned} &\left| \frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} \right| > 1, \\ &\left| \frac{(1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta))}{(1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)} \right| \\ &< \left| 1 - \frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} \right|. \end{aligned} \tag{13}$$

(iii) $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is a saddle iff

$$\begin{aligned} &\left(\frac{(1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta))}{(1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)} \right)^2 \\ &+ 4\left(1 - \frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} \right) > 0, \\ &\left| \frac{(1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta))}{(1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)} \right| \\ &> \left| 1 - \frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} \right|. \end{aligned} \tag{14}$$

(iv) $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is nonhyperbolic iff

$$\begin{aligned} &\left| \frac{(1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta))}{(1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)} \right| \\ &= \left| 1 - \frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} \right|. \end{aligned} \tag{15}$$

Proof. Follows from Theorem 1.1.1 of [6].

Hereafter in the next lemma, we will explore local dynamics along with topological classification about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ if $\Delta = (((1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta)))/((1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)))^2 - 4((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta))/((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) < 0$, of the glycolytic oscillator model (4). \square

Lemma 3. If $\Delta = (((1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta)))/((1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)))^2 - 4((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta))/((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) < 0$, then for $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$, the following holds:

(i) $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is a locally asymptotically stable focus if

$$\frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} < 1. \tag{16}$$

(ii) $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is an unstable focus if

$$\frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} > 1. \tag{17}$$

(iii) $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is nonhyperbolic if

$$\frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} = 1. \tag{18}$$

Hereafter, we will establish that $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of (4) is the periodic point of prime period -1 . Moreover, equilibrium $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ is the periodic point having period $-2, 3, \dots, n$.

4. Existence of Prime Period and Periodic Points of (4)

Theorem 1. $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the model (4) is a periodic point of prime period -1 .

Proof. From (4), we define

$$F(x, y) = (f(x, y), g(x, y)), \tag{19}$$

where

$$f(x, y) = \frac{\alpha h + x}{1 + h\beta + hy^2}, \tag{20}$$

$$g(x, y) = \frac{y + h\beta x + hxy^2}{1 + h}.$$

After manipulation, from (19), one gets

$$F_{P_{xy}^+(\alpha/(\beta+\alpha^2),\alpha)} = P_{xy}^+\left(\frac{\alpha}{\beta+\alpha^2}, \alpha\right). \quad (21)$$

From (21), one can conclude that $P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)$ of (4) is a periodic point of prime period -1 . \square

Theorem 2. $P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)$ of the glycolytic oscillator model (4) is a periodic point of period $-2, 3, \dots, n$.

Proof. From (19), one has

$$\begin{aligned} F^2(x, y) &= \left(\frac{\alpha h + f(x, y)}{1 + h\beta + h(g(x, y))^2}, \frac{g(x, y) + h\beta f(x, y)}{1 + h} + \frac{h(f(x, y)g(x, y)^2)}{1 + h} \right) \\ &\implies F_{P_{xy}^+(\alpha/(\beta+\alpha^2),\alpha)}^2 = P_{xy}^+\left(\frac{\alpha}{\beta+\alpha^2}, \alpha\right), \\ F^3(x, y) &= \left(\frac{\alpha h + f^2(x, y)}{1 + h\beta + h(g^2(x, y))^2}, \frac{g^2(x, y)^2 + h\beta f^2(x, y)}{1 + h} + \frac{h(f^2(x, y)(g^2(x, y))^2)}{1 + h} \right) \\ &\implies F_{P_{xy}^+(\alpha/(\beta+\alpha^2),\alpha)}^3 = P_{xy}^+\left(\frac{\alpha}{\beta+\alpha^2}, \alpha\right), \\ &\vdots \\ F^i(x, y) &= \left(\frac{\alpha h + f^{i-1}(x, y)}{1 + h\beta + h(g^{i-1}(x, y))^2}, \frac{g^{i-1}(x, y) + h\beta f^{i-1}(x, y)}{1 + h} + \frac{h(f^{i-1}(x, y)g^{i-1}(x, y)^2)}{1 + h} \right) \\ &\implies F_{P_{xy}^+(\alpha/(\beta+\alpha^2),\alpha)}^i = P_{xy}^+\left(\frac{\alpha}{\beta+\alpha^2}, \alpha\right). \end{aligned} \quad (22)$$

From (22), $P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)$ of (4) is a periodic point of period $-2, 3, \dots, n$. \square

5. Bifurcation Analysis about

$P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)$ of the Glycolytic Oscillator Model (4)

Recall that if (18) holds, then $|\lambda_{1,2}|_{(18)} = 1$, which implies that the glycolytic oscillator model (4) undergoes N-S bifurcation if parameters h, α , and β go through the curve:

$$\text{N-SB}_{P_{xy}^+(\alpha/(\beta+\alpha^2),\alpha)} = \left\{ (h, \alpha, \beta) : \frac{\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)}{(1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))} = 1, h, \alpha, \beta > 0 \right\}. \quad (23)$$

Hereafter, we will give detailed N-S bifurcation about $P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)$ when parameters h, α , and β go through the curve, which is depicted in (23), by using bifurcation theory [7, 8]. For given parameters h, α , and β , let

$$\begin{aligned} F(h, \alpha, \beta) &= \alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta) \\ &\quad - (1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)). \end{aligned} \quad (24)$$

After simplifying (24), one gets

$$F(h, \alpha, \beta) = \alpha^2 + h\alpha^4 - \beta^2 - 2\alpha^2\beta - \beta - \beta^2h - \alpha^4. \quad (25)$$

From (25), one gets $\partial F(h, \alpha, \beta)/\partial\beta < 0$. So, by implicit function theorem, one can obtain $\beta = \beta(h, \alpha)$ such that $F(h, \alpha, \beta(h, \alpha)) = 0$ and hence choose β as a bifurcation parameter. Now, consider β in a small neighborhood of β^* , i.e., $\beta = \beta^* + \varepsilon$, where $\varepsilon \ll 1$ and hence the glycolytic oscillator model (4) becomes

$$\begin{aligned} x_{n+1} &= \frac{\alpha h + x_n}{1 + h(\beta^* + \varepsilon) + h y_n^2}, \\ y_{n+1} &= \frac{y_n + h(\beta^* + \varepsilon)x_n + h x_n y_n^2}{1 + h}, \end{aligned} \quad (26)$$

where $P_{xy}^+(\alpha/(\beta^* + \varepsilon + \alpha^2), \alpha)$ be its unique positive fixed point. Moreover, $J|_{P_{xy}^+(\alpha/(\beta^* + \varepsilon + \alpha^2), \alpha)}$ about the fixed point $P_{xy}^+(\alpha/(\beta^* + \varepsilon + \alpha^2), \alpha)$ of (26) is

$$J|_{P_{xy}^+(\alpha/(\beta^* + \varepsilon + \alpha^2), \alpha)} = \begin{pmatrix} \frac{1}{1+h(\beta^* + \varepsilon + \alpha^2)} - \frac{2\alpha^2 h}{(\alpha^2 + \beta^* + \varepsilon)(1+h(\beta^* + \varepsilon + \alpha^2))} & \\ \frac{h(\beta^* + \varepsilon + \alpha^2)}{1+h} & \frac{\alpha^2 + \beta^* + \varepsilon + 2h\alpha^2}{(\beta^* + \varepsilon + \alpha^2)(1+h)} \end{pmatrix}. \quad (27)$$

