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CONTINUED-FRACTION EXPANSIONS FOR THE RIEMANN ZETA FUNCTION AND POLYLOGARITHMS

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Abstract. It appears that the only known representations for the Riemann zeta function $\zeta(z)$ in terms of continued fractions are those for $z = 2$ and 3. Here we give a rapidly converging continued-fraction expansion of $\zeta(n)$ for *any* integer $n \geq 2$. This is a special case of a more general expansion which we have derived for the polylogarithms of order n, $n \geq 1$, by using the classical Stieltjes technique. Our result is a generalisation of the Lambert-Lagrange continued fraction, since for $n = 1$ we arrive at their well-known expansion for $\log(1 + z)$. Computation demonstrates rapid convergence. For example, the 11th approximants for all $\zeta(n)$, $n \geq 2$, give values with an error of less than 10^{-9}.

1. Introduction

Many analytic functions are known to have continued-fraction representations, a majority of them obtained by Euler, Gauss, Stieltjes and Ramanujan [1, Chapter 12]; [2, Vol. 2, Chapters 3 and 4]; [3, Chapters 18 and 19]; [4, Chapter 6]; [5, Appendix]. It appears that little is known about continued-fraction expansions of the Riemann zeta function ζ, and only the continued fractions for $\zeta(2)$ and $\zeta(3)$ are listed in the literature [1, pp. 150, 153 and 155]. See also the papers by Bradshaw [6, p. 390] and Nesterenko [7, Theorem 2, p. 868]. The expansion [1, p. 155]

$$\zeta(3) = 1 + \frac{1}{2 \times 2+}\,\frac{1^3}{1\ +}\,\frac{1^3}{6 \times 2+}\,\frac{2^3}{1\ +}\,\frac{2^3}{10 \times 2+}\cdots$$

found by Stieltjes also follows from the work of Apéry and was crucial to his famous proof that $\zeta(3)$ is irrational. In this work we show that the classical Stieltjes technique enables us to derive a continued-fraction expansion for the polylogarithms. Continued fractions for $\zeta(n)$, where $n \geq 2$ is an integer, then follow as a special case.

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2. Statement of the results

In what follows, a function depending on the parameter ν and defined by the Dirichlet power series

$$Li_\nu = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \tag{1}$$

is referred to as a polylogarithm. The series in (1) converges absolutely for all ν if $|z| < 1$ for Re$\nu > 0$ if $|z| = 1$ and $z \neq 1$, and for Re$\nu > 1$ if $z = 1$. It is known that the polylogarithm can be extended to the whole $\nu-$plane by means of a contour integral [8, p. 236, Eq. 7.189]. When $Li_\nu(z)$ is continued onto the $z-$plane, it has three branch points: $0, 1$ and ∞. Note that

$$Li_1(z) = -\log(1-z) \tag{2a}$$

and

$$Li_\nu(1) = \zeta(\nu), \qquad Li_\nu(-1) = (2^{1-\nu} - 1)\zeta(\nu) \tag{2b}$$

for $\nu \neq 1$.

The polylogarithm Li_n of order $n \geq 1$ is thoroughly covered in Lewin's standard text [8], while many formulae involving $Li_\nu(z)$ can be found in Vol. 3 of "Integrals and Series" by Prudnikov, Brychkov and Marichev [9, Vol. 3, Appendix II.5, pp. 762-763].

An (infinite) *continued fraction*

$$K(a_k/b_k) = \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k} = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}} \tag{3a}$$

is said to converge if the sequence of its approximants

$$F_n = \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k}{b_k} = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots \frac{a_n}{+ b_n} = \frac{A_n}{B_n} \qquad \text{for } n = 1, 2, 3, \ldots \tag{3b}$$

(where F_n is nth *approximant* while A_n and B_n are the nth *numerator* and *denominator*, respectively) converges in $C \cup \{\infty\}$. The value of the continued fraction is then $F = \lim_{n\to\infty} F_n$. We say that $K(a_k/b_k)$ diverges if this limit does not exist. The numbers $a_k \in C \setminus \{0\}$ and $b_k \in C$ for all k are known as the kth *partial numerator* and *denominator*, or simply as *elements* of the continued fraction.

Classical texts on the analytical theory of continued fractions are those of Perron [2] and Wall [3], while a modern treatment of the topic can be found in Jones and Thron [4], Henrici [10, Chapter 12] and Lorentzen and Waadeland [5]. A good review of the theory is given by Baker [11, Chapters 4 and 5], while Chapter 12 of Berndt's treatise [1] can serve as excellent introductory text and an encyclopaedic source.

Our results are as follows.