The auxiliary equation of $J|_{P_{xy}^+(\alpha/(\beta^* + \varepsilon + \alpha^2), \alpha)}$ about $P_{xy}^+(\alpha/(\beta^* + \varepsilon + \alpha^2), \alpha)$ is

$$\lambda^2 - p(\varepsilon)\lambda + q(\varepsilon) = 0, \quad (28)$$

where

$$p\left(\frac{\alpha}{\beta^* + \varepsilon + \alpha^2}, \alpha\right) = \frac{(1+h)(\beta^* + \varepsilon + \alpha^2) + (\alpha^2 + \beta^* + \varepsilon + 2h\alpha^2)(1+h(\alpha^2 + \beta^* + \varepsilon))}{(1+h(\alpha^2 + \beta^* + \varepsilon))(1+h)(\beta^* + \varepsilon + \alpha^2)}, \quad (29)$$

$$q\left(\frac{\alpha}{\beta^* + \varepsilon + \alpha^2}, \alpha\right) = \frac{\alpha^2 + \beta^* + \varepsilon + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta^* + \varepsilon)}{(1+h)(\alpha^2 + \beta^* + \varepsilon)(1+h(\beta^* + \varepsilon + \alpha^2))}.$$

The zeroes of (28) are

$$\lambda_{1,2} = \frac{p(\varepsilon) \pm \iota \sqrt{4q(\varepsilon) - p^2(\varepsilon)}}{2}$$

$$= \frac{((1+h)(\beta^* + \varepsilon + \alpha^2) + (\alpha^2 + \beta^* + \varepsilon + 2h\alpha^2)(1+h(\alpha^2 + \beta^* + \varepsilon)))/((1+h(\alpha^2 + \beta^* + \varepsilon))(1+h)(\beta^* + \varepsilon + \alpha^2))}{2} \quad (30)$$

$$\pm \frac{\iota}{2} \sqrt{\Delta^*},$$

where

$$\Delta^* = 4\left(\frac{\alpha^2 + \beta^* + \varepsilon + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta^* + \varepsilon)}{(1+h)(\alpha^2 + \beta^* + \varepsilon)(1+h(\beta^* + \varepsilon + \alpha^2))}\right) - \left(\frac{(1+h)(\beta^* + \varepsilon + \alpha^2) + (\alpha^2 + \beta^* + \varepsilon + 2h\alpha^2)(1+h(\alpha^2 + \beta^* + \varepsilon))}{(1+h(\alpha^2 + \beta^* + \varepsilon))(1+h)(\beta^* + \varepsilon + \alpha^2)}\right)^2, \quad (31)$$

$$|\lambda_{1,2}| = (q(\varepsilon))^{1/2},$$

$$\left. \frac{d|\lambda_{1,2}|}{d\varepsilon} \right|_{\varepsilon=0} \neq 0. \quad (32)$$

Additionally, it is required that $\lambda_{1,2}^m \neq 1, m = 1, \dots, 4$, which is equivalent to $p(0) \neq -2, 0, 1, 2$, and hence by manipulation, it is true. If $u_n = x_n - x^*, v_n = y_n - y^*$, then $P_{xy}^+(\alpha/(\beta^* + \varepsilon + \alpha^2), \alpha)$ of the glycolytic oscillator model (26) transforms into $P_{00}(0, 0)$. So

$$u_{n+1} = \frac{\alpha h + u_n + x^*}{1+h(\beta^* + \varepsilon) + h(v_n + y^*)^2} - x^*, \quad (33)$$

$$v_{n+1} = \frac{v_n + y^* + h(u_n + x^*)(\beta^* + \varepsilon + (v_n + y^*)^2)}{1+h} - y^*,$$

where $x^* = \alpha/(\beta + \alpha^2)$ and $y^* = \alpha$. Hereafter, the normal form of (33) is studied if $\varepsilon = 0$. By Taylor series expansion about $(u_n, v_n) = (0, 0)$, from (33), one gets

$$\begin{aligned} u_{n+1} &= a_{11}u_n + a_{12}v_n + a_{13}u_nv_n + a_{14}v_n^2 + o\left(\left(|u_n| + |v_n|\right)^2\right), \\ v_{n+1} &= a_{21}u_n + a_{22}v_n + a_{23}u_nv_n + a_{24}v_n^2 + o\left(\left(|u_n| + |v_n|\right)^2\right), \end{aligned} \quad (34)$$

where

$$\begin{aligned} a_{11} &= \frac{1}{1 + h(\beta^* + y^{*2})}, \\ a_{12} &= \frac{-2hy^*(x^* + h\alpha)}{1 + h(\beta^* + y^{*2})}, \\ a_{13} &= \frac{-2hy^*}{(1 + h(\beta^* + y^{*2}))^2}, \\ a_{14} &= \frac{h(x^* + h\alpha)(-(1 + h\beta^* + hy^{*2}) + 4y^{*2}h)}{(1 + h(\beta^* + y^{*2}))^3}, \\ a_{21} &= \frac{h(\beta^* + y^{*2})}{1 + h}, \\ a_{22} &= \frac{1 + 2hx^*y^*}{1 + h}, \\ a_{23} &= \frac{2hy^*}{1 + h}, \\ a_{24} &= \frac{hx^*}{1 + h}. \end{aligned} \quad (35)$$

Hereafter, one contracts the following invertible matrix T that puts the linear part of (34) into the canonical form:

$$T := \begin{pmatrix} a_{12} & 0 \\ \eta - a_{11} & -\zeta \end{pmatrix}, \quad (36)$$

where

$$\begin{aligned} \eta &= \frac{(1 + h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1 + h(\alpha^2 + \beta))}{2(1 + h(\alpha^2 + \beta))(1 + h)(\beta + \alpha^2)}, \\ \zeta &= \frac{1}{2}\sqrt{\Delta^*}, \end{aligned} \quad (37)$$

and Δ^* is depicted in (31). Hence, (34) then implies

$$\begin{aligned} X_{n+1} &= \eta X_n - \zeta Y_n + \bar{P}, \\ Y_{n+1} &= \zeta X_n + \eta Y_n + \bar{Q}, \end{aligned} \quad (38)$$

where

$$\begin{aligned} \bar{P}(X_n, Y_n) &= l_{11}X_n^2 + l_{12}X_nY_n + l_{13}Y_n^2 + o\left(\left(|X_n| + |Y_n|\right)^2\right), \\ \bar{Q}(X_n, Y_n) &= l_{21}X_n^2 + l_{22}X_nY_n + l_{23}Y_n^2 + o\left(\left(|X_n| + |Y_n|\right)^2\right), \end{aligned} \quad (39)$$

in which

$$\begin{aligned} l_{11} &= \frac{\eta - a_{11}}{a_{12}} [a_{12}a_{13} + a_{14}(\eta - a_{11})], \\ l_{12} &= -\frac{\zeta}{a_{12}} [a_{12}a_{13} + 2a_{14}(\eta - a_{11})], \\ l_{13} &= \frac{a_{14}}{a_{12}} \zeta^2, \\ l_{21} &= \frac{\eta - a_{11}}{\zeta a_{12}} [(\eta - a_{11})(a_{12}(a_{13} - a_{24}) - a_{14}a_{24}) - a_{12}^2 a_{23}], \\ l_{22} &= -\frac{\eta - a_{11}}{a_{12}} [2a_{14}(\eta - a_{11}) + a_{12}(a_{13} + 2a_{24}) + a_{23}a_{12}^2], \\ l_{23} &= \frac{\zeta}{a_{12}} [a_{14}\zeta(\eta - a_{11}) - a_{12}a_{24}], \end{aligned} \quad (40)$$

by

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} := \begin{pmatrix} a_{12} & 0 \\ \eta - a_{11} & -\zeta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}. \quad (41)$$

From (39), one gets

$$\begin{aligned} \bar{P}_{X_n X_n} \Big|_{P_{00}(0,0)} &= 2l_{11}, \\ \bar{P}_{X_n Y_n} \Big|_{P_{00}(0,0)} &= l_{12}, \\ \bar{P}_{Y_n Y_n} \Big|_{P_{00}(0,0)} &= 2l_{13}, \\ \bar{Q}_{X_n X_n} \Big|_{P_{00}(0,0)} &= 2l_{21}, \\ \bar{Q}_{X_n Y_n} \Big|_{P_{00}(0,0)} &= l_{22}, \\ \bar{Q}_{Y_n Y_n} \Big|_{P_{00}^+(0,0)} &= 2l_{23}. \end{aligned} \quad (42)$$

In order to ensure the N-S bifurcation for (38), it is required that the following discriminatory quantity should be nonzero ([7–14]):

$$\Omega = -\operatorname{Re} \left[\frac{(1 - 2\bar{\lambda})\bar{\lambda}^2}{1 - \lambda} \xi_{11} \xi_{20} \right] - \frac{1}{2} \|\xi_{11}\|^2 - \|\xi_{02}\|^2 + \operatorname{Re}(\bar{\lambda} \xi_{21}), \quad (43)$$

where

$$\begin{aligned} \xi_{02} &= \frac{1}{8} \left[\bar{P}_{X_n X_n} - \bar{P}_{Y_n Y_n} + 2\bar{Q}_{X_n Y_n} + \iota(\bar{Q}_{X_n X_n} - \bar{Q}_{Y_n Y_n} + 2\bar{P}_{X_n Y_n}) \right] \Big|_{P_{00}(0,0)}, \\ \xi_{11} &= \frac{1}{4} \left[\bar{P}_{X_n X_n} + \bar{P}_{Y_n Y_n} + \iota(\bar{Q}_{X_n X_n} + \bar{Q}_{Y_n Y_n}) \right] \Big|_{P_{00}(0,0)}, \\ \xi_{20} &= \frac{1}{8} \left[\bar{P}_{X_n X_n} - \bar{P}_{Y_n Y_n} + 2\bar{Q}_{X_n Y_n} + \iota(\bar{Q}_{X_n X_n} - \bar{Q}_{Y_n Y_n} - 2\bar{P}_{X_n Y_n}) \right] \Big|_{P_{00}(0,0)}, \\ \xi_{21} &= \frac{1}{16} \left[\bar{P}_{X_n X_n X_n} + \bar{P}_{X_n Y_n Y_n} + \bar{Q}_{X_n X_n Y_n} + \bar{Q}_{Y_n Y_n X_n} + \iota(\bar{Q}_{X_n X_n X_n} + \bar{Q}_{X_n Y_n Y_n} - \bar{P}_{X_n X_n Y_n} - \bar{P}_{Y_n Y_n X_n}) \right] \Big|_{P_{00}(0,0)}. \end{aligned} \tag{44}$$

After manipulation, one gets

$$\begin{aligned} \xi_{02} &= \frac{1}{8} [2l_{11} - l_{12} + 2l_{22} + 2\iota(l_{21} + l_{12} - l_{23})], \\ \xi_{11} &= \frac{1}{2} [l_{11} + l_{13} + \iota(l_{21} + l_{23})], \\ \xi_{20} &= \frac{1}{4} [l_{11} - l_{13} + l_{22} + \iota(l_{21} - l_{23} - l_{12})], \\ \xi_{21} &= 0. \end{aligned} \tag{45}$$

From the analysis, one can conclude the following.

Theorem 3. *If $\Omega \neq 0$, then the glycolytic oscillator model (4) undergoes N-S bifurcation about the fixed point: $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ as parameters h, α , and β go through the curve, which is depicted in (23). Additionally, attracting (resp., repelling) closed curve bifurcates appear from $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ if $\Omega < 0$ (resp., $\Omega > 0$).*

According to bifurcation theory, the bifurcation is supercritical (resp., subcritical) N-S bifurcation if $\Omega < 0$ (resp., $\Omega > 0$). In the subsequent section, simulations guarantee that the glycolytic oscillator model (4) undergoes the supercritical N-S bifurcation if h, α , and β go through (23).

Remark. The discrete-time glycolytic oscillator model (4) cannot undergo any bifurcation, except N-S bifurcation, about the positive fixed point $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$.

6. Numerical Simulations

Theoretical results are verified numerically in this section. For instance, if $\alpha = 0.6, h = 0.7$, then from (18), one gets $\beta = 0.16105922820694768$. Theoretically, equilibrium $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the glycolytic oscillator model (4) is a locally asymptotically stable focus if $\beta > 0.16105922820694768$, i.e., condition (16) holds. To see this, if $\beta = 0.4 >$

0.16105922820694768 , then the unique positive equilibrium point $P_{xy}^+(0.7894736842105263, 0.6)$ of (4) is a locally asymptotically stable focus (see Figure 1(a)). Similarly for other values of β , if $\beta > 0.16105922820694768$, then $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of (4) is a locally asymptotically stable focus (see Figures 1(b)–1(l)). But if $\beta < 0.16105922820694768$, then positive equilibrium $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ becomes an unstable focus, i.e., condition (17) holds. Precisely, if $\beta < 0.16105922820694768$, then positive equilibrium $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ loses its stability and meanwhile an attracting closed curve appears. The appearance of this closed curve indicates that the glycolytic oscillator model (4) undergoes a supercritical N-S bifurcation if α, β , and h vary in a neighborhood of $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$. To see this, if $\beta = 0.16 < 0.16105922820694768$, then eigenvalues of $J|_{P_{xy}^+\{(1.1538461538461537, 0.6)\}}$ about $(1.1538461538461537, 0.6)$ are

$$\lambda_{1,2} = 0.615799684186781 \pm 0.3270196350217114\iota. \tag{46}$$

After some manipulation, from (43), one gets

$$\begin{aligned} \xi_{02} &= -0.007611881686547611 + 0.024203687145576566\iota, \\ \xi_{11} &= 0.03740896019401933 - 0.1966375637334364\iota, \\ \xi_{20} &= 0.009013748156314753 - 0.0593513169940678\iota, \\ \xi_{21} &= 0. \end{aligned} \tag{47}$$