Theorem. *Suppose that r is a non-negative integer and m and n are positive integers. For any fixed r, m and n, define $A_m^{(r)}(n)$ as the determinant of an $m \times m$ matrix*

$$A_m^{(r)}(n) = \det \left\| \frac{(-1)^{i+j+r}}{(r+i+j-1)^n} \right\|_{1 \leq i,j \leq m}. \tag{4}$$

It is assumed that $A_0^{(r)}(n) = 1$*. Let* $Li_n(z)$ *and* $\zeta(z)$ *be the polylogarithm of order* n *and the Riemann zeta function, respectively.*

We have

(5a) $$-Li_n(-z) = \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_{n,k}z}{1} = \frac{a_{n,1}z}{1} {}_{+} \frac{a_{n,2}z}{1} {}_{+} \frac{a_{n,3}z}{1} {}_{+} \cdots$$

where the continued fraction converges and represents a single-valued branch of the analytic function on the left for all complex z outside the cut $(-\infty, -1]$. The elements of the continued fraction are given by

(5b)
$$a_{n,1} = 1, \quad a_{n,2m} = -\frac{A_m^{(1)}(n)A_{m-1}^{(0)}(n)}{A_m^{(0)}(n)A_{m-1}^{(1)}(n)}, \quad a_{n,2m+1} = -\frac{A_{m-1}^{(1)}(n)A_{m+1}^{(0)}(n)}{A_m^{(0)}(n)A_m^{(1)}(n)}.$$

In particular, when $z = 1$ we have

(5c)
$$\log 2 = \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_{1,k}}{1} \qquad \text{and} \qquad \zeta(n) = \frac{1}{(1-2^{1-n})} \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_{n,k}}{1}, \qquad n \geq 2.$$

Note. It is easily seen from the proof of the Theorem (Section 3) that it can be extended to any real $n \geq 1$. Moreover, it can be extended to the Lerch and the Hurwitz zeta functions. However, in this work we limit ourselves only to the case given in the Theorem.

3. Proof of the Theorem

Throughout this section, for each pair (a, b), such that $-\infty \leq a \leq b \leq \infty$, we let $\Phi(a, b)$ denote the family of all real-valued, bounded, monotone non-decreasing functions $\phi(t)$ with infinitely many points of increase on $a \leq t \leq b$.

In order to prove the Theorem we shall use the result due to Markov (1895), here stated without proof, in the form given by Jones and Thron [4, p. 344]. The proof can be found in Perron [2, Vol. 2, pp. 198-202]. This result conveniently sums up the classical analytical theory of continued fractions, developed mainly by Stieltjes. Certain necessary details related to the Markov Theorem are further discussed in the Notes below. First, preliminary definitions are required.

Two continued fractions $K(a_k/b_k)$ and $K(a_k^*/b_k^*)$ with nth approximants F_n and F_n^*, respectively, are said to be *equivalent*, which is denoted by $K(a_k/b_k) \cong K(a_k^*/b_k^*)$, if $F_n = F_n^*$ for $n = 1, 2, \ldots$, i.e. if they have the same sequence of approximants.

A *regular C-fraction* (regular corresponding fraction) is a continued fraction of the form

(6) $$\frac{a_1 z}{1} {}_{+} \frac{a_2 z}{1} {}_{+} \frac{a_3 z}{1} {}_{+} \frac{a_4 z}{1} {}_{+} \cdots, \qquad a_k \neq 0 \qquad \text{for } k = 1, 2, \ldots.$$

If $a_k > 0$ for all k, then (6) is called the *S-fraction* (or the Stieltjes fraction).

A *modified regular C-fraction* is a continued fraction of the form

(7) $$\frac{a_1}{1} {}_{+} \frac{a_2}{1} {}_{+} \frac{a_3}{1} {}_{+} \frac{a_4}{1} {}_{+} \cdots, \qquad a_k \neq 0 \qquad \text{for } k = 1, 2, \ldots.$$

If $a_k > 0$ for all k, then (7) is called the *modified S-fraction.*

For a given sequence $\{c_k\}_{k=0}^{\infty}$ the *Hankel determinants* $H_m^{(r)}$ (of dimension m, $m = 1, 2, 3, ...$) *associated* with sequence are defined by

(8)

$$H_0^{(r)} = 1, \qquad H_m^{(r)} = \begin{vmatrix} c_r & c_{r+1} & \cdots & c_{r+m-1} \\ c_{r+1} & c_{r+2} & \cdots & c_{r+m} \\ & & & \\ c_{r+m-1} & c_{r+m} & \cdots & c_{r+2m-2} \end{vmatrix} \qquad (r = 0, 1, 2, ...).$$