In view of (46) and (47), from (43), one gets $\Omega = -0.029495873864781315 < 0$. Hence, if $\beta = 0.16 < 0.16105922820694768$, then model (4) undergoes a supercritical N-S bifurcation and so a stable curve appears, which is depicted in Figure 2(a). Also for different choices of the bifurcation parameter, if $\beta < 0.16105922820694768$, then one gets $\Omega < 0$ (see Table 2), and corresponding closed curves are depicted in Figures 2(b)–2(l). Moreover, the bifurcation diagrams along with the maximum Lyapunov exponent are plotted and drawn in Figure 3. Finally, 3D bifurcation diagrams are plotted and drawn in Figure 4.

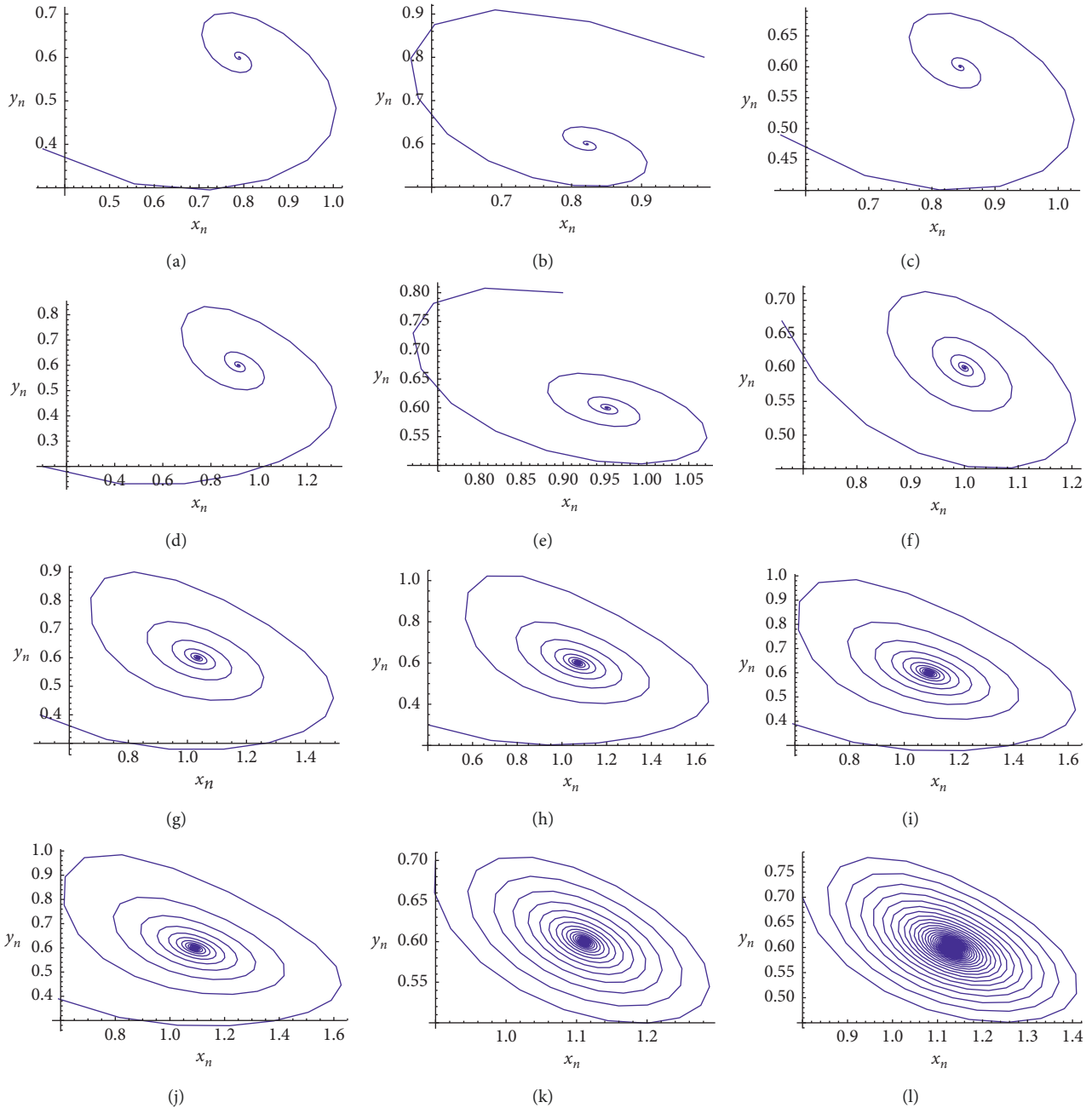


FIGURE 1: Trajectories of the discrete-time glycolytic oscillator model (4) about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$: (a) $\beta = 0.4$ with $(0.35, 0.39)$; (b) $\beta = 0.37$ with $(0.99, 0.80)$; (c) $\beta = 0.35$ with $(0.56, 0.49)$; (d) $\beta = 0.2965$ with $(0.1, 0.2)$; (e) $\beta = 0.27$ with $(0.9, 0.8)$; (f) $\beta = 0.24$ with $(0.66, 0.67)$; (g) $\beta = 0.22$ with $(0.5, 0.4)$; (h) $\beta = 0.2$ with $(0.4, 0.3)$; (i) $\beta = 0.19$ with $(0.59, 0.39)$; (j) $\beta = 0.199$ with $(0.9, 0.7)$; (k) $\beta = 0.18$ with $(0.9, 0.7)$; (l) $\beta = 0.17$ with $(0.8, 0.7)$.

7. Control of N-S Bifurcation about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the Glycolytic Oscillator Model (4)

The hybrid control strategy is applied to control N-S bifurcation about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the glycolytic oscillator model (4) motivated from the work of [15–18]. The control system corresponding to the glycolytic oscillator model (4) is

$$x_{n+1} = c \left(\frac{\alpha h + x_n}{1 + h\beta + h y_n^2} \right) + (1 - c)x_n, \quad (48)$$

$$y_{n+1} = c \left(\frac{y_n + h\beta x_n + h x_n y_n^2}{1 + h} \right) + (1 - c)y_n.$$

$J|_{P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)}$ about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$ of the controlled system, which is depicted in (48), is

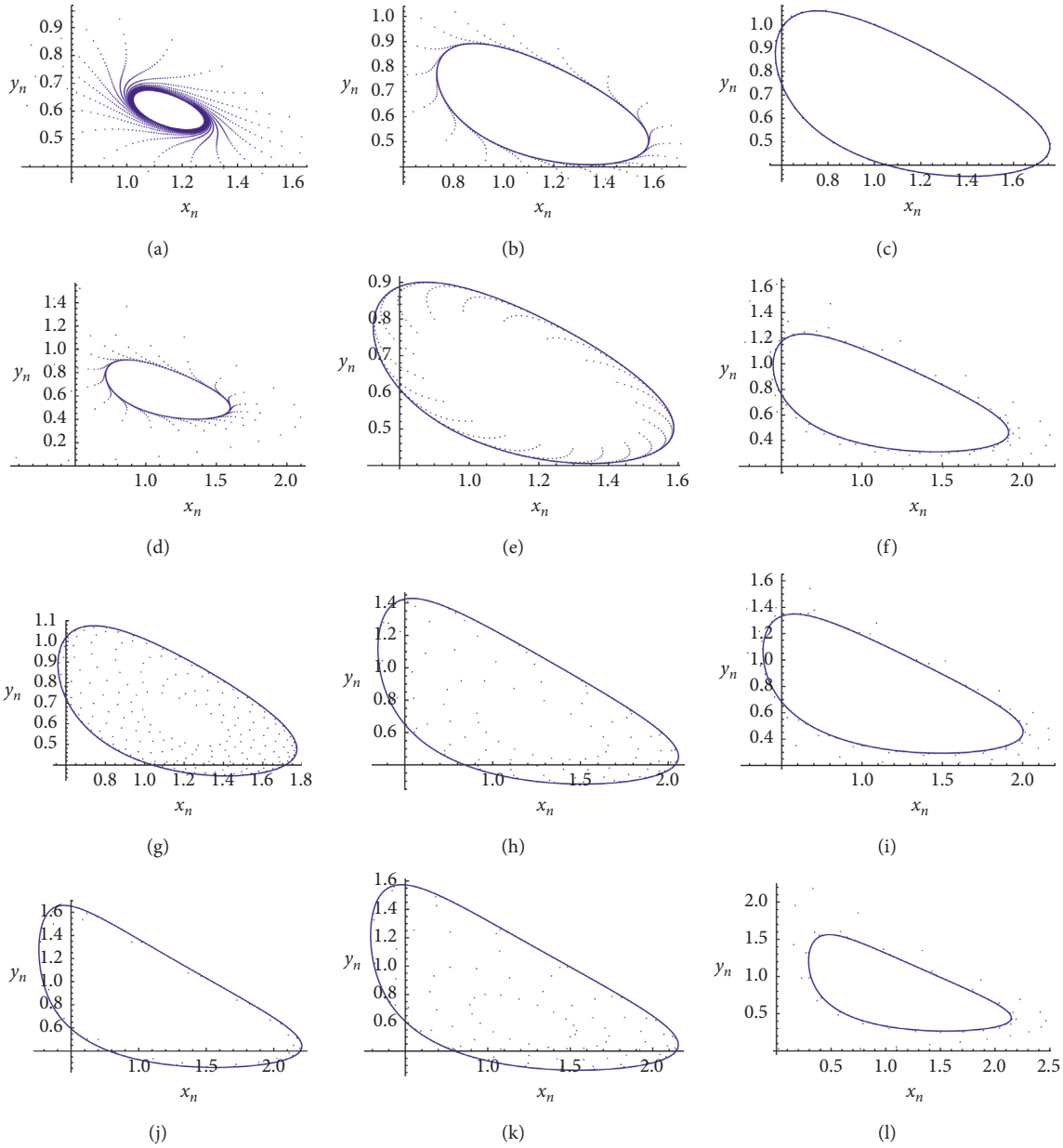


FIGURE 2: Supercritical N-S bifurcation of the discrete-time glycolytic oscillator model (4) about $P_{xy}^+(\alpha/(\beta + \alpha^2), \alpha)$: (a) $\beta = 0.16$ with (0.64, 0.62); (b) $\beta = 0.15$ with (0.64, 0.62); (c) $\beta = 0.14$ with (0.69, 0.6); (d) $\beta = 0.149$ with (0.081, 0.072); (e) $h = 0.1495$ with (0.9, 0.8); (f) $\beta = 0.13$ with (0.98, 0.19); (g) $\beta = 0.139$ with (0.99, 0.67); (h) $\beta = 0.12$ with (0.91, 0.72); (i) $\beta = 0.1239$ with (0.59, 0.35); (j) $\beta = 0.11$ with (0.7, 0.5); (k) $\beta = 0.1135$ with (1.5, 0.5); (l) $\beta = 0.114$ with (0.02, 0.03).

$$J|_{P_{xy}^+(\alpha/(\beta+\alpha^2),\alpha)} = \begin{pmatrix} \frac{c}{1+h\beta+h\alpha^2} + 1 - c & \frac{-2hc\alpha^2}{(\alpha^2 + \beta)(1+h(\beta+y^2))^2} \\ \frac{hc(\alpha^2 + \beta)}{1+h} & \frac{c(\alpha^2 + \beta + 2h\alpha^2)}{(1+h)(\beta + \alpha^2)} + 1 - c \end{pmatrix}. \quad (49)$$

The rest of this section deals with the study of N-S bifurcation of the controlled system (48) at $h = 0.7$, $\beta = 0.1601 < 0.16105922820694768$, and $\alpha = 0.6$. In this case, (49) becomes

$$J|_{P_{xy}^+(1.15362,0.6)} = \begin{pmatrix} 1 - 0.2669c & -0.710407c \\ 0.214159c & 1 + 0.158261c \end{pmatrix}, \quad (50)$$

whose auxiliary equation is

TABLE 2: Numerical values of Ω if $\beta < 0.16105922820694768$.

Value of β if $q > 1$	Numerical value of Ω
0.16	-0.029495873864781315 < 0
0.15	-0.024558009135300485 < 0
0.14	-0.025661100529415302 < 0
0.149	-0.02516944528172212 < 0
0.1495	-0.02461170028243595 < 0
0.13	-0.026828812841863014 < 0
0.139	-0.025774885406930285 < 0
0.12	-0.028066167307452507 < 0
0.1239	-0.027574969477341914 < 0
0.11	-0.029378827673349428 < 0
0.1135	-0.028910444972707013 < 0
0.114	-0.028844333111783357 < 0

$$\lambda^2 - p\lambda + q = 0, \quad (51)$$

where $p = 2 - 0.108492c$ and $q = 1 - 1.150755c + 1.152135c^2$. In view of Lemma 1 of [15], one gets

$$\begin{aligned} 1 - (2 - 0.108492c) + 1 - 1.150755c + 1.152135c^2 &> 0, \\ 1 - 1.150755c + 1.152135c^2 &= 1, \\ (2 - 0.108492c)^2 - 4(1 - 1.150755c + 1.152135c^2) &< 0. \end{aligned} \quad (52)$$