A continued fraction

$$\mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k(z)}{b_k(z)} = \frac{a_1(z)}{b_1(z)+}\frac{a_2(z)}{b_2(z)+}\frac{a_3(z)}{b_3(z)+}\cdots$$

with the nth approximant $F_n(z)$, is said to correspond to the series $\sum_{k=0}^{\infty} c_k z^{-k}$ at $z = \infty$, if the following formal power series expansions are valid:

$$F_n(z) - \sum_{p=0}^{\lambda_n} c_p z^{-p} = \text{const } z^{-(\lambda_n+1)} + ... \qquad (n = 1, 2, 3, ...)$$

where $\lambda_n \to \infty$ as $n \to \infty$.

The Markov Theorem. *If $\phi \in \Phi(0, a), a > 0$, then the modified S-fraction (see Eq. 7 with $a_k > 0$) which corresponds to the series*

$$\text{(9a)} \qquad \sum_{k=0}^{\infty} (-1)^k \mu_k z^{-k} \qquad \textit{with} \qquad \mu_k = \int_0^a t^k d\phi(t)$$

(where μ_k is the kth moment of ϕ) at $z = \infty$, converges to the function

$$\text{(9b)} \qquad \int_0^a \frac{z d\phi(t)}{z + t}$$

for all $z \in C$ such that $z \notin [w : -a \leq w \leq 0]$.

Notes (a) The integral of the form

$$\int_a^b f(x) du(x)$$

(see Eq. 9(a) and (b)) is called the Stieltjes-Riemann integral of f with respect to u. The properties of such integral and the various conditions under which it exists are discussed in detail by, for instance, Rudin [12, Chapter 6].

(b) Let $\phi \in \Phi(0, \infty)$. Then the following Stieltjes-Riemann integral

$$f(z) = \int_0^{\infty} \frac{d\phi(t)}{z + t}$$

always exists, and f is analytic in the cut $z-$plane with the cut along the real axis from $-\infty$ to 0. It is said that f and ϕ form the Stieltjes transform pair (or that f is the *Stieltjes transform* of ϕ), which is denoted by $f = \mathcal{S}\{\phi\}$. If ϕ is constant on $a \leq t < \infty$, then $f = \mathcal{S}\{\phi\}$ is given by [10, p. 160]

$$f(z) = \int_0^a \frac{d\phi(t)}{z + t}, \qquad z \notin [w : -a \leq w \leq 0].$$

(c) The phrase "with infinitely many points of increase" (or "taking infinitely many different values" in the definition of $\Phi(a, b)$ is used in order to exclude the case when ϕ is a piecewise constant function on $[a,b]$, i.e. it "takes finitely many different values". It is thus ensured that $f = \mathcal{S}\{\phi\}$ cannot be a rational function.

Proof of the Theorem. The proof is based on the Markov Theorem. First, we assert that all ϕ_n defined by

(10)
$$\phi_n(t) = \begin{cases} 0, & t = 0, \\ \frac{1}{(n-1)!}\int_0^t \{\log(1/x)\}^{n-1}\,dt, & 0 < t \le 1, \\ \phi_n(1) = 1, & t > 1, \end{cases} \qquad (n = 1,2,3,...),$$

belong to the class $\Phi(0,\infty)$. Clearly, the integral in (10) exists for $n = 1$. When $n \ge 2$, we have [9, Vol. 1, p. 241, Entry 1.6.1.14]

$$\int_\varepsilon^t \{\log(1/x)\}^{n-1}\,dx = \sum_{k=0}^{n-1}(-1)^k\frac{(n-1)!}{k!}\left(t\log^k t - \varepsilon\log^k\varepsilon\right)$$

for any real ε and $t, 0 < \varepsilon < t$. Since

$$\lim_{\varepsilon\to 0}\varepsilon\log^k\varepsilon = 0$$

the existence of the integral in (10) is assured. Note that all $\phi_n(t)$ are continuous on the right at 0. It now follows without difficulty that all $\phi_n(t)$, when $t \in [0,1]$, are strictly increasing and that $0 \le \phi_n(t) \le 1$. Further, the condition concerning taking infinitely many different values is met. Thus, $\phi_n(t) \in \Phi(0,1)$. On $[0,\infty)$ all ϕ_n are non-decreasing and bounded and $\phi_n \in \Phi(0,\infty)$, as stated.