Let $c = 0.998$, then from (52), one gets

$$\begin{aligned} 1 - (2 - 0.108492c) + 1 - 1.150755c + 1.152135c^2 &= 0.107354 > 0, \\ 1 - (1.150755)c + (1.152135)c^2 &= 1, \\ (2 - (0.108492)c)^2 - 4(1 - (1.150755)c + (1.152135)c^2) &= -0.417687 < 0, \end{aligned} \quad (53)$$

which hold true. Also, $J|_{P_{xy}^+(1.15362, 0.6)} = (1.15362, 0.6)$ in terms of the bifurcation parameter β becomes

$$J|_{P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)} = \begin{pmatrix} 0.002 + \frac{0.998}{0.252 + 0.7\beta} & \frac{0.83832(0.6 + 0.42(0.36 + \beta))}{(1.252 + 0.7\beta)^2(0.36 + \beta)} \\ 0.410941(0.36 + \beta) & 0.002 + \frac{0.587059(0.864 + \beta)}{0.36 + \beta} \end{pmatrix}, \quad (54)$$

whose roots are

$$\lambda_{1,2} = \frac{1.69341 + 3.55991\beta + 0.70336\beta^2 \pm \sqrt{\Delta^{**}}}{2(0.766224 + 2.5568\beta + 1.19\beta^2)}, \quad (55)$$

where

$$\begin{aligned} \Delta^{**} &= (-1.69341 - 3.55991\beta - 0.70336\beta^2)^2 \\ &\quad - 4(0.224208 - 1.2023\beta - 1.1886\beta^2) \\ &\quad \cdot (0.766224 + 2.5568\beta + 1.19\beta^2). \end{aligned} \quad (56)$$

If $\beta = 0.1601$, then from (32), one gets

$$\left. \frac{d|\lambda_{1,2}|}{d\varepsilon} \right|_{\varepsilon=0} = 3.82461 > 0. \quad (57)$$

From (57), the nondegenerate condition holds, and hence, (48) undergoes N-S bifurcation at $\beta = 0.1601$. In terms of above chosen parametric values: $\beta = 0.1601$, $\alpha = 0.6$, $h = 0.7$, and $c = 0.998$, the controlled system (48) becomes

$$\begin{aligned} x_{n+1} &= 0.002x_n + \frac{(0.41916 + 0.998x_n)}{(1.11207 + 0.7y_n^2)}, \\ y_{n+1} &= 0.002y_n + 0.588235(0.6986x_n + 0.998y_n + 0.7x_ny_n^2), \end{aligned} \quad (58)$$

whose fixed point is $P_{xy}^+(1.15362, 0.6)$. Now, we transform $P_{xy}^+(1.15362, 0.6)$ into $P_{00}(0, 0)$ using the following transformation:

$$\begin{aligned} u_n &= x_n - 1.15362, \\ v_n &= y_n - 0.6, \end{aligned} \quad (59)$$

where (58) becomes

$$\begin{aligned} u_{n+1} &= 0.733634u_n - 0.967106v_n - 0.450543u_nv_n - 0.154225v_n^2 \\ &\quad + o\left(\left(|u_n| + |v_n|\right)^2\right), \\ v_{n+1} &= 0.213731u_n + 1.15794v_n + 0.493129u_nv_n + 0.474072v_n^2 \\ &\quad + o\left(\left(|u_n| + |v_n|\right)^2\right). \end{aligned} \quad (60)$$

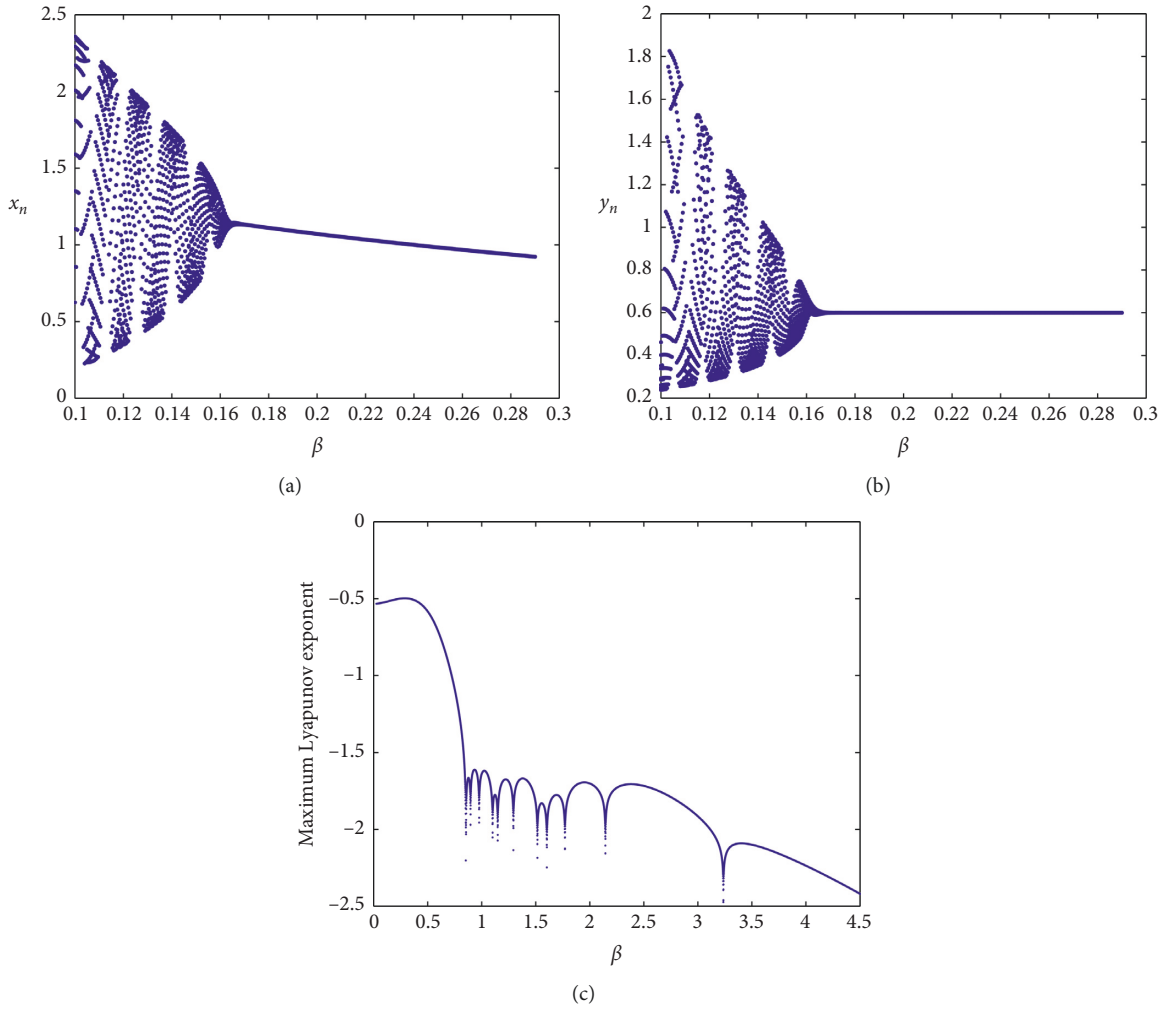


FIGURE 3: Bifurcation diagram and its corresponding maximum Lyapunov exponent of the model (4) about P_{xy}^+ ($\alpha/(\beta + \alpha^2)$, α). (a, b) Bifurcation diagram of the model if $\beta \in [0.1, 0.29]$ and initial condition (0.74, 0.62). (c) Maximum Lyapunov exponent corresponding to (a) and (b).

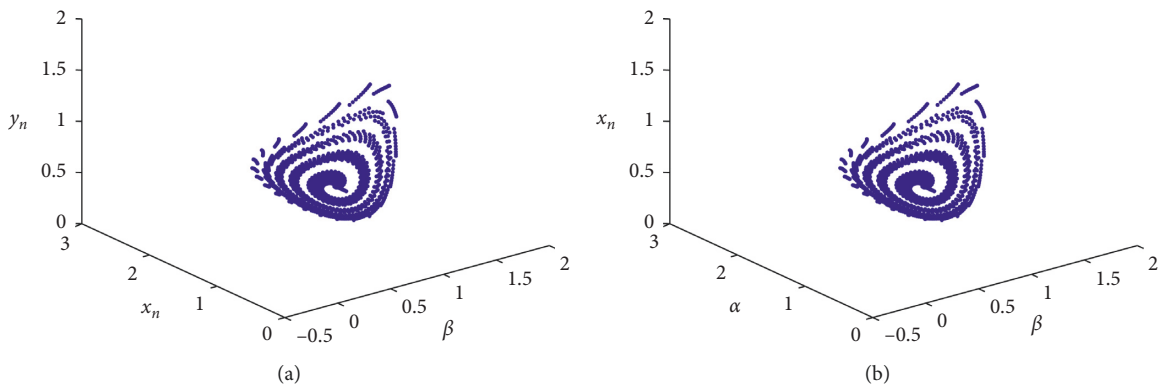


FIGURE 4: Continued.

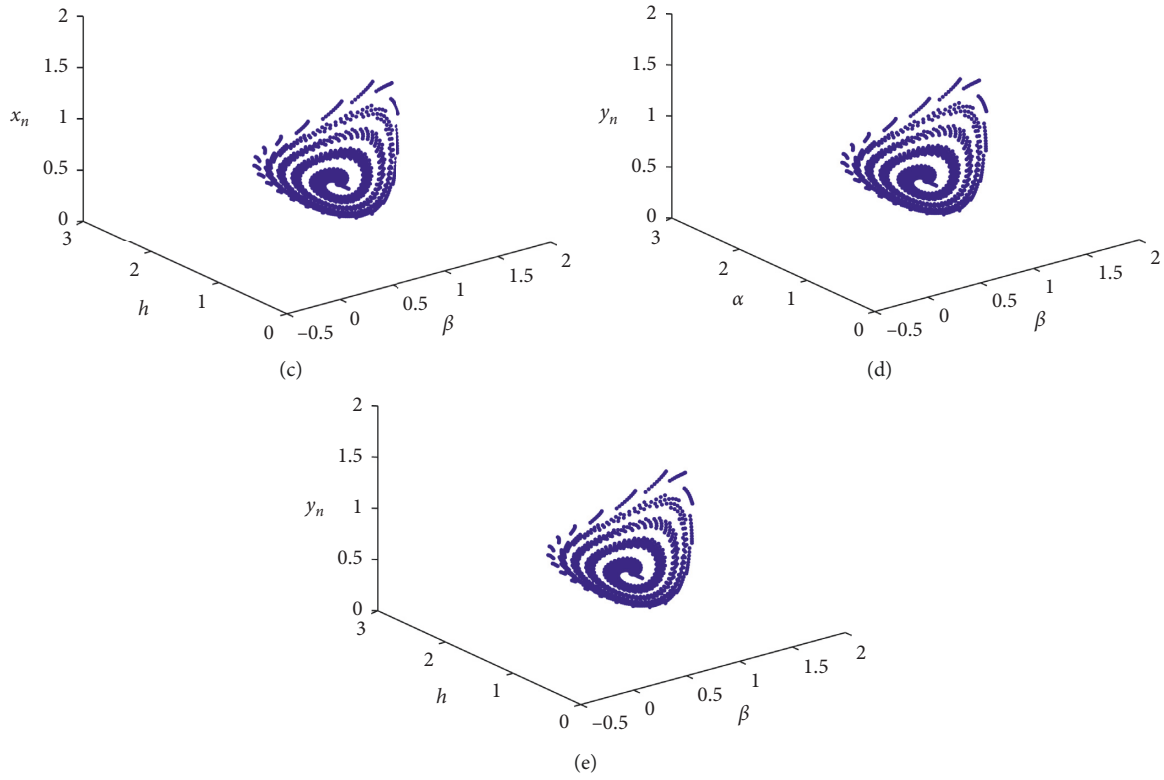


FIGURE 4: 3D bifurcation diagrams of the model (4).