Second, let ϕ_n be given by (10). Then f_n, the Stieltjes transform of ϕ_n,

(11)
$$\begin{aligned} f_n(z) &= \int_0^\infty \frac{d\phi_n(t)}{z+t} = \frac{1}{(n-1)!}\int_0^1 \frac{\{\log(1/t)\}^{n-1}\,dt}{z+t} \\ &= -Li_n(-1/z) \qquad (n = 1,2,3,...) \end{aligned}$$

is valid for all $z \in C$ such that $z \notin [w : -1 \le w \le 0]$. Indeed, the transform of ϕ_n exists and the Stieltjes-Riemann integral reduces to the ordinary Riemann integral. This integral, on making use of the substitution $x = \log(1/t)$ in [8, p. 312, Entry A.3.8. (2)],

$$Li_n(z) = \frac{z}{(n-1)!}\int_0^\infty \frac{x^{n-1}dx}{e^x - z} \qquad (n = 1,2,...; |\arg(1-z)| < \pi)\,,$$

can be easily evaluated. Note that the Stieltjes pair in (11) is not listed either in the Table of Stieltjes transforms in Erdélyi et al. [13, Chapter 14] or in Lewin's book [8, Appendix 3, p. 303].

Third, let f_n be the Stieltjes transform of ϕ_n given by (11). Then by virtue of (9a) we have

(12a)
$$zf_n(z) = \sum_{k=0}^\infty c_{n,k}(1/z)^k, \qquad |z| > 1 \qquad (n = 1,2,3,...)$$

with

(12b)
$$c_{n,k} = \frac{(-1)^k}{(k+1)^n} = (-1)^k\mu_{n,k} = (-1)^k\frac{1}{(n-1)!}\int_0^1 t^k\{\log(1/t)\}^{n-1}\,dt$$

where the $\mu_{n,k}$ are the moments of ϕ_n. This follows trivially from the series representation of the polylogarithm in (1).

Fourth, let $a_{n,k}$ be given by (5b) and (4). Then

$$-zLi_n(-1/z) = \frac{a_{n,1}}{1} \underset{+}{} \frac{a_{n,2}}{z} \underset{+}{} \frac{a_{n,3}}{1} \underset{+}{} \frac{a_{n,4}}{z} \underset{+}{} ..., \qquad a_{n,k} > 0, \tag{13}$$

is valid for $z \in C$ such that $z \notin [w : -1 \leq w \leq 0]$. In order to show this, recall that, in general, if for given a formal power series $\sum_{k=0}^{\infty} c_k z^{-k}$ ("formal", since the series may not converge anywhere except at $z = \infty$) with the sequence of coefficients $\{c_k\}_{k=0}^{\infty}$ there exists a modified C-fraction (see Eq. 7) which corresponds to the series (at $z = \infty$), then the elements a_k are given by [4, p. 226]

$$a_1 = c_0, \qquad a_{2m} = -\frac{H_m^{(1)} H_{m-1}^{(0)}}{H_m^{(0)} H_{m-1}^{(1)}}, \qquad a_{2m+1} = -\frac{H_{m-1}^{(1)} H_{m+1}^{(0)}}{H_m^{(0)} H_m^{(1)}}, \qquad (m = 1, 2, 3, ...)$$

where the $H_m^{(r)}$ are the Hankel determinants associated with the sequence $\{c_k\}_{k=0}^{\infty}$ (see Eq. 8). In our case, there exists a modified C (in fact S, i.e. $a_{n,k} > 0$) fraction (the existence and convergence are assured by the Markov Theorem) which corresponds to the power series in (12). Further, the Hankel determinants associated with $\{c_k\}_{k=0}^{\infty}$ in (12b) are exactly those defined by (4). In this way, the representation in (13) follows.

Finally, knowing that the following two continued fractions are equivalent [4, pp. 386-387]

$$\frac{a_1(1/z)}{1} \underset{+}{} \frac{a_2(1/z)}{1} \underset{+}{} \frac{a_3(1/z)}{1} \underset{+}{} \frac{a_4(1/z)}{1} \underset{+}{} ... \cong (1/z) \frac{a_1}{1} \underset{+}{} \frac{a_2}{z} \underset{+}{} \frac{a_3}{1} \underset{+}{} \frac{a_4}{z} \underset{+}{} ...,$$

it is easy to obtain from (13) the proposed formula in (5a). The particular case (5c) involving the logarithm and the Riemann zeta function is a straightforward conseqence of (2) and (5a) for $z = 1$. This completes the proof of the Theorem. $\square$

4. Concluding remarks

Since C-fraction expansions are unique, for $n = 1$ our continued fraction in (5a) must be [10, p. 534]