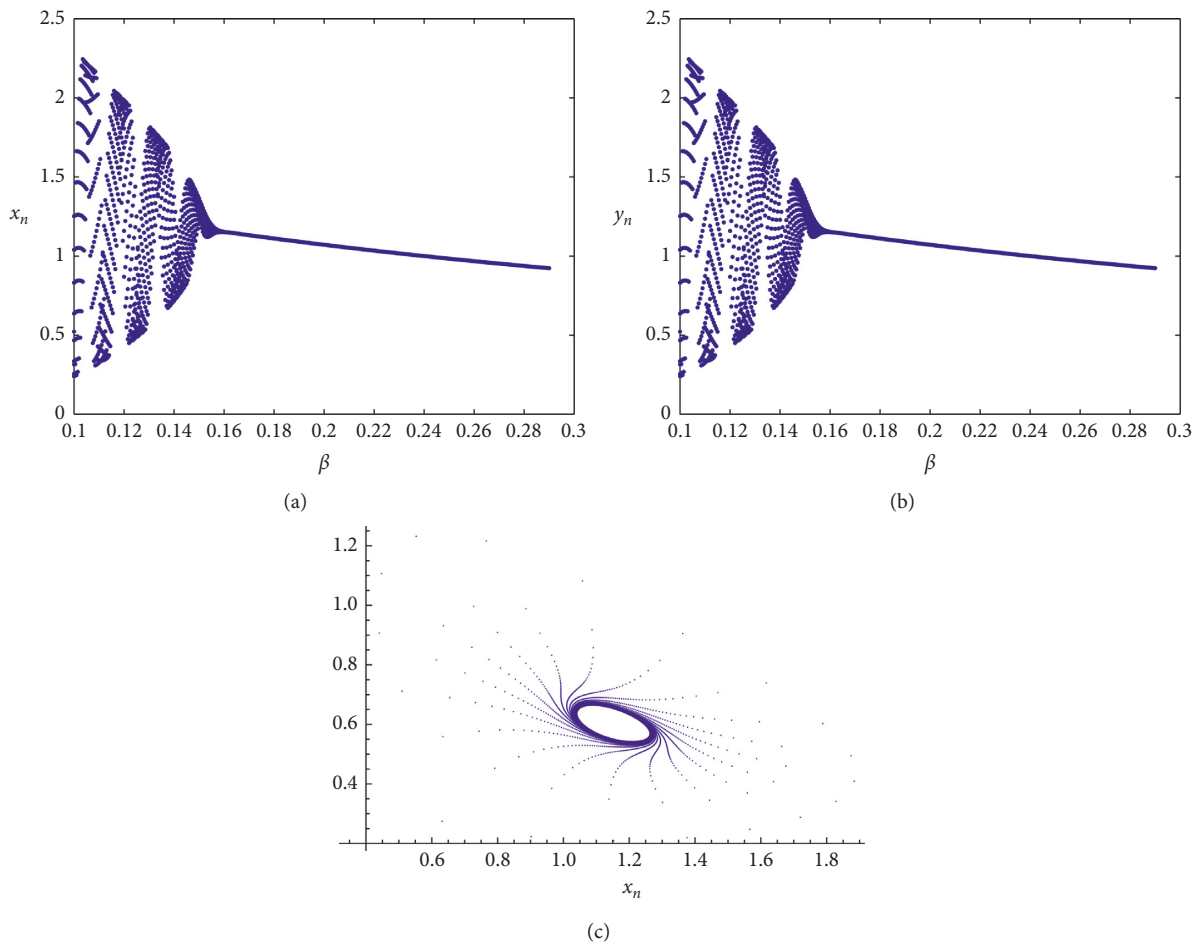


FIGURE 5: Control results of N-S bifurcation of the glycolytic oscillator model (4) about P_{xy}^+ ($\alpha/(\beta + \alpha^2)$, α) if $\alpha = 0.6$, $h = 0.7$, $c = 0.998$, and $\beta \in [0.1, 0.29]$.

TABLE 3: Fixed point P_{xy}^+ ($\alpha/(\beta + \alpha^2)$, α) along with qualitative behavior of the glycolytic oscillator model (4).

Equilibrium point	Corresponding behavior
P_{xy}^+ ($\alpha/(\beta + \alpha^2)$, α)	Sink iff $ (1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta)) / ((1+h(\alpha^2 + \beta))(1+h)(\beta + \alpha^2)) < 1 - ((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) < 2$
	Repeller iff $ (1+h)(\beta + \alpha^2) < 1 - ((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) > 1$ and $ (1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta)) / ((1+h)(\beta + \alpha^2)) > 1$
	Saddle iff $((1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta))) / ((1+h)(\beta + \alpha^2)) > 4(1 - ((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) > 0$ and $ (1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta)) / ((1+h)(\beta + \alpha^2)) > 1 - ((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) $
	Nonhyperbolic iff $ (1+h)(\beta + \alpha^2) + (\alpha^2 + \beta + 2h\alpha^2)(1+h(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))) = 1 - ((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) $
	Locally asymptotically stable focus if $((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) < 1$
	Unstable focus if $((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) > 1$
	Nonhyperbolic (for complex eigenvalues) if $((\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta)) / ((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2)))) = 1$

Since the linear part of (60) is just as $J|_{P_{xy}^+(\alpha/(\beta+\alpha^2),\alpha)} = (1.15362, 0.6)$ with eigenvalues,

$$\lambda_{1,2} = 0.6185299315830337 \pm 0.4140823777730747i. \quad (61)$$

Now, (61) takes the following form:

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.6185299315830337 & -0.4140823777730747 \\ 0.4140823777730747 & -0.6185299315830337 \end{pmatrix} \cdot \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \bar{P}(X_n, Y_n) \\ \bar{Q}(X_n, Y_n) \end{pmatrix}, \quad (62)$$

where

$$\begin{aligned} \bar{P}(X_n, Y_n) &= 0.0539722X_n^2 + 0.201764X_nY_n \\ &\quad + 0.0273435Y_n^2 + o\left(\left(|X_n| + |Y_n|\right)^2\right), \\ \bar{Q}(X_n, Y_n) &= -0.17732X_n - 0.00184395X_nY_n \\ &\quad - 0.203906Y_n^2 + o\left(\left(|X_n| + |Y_n|\right)^2\right), \end{aligned} \quad (63)$$

by

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} := \begin{pmatrix} -0.9671063257065947 & 0 \\ -0.11510407546939039 & -0.4140823777730747 \end{pmatrix} \cdot \begin{pmatrix} X_n \\ Y_n \end{pmatrix}. \quad (64)$$

In view of (63), from (45), one gets

$$\begin{aligned} \xi_{02} &= -0.012188390118933261 + 0.05708718392199485i, \\ \xi_{11} &= 0.04065784974095253 - 0.190612959926066i, \\ \xi_{20} &= 0.006196180814291302 - 0.04379461343161802i, \\ \xi_{21} &= 0. \end{aligned} \quad (65)$$

In view of (61) and (65), from (43), one gets $\Omega = -0.028844333111783357 < 0$. Finally, the controlled results of N-S bifurcation are plotted and drawn in Figure 5. For more interesting results for solving nonlinear equations using the multipoint iterative method, we refer the reader to [19] and references cited therein.

8. Conclusion

In the proposed work, we have explored the local dynamics, N-S bifurcation, and hybrid control in a discrete-time glycolytic oscillator model, which is depicted in (4), in the interior of \mathbb{R}_+^2 . It is proved that $\forall \alpha$ and β , the glycolytic oscillator model (4) has a unique +ve equilibrium point $P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)$. By the method of linearization, we have explored the local dynamics along with topological classification about $P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)$ of the discrete-time glycolytic oscillator model (4), and conclusions are presented in Table 3. We have also investigated the prime period and periodic points for the model (4). Furthermore, we have explored the existence of bifurcation about $P_{xy}^+(\alpha/(\beta+$

$\alpha^2), \alpha)$ and proved that the glycolytic oscillator model (4) undergoes no bifurcation, except N-S bifurcation, when h, α , and β go through the curve $N-SB_{P_{xy}^+}(\alpha/(\beta+\alpha^2), \alpha) = \{(h, \alpha, \beta): (\alpha^2 + \beta + 2h\alpha^2 + 2h^2\alpha^2(\alpha^2 + \beta))/((1+h)(\alpha^2 + \beta)(1+h(\beta + \alpha^2))) = 1\}$. We have performed the N-S bifurcation about $P_{xy}^+(\alpha/(\beta+\alpha^2), \alpha)$ by bifurcation theory. Moreover, the hybrid control strategy is applied to control the N-S bifurcation in the glycolytic oscillator model (4). Finally, theoretical results are verified numerically.

Data Availability

All the data utilized in this article have been included, and the sources from where they were adopted were cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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