(14a)

$$-Li_1(-z) = \log(1 + z) = \frac{z}{1} \underset{+}{} \frac{z/2}{z} \underset{+}{} \frac{z/6}{1} \underset{+}{} \frac{2z/6}{1} \underset{+}{} \frac{2z/10}{1} \underset{+}{} \frac{3z/10}{z} \underset{+}{} ...$$

which by an equivalence transformation may be rewritten in the form

$$\log(1 + z) = \frac{z}{1} \underset{+}{} \frac{1^2 z}{2} \underset{+}{} \frac{1^2 z}{3} \underset{+}{} \frac{2^2 z}{4} \underset{+}{} \frac{2^2 z}{5} \underset{+}{} \frac{3^2 z}{6} \underset{+}{} ... . \tag{14b}$$

This is a well-known expansion which goes back to Lambert (1770) and Lagrange (1776) [3, p. 342]. Note that the expression for $\log(1+z)$ equivalent to (14b) given by Jones and Thron [4, p. 203] contains a misprint.

It is easy to demonstrate that the continued fraction in (14a) follows from our equations in (5). First, the following

$$\mathcal{H}_m^{(r)} = \det \left\| \frac{1}{r + i + j - 1} \right\|_{1 \leq i,j \leq m} = \prod_{k=0}^{m-1} \frac{(k!)^2}{(k + r + 1)_m} \quad \neq -1, -2, ..., -(2m - 1)$$

where $(...)_m$ stands for the Pochhammer symbol is the well-known determinant of the *generalised Hilbert matrix* [14, pp. 98-99 and 300]. Next, let the (i, j)

entry of two $m \times m$ matrices be α_{ij} and $\beta_{ij} = (-1)^{i+j}\alpha_{ij}$, respectively. Then, $\det\|\beta_{ij}\| = \det\|\alpha_{ij}\|$ [15, p. 8, Entry 1.23 and p. 29]. Now, since it is easily seen that there exists the relation $A_m^{(r)}(1) = (-1)^{rm}\mathcal{H}_m^{(r)}$, we have

$$A_m^{(0)}(1) = \frac{[1!2!\cdots(m-1)!]^4}{1!2!\cdots(2m-1)!}, \qquad A_m^{(1)}(1) = \frac{[1!2!\cdots(m-1)!]^4\,[m!]^2}{1!2!\cdots(2m-1)!(2m)!}.$$

By (5b) the elements $a_{1,k}$ are

$$a_{1,1} = 1, \qquad a_{1,2m} = \frac{m}{2(2m-1)}, \qquad a_{n,2m+1} = \frac{m}{2(2m+1)},$$

and we arrive at (14a).

Unfortunately, it appears that when $n \geq 2$ the closed-form evaluation of the determinants $A_m^{(r)}(n)$ is unknown, and thus our $a_{n,k}$ are not given explicitly. This is very similar to the case of the Stieltjes expansion for the gamma function [4, pp. 348-350] where the elements in the continued fraction are also unknown.However, this difficulty can be overcome numerically. First, the various programs for symbolic manipulation and computation, such as Mathematica (Wolfram Research), enable an easy computation of these determinants up to order 10–15. We have used Mathematica to compute the elements given in Table 1. Second, the determinants can be avoided altogether by making use of the "qd algorithm" [4, p. 227]. Thus for any n the corresponding sequences $\left\{q_m^{(0)}\right\}_{m=1}^{\infty}$ and $\left\{e_m^{(0)}\right\}_{m=1}^{\infty}$ in

$$a_{n,1} = 1, \qquad a_{n,2m} = -q_m^{(0)}, \qquad a_{n,2m+1} = -e_m^{(0)}$$

can be computed by the rhombus rules.

Table 1. Elements a_{nk} of the continued fraction in (5a) for several values of n (rows) and k (columns).

n\k	**1**	**2**	**3**	**4**	**5**	**6**
1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{3}{10}$
2	1	$\frac{1}{4}$	$\frac{7}{36}$	$\frac{17}{63}$	$\frac{647}{2975}$	$\frac{294777}{1099900}$
3	1	$\frac{1}{8}$	$\frac{37}{216}$	$\frac{217}{999}$	$\frac{30271}{143375}$	$\frac{1566514917}{6568807000}$
4	1	$\frac{1}{16}$	$\frac{175}{1296}$	$\frac{493}{2835}$	$\frac{2081687}{10784375}$	$\frac{10844839670553}{51313584550000}$
5	1	$\frac{1}{32}$	$\frac{781}{7776}$	$\frac{26281}{189783}$	$\frac{10916749081}{64142065625}$	$\frac{764501700472728669}{4098615465682300000}$

Preliminary numerical experiments with continued fractions in (5) show rapid convergence. For example, the 11th approximants for all $\zeta(n), n \geq 2$, give values with an error of less than 10^{-9}.

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